UNIVERSIDADE FEDERAL DE ITAJUBÁ - UNIFEI
GRADUATION PROGRAM IN PHYSICS

Analogue models of classical and semiclassical gravity

Lucas Tobias de Paula

Itajubá, September 28, 2022

UNIVERSIDADE FEDERAL DE ITAJUBÁ - UNIFEI
GRADUATION PROGRAM IN PHYSICS

Lucas Tobias de Paula

Analog models of classical and semiclassical gravity

Dissertation submitted to the Physics Graduation Program as a requirement for the Master's of Science in Physics Degree.

Major: Quantum Field Theory, Cosmology and Gravitation

Supervisor: Prof. PhD Vitorio Alberto De Lorenci
Co-supervisor: Prof. PhD Alessandro Fabbri

September 28, 2022
Itajubá

Acknowledgements

I am thankful to my family, for continuing to be my refuge in my moments of difficulty.

I thank my great master Prof. Altamiro Franco Filho, also known as Milo, without whom I would never have found the right path to follow on my journey. I also thank my supervisor Prof. Vitorio De Lorenci, for believing in my potential from the beginning and for continuing to help me, with patience and wisdom, even in my moments of fragility in the last two years. Last but not least, I thank my co-supervisor Prof. Alessandro Fabbri for accepting to receive me on my first adventure beyond the Atlantic ocean and for his kind hospitality in the lands of Dom Quixote. This period contributed to many discoveries, not only in the scientific field, but also in the personal sphere.

Special thanks to the professors of this graduation program: Prof. Gabriel Hidalgo, Prof. Eduardo Bittencourt and Prof. Fabrício Barone. Their professionalism at the pandemic times was proof of their vocation.

This project was financially supported by CAPES and by the Federal University of Itajubá, UNIFEI.

"Lend me your ears and I'll sing you a song, and I'll try not to sing out of key..."

Lennon-McCartney

Abstract

Formal analogies between gravitational and acoustic or optical phenomena have been a subject of study for over a century, leading to interesting scenarios for testing kinematic aspects of general relativity in terrestrial laboratories. Here, some aspects about analog models of gravity obtained from the description of these two different kind of systems are analysed. First, light propagation in linear magnetoeletric media is examined. In particular, it is shown that this effect produces mixed time-space terms in the effective metric that depend only on the antisymmetric part of the generally non-symmetric magnetoelectric coefficient. Furthermore, the dispersion relation related to the linear effect motivates the analysis of an idealised exact model presenting an analog event horizon. Then, a short discussion comparing different ways of constructing analog models is provided. Subsequently, motivated by the results obtained in the optical context, we make a bibliographic review about those analog models obtained from moving media, establishing an equivalence between the propagation of acoustic perturbations in such a background and the propagation of free scalar fields near a Schwarschild black hole. This last aspect drives us to analyse the particle production in this scenario, a result that was first addressed by Stephen Hawking [1, 2], which yields to the the description of the so called Hawking radiation. When treating a non-stationary spacetime, particularly those presenting a gravitational collapse, we can extend the description of quantum fields to curved spacetimes by splitting the metric into two asymptotically stationary regions, with that we show that the presence of the horizon is fundamental for the creation of particles. Finally, it is also shown that the thermal distribution of this particle emission is identical to the Planck distribution for bosons, and because of that the Hawking temperature appears to be very small when we consider astrophysical scenarios.

Key-words: Linear magnetoeletrics, Analogue event horizon, Acoustic black holes, Quantum field theory in curved spacetime, Hawking radiation.

List of Figures

Table of Contents

Introduction

analogue models of General Relativity (GR) solutions have been a subject of investigation since the beginning of the 20th century, when Gordon originally studied light propagation in material media and reinterpreted the refractive index of a medium by means of an effective geometric description [3]. Throughout the years, the possibility of creating analogs for GR spacetime geometries in laboratory was extensively studied not only in the realm of electromagnetism [4, 5, 6, 7], but also in the context of acoustic waves and condensed matter systems [8, 9]. Models containing an event horizon in Bose-Einstein condensates have also been frequently examined [10, 11], which includes the analysis of analogue Hawking's radiation phenomenon. In addition, analogue models seem to be an interesting tool to test GR metric solutions related to controversial predictions, such as those containing closed time-like curves [12, 13] (even though quantum physics suggests that such possibilities are forbidden [14, 15]).

Solutions for the propagation of a light ray in a material medium are obtained from Maxwell's equations together with certain constitutive relations. Such relations depend on each specific medium and are related to the way the material is polarized or magnetized by means of external applied fields. The fundamental equation governing the propagation of the light rays is the dispersion relation connecting the wavevector to the frequency of the propagating wave. From this relation, formal analogies between light propagating in the optical material and in a curved spacetime can be established.

Recent advances in the science and technology of optical materials, which includes magnetoelectrics [16, 17, 18, 19] and metamaterials [20], have opened a new window to investigate analogue models based on electromagnetism. In particular, in a magnetoelectric material, the polarization phenomenon can be induced by a magnetic field, and magnetization can be induced by an electric field, or both together. In this project we begin by investigating some aspects of analogue models based on light propagation in material media, with particular interest in linear magnetoelectric media. The analysis is restricted to the regime of lossless and dispersionless systems, which consist of materials whose delay in their response to external electromagnetic perturbations is negligible. It is assumed that the total electromagnetic fields can be split in two contributions: a strong and slowly varying part, mainly responsible for activating the polarization and magnetization of the material, and a weak and rapidly varying field, which is the one that propagates in the medium. analogue models based on materials whose magnetoelectric coefficient is generally non-symmetric are thus constructed and effective geometries describing curved spacetimes are obtained. In particular, metrics with nonzero time-space components, g_{0i}, are investigated. Motivated by the behavior of light rays in a model based on a linear magnetoelectric system, an idealized toy model exhibiting an analogue event horizon is discussed.

We also investigate analogue models obtained from fluid motion, which are directly related to the Hawking radiation effect. Since the first description of the evaporation process of black holes made by Stephen Hawking [1, 2], acoustic black holes became an effervescent area of investigation. Beginning with the early studies of Unruh about experiments measuring Hawking radiation in the laboratory [21], many other theoretical studies have been done about this issue [22, 23], together with its experimental analysis [24]. The difficulty of measuring this phenomenon in astrophysical scenarios led us to many attempts of constructing "analogue black holes". The fundamental aspect behind the so called "analogue models" is to establish an equivalence between the propagation of perturbations near astrophysical structures, as for instance, black holes (but also spinning strings, wormholes, etc), and the propagation of sound waves or light rays in fluids or optical materials, respectively. The comparison in fluid motion is done between sound waves in acoustic systems and quantum fields propagating near a Schwarzschild black hole. Among the subjects addressed in this project are the analogue models constructed with acoustic systems, with particular interest in the relationship between them and Hawking radiation. We analyse the behavior of linear perturbations in a homentropic and irrotational flow, and with that we establish the connection with a massless scalar field propagating freely near a Schwarzschild black hole, written in a special set of coordinates, called Painlevè-Gullstrand (PG) coordinates [25, 26]. Henceforth, we turn our attention to the quantum aspects involving these systems, which leads to the description, by means of the tools of Quantum Field Theory (QFT) in curved spacetimes, of the Hawking radiation effect.

In the next Chapter we analyse the analogue models obtained from light propagation in linear magnetoelectric materials. In particular, the analysis based on first-order linear magnetoelectric effect leads to a dispersion relation that motivates the investigation of an exact model for which a variety of optical effects are present. Moreover, the behavior of the light rays in such a hypothetical system anticipates the existence of an analogue event horizon solution. analogue models based on linear magnetoelectric effects are thus constructed. It is shown that, when only the first order contribution to the effect is considered, the solution for the phase velocity leads to an effective metric having mixed time-space components, which includes stationary solutions of GR. Such mixed components are essentially related to the magnetoelectric properties of the medium. In this case it is shown that only the antisymmetric part of the magnetoelectric coefficient α_{ij} takes place in the effective geometry, and it appears only in its mixed time-space sector. Additionally, we examine a toy model for an optical event horizon based on the idealized model.

In Chap. 2 a selfcontented review on acoustic black holes is adressed. It is shown for linear perturbations that the equations of motion (EOM) describing these perturbative quantities can be written in a form identical to the d'Alembert equation for propagation of a free massless scalar field, by means of a background acoustic metric. Depending on the fluid motion one can construct a *sonic event horizon*, which behaves exactly like an event horizon of a Schwarzschild black hole. In Chap. 3 we describe the particle production in a simple system where the formation of the black hole is driven by collapse of incoming radiation. Then, using tools of QFT in curved spacetimes, such as the Bogoliubov transformation, we are able to describe the different field modes, one related to the Minkowski geometry (infinite past region) and the other related to the Schwarzschild black hole (infinite future region). It is shown that the vacuum states of each region are different, and this difference yields a process of particle emission during the transition of the metric. These particles are then shown to follow the *Planck thermal distribution* and the Hawking temperature is derived. Finally, we discuss the results and the consequences of the phenomenon of particle emission by a black hole, and how it is related to the investigation of analogue models in a variety of contexts, such as optical materials and Bose-Einstein condensates.

Throughout the text Greek indices α, β, γ... run from 0 to 3 (spacetime indices) while Latin indices i, j, k... run from 1 to 3 (the three spatial directions) and the Einstein convention for sum is used, i.e., repeated indices in a monomial indicate summation. Partial and covariant derivatives with respect to coordinate x^μ is denoted, respectively, by a comma and a semicolon followed by the corresponding μ index. In Galilean coordinates, the three-dimensional Levi-Civita symbol ϵ_{ijk} is a completely antisymmetric object defined by $\epsilon_{123} = 1$. The components of the identity matrix (the Kronecker delta) is represented by δ_{ij}. Parentheses encompassing two indices mean symmetrization, whether or not those indices belong to the same object. For instance, $t_{(\mu\nu)} = (1/2)\,(t_{\mu\nu} + t_{\nu\mu})$, for any rank-2 tensor $t_{\mu\nu}$. Similarly, square brackets will be used to indicate antisymmetrization as, for instance, $t_{[\mu\nu]} = (1/2)\,(t_{\mu\nu} - t_{\nu\mu})$. Natural units $c = 1$ and $G = 1$ are used throughout the text.

1 Optical analog models

In this chapter we analysed the propagation of light rays through material media and show how it can be connected with an effective geometry. First, we present a review on how to construct analog models based on the propagation of light rays in a material medium. We follow then with a method proposed by Hadamard [27] for the description of such phenomena. Furthermore, we restrict ourselves to the analysis of a linear magnetoelectric media from which we obtain the analog model.

1.1 Effective geometry

As it is well known from classic electrodynamics, a monochromatic light wave of angular frequency ω and wave vector $\vec{q}$, propagating in vacuum, is described by the dispersion relation $k^2 = 0$, i.e.,

$$\eta^{\mu\nu} k_\mu k_\nu = 0,$$

where we defined the wave 4-vector $k_\mu \doteq (\omega, -\vec{q})$, such that $k^2 = \omega^2 - q^2$, with $q = \|\vec{q}\|$. In other words, k_μ is a null-like vector in the Minkowski spacetime whose metric is $\eta_{\mu\nu}$. In a curved spacetime the metric will be a solution of GR, and the wave vector will still be a null vector, but now in the curved metric, i.e., $g^{\mu\nu}_{\text{(GR)}} k_\mu k_\nu = 0$.

In an optical material the dispersion relation is a bit more elaborated. New terms related to specific optical properties of the medium are added in such a way that the dispersion relations generalizes to $(\eta^{\mu\nu} + \theta^{\mu\nu}) k_\mu k_\nu = 0$, where $\theta^{\mu\nu}$ is related to the susceptibilities coefficients of the medium and possibly to external fields that couple to such coefficients, as it is the case for nonlinear materials. As a consequence, the magnitude of the phase velocity of light in a material medium will be generally dependent on its optical susceptibilities, the applied fields and also the direction of wave propagation. This expression can be presented in the suggestive form,

$$\bar{g}^{\mu\nu} k_\mu k_\nu = 0, \tag{1.1}$$

where it was defined the rank-2 tensor field $\bar{g}^{\mu\nu} = \eta^{\mu\nu} + \theta^{\mu\nu}$.

Let us define a new tensor field $g_{\mu\nu}$ as the inverse of $\bar{g}^{\mu\nu}$ such that

$$\bar{g}^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu. \tag{1.2}$$

It is worth emphasising that the background metric is the Minkowski one, $\eta_{\mu\nu}$. In this sense, a covariant tensor $\bar{g}_{\mu\nu}$ associated with $\bar{g}^{\mu\nu}$ is obtained by means of $\eta_{\mu\nu}$ as $\bar{g}_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}\bar{g}^{\alpha\beta}$. Thus, $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are generally quite different objects. They will coincide only when light is propagating in the Minkowski empty space.

It can be shown [28, 29, 30] that Eq. (1.1) allows an interpretation that $g_{\mu\nu}$, whose inverse is $\bar{g}^{\mu\nu}$, is in fact an effective metric for the wave 4-vector k_μ. Thus, light propagation in material media is equivalent to light propagation in curved spacetimes, and formal analogies between these two different scenarios are possible. For completeness and future reference, this result [29] shall be revisited below in details.

We start by taking the derivative of Eq. (1.1) with respect to the coordinate x^γ, yielding

$$\bar{g}^{\mu\nu}{}_{,\gamma}\, k_\mu k_\nu + 2\bar{g}^{\mu\nu} k_{\gamma,\mu} k_\nu = 0, \tag{1.3}$$

where it was used the fact that the wave 4-vector k_μ is a gradient field, i.e., $k_\mu = \partial_\mu \Phi$, where Φ is the phase of the fields, which implies that $k_{\mu,\nu} = k_{\nu,\mu}$. Now, taking the derivative of Eq. (1.2) with respect to x^γ, one gets $g_{\mu\alpha}\bar{g}^{\mu\nu}{}_{,\gamma} = -g_{\mu\alpha,\gamma}\bar{g}^{\mu\nu}$, which, after contraction with $\bar{g}^{\alpha\beta}$, results in,

$$\begin{aligned}\bar{g}^{\beta\nu}{}_{,\gamma} =& -\bar{g}^{\alpha\beta} g_{\mu\alpha,\gamma}\bar{g}^{\mu\nu} \\ =& -\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}(g_{\mu\alpha,\gamma} + g_{\gamma\alpha,\mu} - g_{\gamma\alpha,\mu}).\end{aligned}$$

Returning this result in Eq. (1.3) and conveniently reorganizing the indices and using the symmetry of $\bar{g}^{\mu\nu}$, one obtains

$$\bar{k}^\sigma \left(k_{\gamma,\sigma} - \Gamma^\mu_{\sigma\gamma} k_\mu \right) = 0, \tag{1.4}$$

where we have defined the contravariant vector $\bar{k}^\sigma \doteq \bar{g}^{\sigma\nu} k_\nu$, and also

$$\Gamma^\mu_{\sigma\gamma} \doteq \frac{1}{2}\bar{g}^{\mu\alpha}\left(g_{\sigma\alpha,\gamma} + g_{\alpha\gamma,\sigma} - g_{\sigma\gamma,\alpha}\right).$$

Looking at Eq. (1.4), it is clearly seen that, if $g_{\mu\nu}$ is regarded as a metric, the expression between brackets should be identified with the covariant derivative of the wave vector, i.e.,

$$k_{\gamma;\sigma} = k_{\gamma,\sigma} - \Gamma^\mu_{\sigma\gamma} k_\mu.$$

In other words, whenever the wave vector k_μ is considered, $g_{\mu\nu}$ effectively works as a curved spacetime metric, whose inverse is given by means of Eq. (1.2). So, $\Gamma^\mu_{\sigma\gamma}$ holds for the connection coefficients associated to this effective metric, which is indeed experienced by the wave vector in an optical medium.

With the above definitions, Eq. (1.4) reads $\bar{k}^\sigma k_{\gamma;\sigma} = 0$. Multiplying this equation by $\bar{g}^{\gamma\beta}$ and using the identity $\bar{g}^{\gamma\beta} k_{\gamma,\sigma} = \bar{k}^\beta{}_{,\sigma} + \bar{k}^\rho \bar{g}^{\gamma\beta} g_{\rho\gamma,\sigma}$, straightforward calculations lead to

$$\bar{k}^\sigma \left(\bar{k}^\beta{}_{,\sigma} + \Gamma^\beta_{\sigma\rho}\bar{k}^\rho \right) = 0.$$

Finally, identifying the covariant derivative of $\bar{k}^\beta$ with respect to the effective metric,

$$\bar{k}^\beta{}_{;\sigma} = \bar{k}^\beta{}_{,\sigma} + \Gamma^\beta_{\sigma\rho}\bar{k}^\rho,$$

results,

$$\bar{k}^{\sigma}\bar{k}^{\beta}{}_{;\sigma} = 0. \tag{1.5}$$

Notice that Eq. (1.5) is the geodesic equation in the spacetime described by the geometry $g_{\mu\nu}$ and it clearly shows that k_μ is a null vector in the effective geometry $g_{\mu\nu}$. Thus, a light ray propagating in a material medium shows complete analogy with a light ray propagating in an empty, but curved spacetime, which may be a solution of general relativity. This is a mathematical equivalence that holds as far as kinematic aspects of GR are considered.

Additionally, as $\bar{k}^\mu$ is a tangent vector along a curve γ that describes the path of light, we may set $\bar{k}^\mu = dx^\mu/du$, where u is an affine parameter along γ. In this way, Eq. (1.5) takes the canonical form

$$\frac{d^2x^\beta}{du^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0.$$

It is worth emphasizing that all the above results did not make use of $g_{\mu\nu}$ as the spacetime metric. It is just an effective one that is experienced only by the wave vector. The true background metric of the spacetime is still the Minkowski one $\eta_{\mu\nu}$.

1.2 Wave propagation in optical materials

The Maxwell's equations, which govern electrodynamics in a material medium, are given by

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0, \qquad \nabla\cdot\vec{B} = 0, \tag{1.6}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \nabla\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}, \tag{1.7}$$

where the source terms are of the form

$$\begin{aligned} \rho &= \rho_F + \rho_P = \rho_F - \nabla\cdot\vec{P}, \\ \vec{J} &= \vec{J}_F + \vec{J}_P + \vec{J}_M = \vec{J}_F + \frac{\partial\vec{P}}{\partial t} + \nabla\times\vec{M}, \end{aligned} \tag{1.8}$$

and the indeces F, P and M indicate, respectively, free sources, polarization sources and magnetization sources. Still, one can rewrite Eqs. (1.6) e (1.7) as

$$\nabla\cdot\vec{D} = \rho_F, \qquad \nabla\cdot\vec{B} = 0, \tag{1.9}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}_F, \tag{1.10}$$

where $\vec{D}$ is the displacement vector and $\vec{H}$ is the induced magnetic field, given by

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \tag{1.11}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}. \tag{1.12}$$

Moreover, one can build the constitutive relations between the fundamental and induced fields, using general permeability and permittivity coefficients, which characterize the medium. In components, these equations are given by

$$D_i = \varepsilon_{ij} E_j + \tilde{\varepsilon}_{ij} B_, \tag{1.13}$$

$$H_i = \mu_{ij}^{(-1)} B_j + \tilde{\mu}_{ij}^{(-1)} E_j, \tag{1.14}$$

where $\tilde{\varepsilon}_{ij}$ and $\tilde{\mu}_{ij}^{(-1)}$ are coupled terms, i.e., in materials presenting these properties, the magnetic field induces polarization and the electric field induces magnetization.

1.2.1 Step method

The method we use here to describe the light propagation in material media is the step method, presented by J. Haddamard in 1903, which makes use of the constitutive relations, (1.13) and (1.14), together with boundary analysis [27]. This will be analysed in what follows.

First, we define a hypersurface Λ, given by $\phi(t, \vec{x}) = 0$. This divides the spacetime into two regions, which we will call Z^+ and Z^-, which represent, respectively, the set of points inside the hypersurface ($P^- \in \phi(t, \vec{x}) < 0$) and outside of it ($P^+ \in \phi(t, \vec{x}) > 0$). These regions are evidently disjoint, i.e., $Z^+ \cap Z^- = \{\emptyset\}$. The discontinuity of a function $f(t, \vec{x})$ at a point P on the boundary of Λ is given by

$$[f(t, \vec{x})]_\Lambda \doteq \lim_{P^\pm \to P} \left[f(P^+) - f(P^-) \right]. \tag{1.15}$$

The electromagnetic fields must be smooth functions on Z^+ and Z^- and continuous along Λ. However, the same cannot be said with respect to their first derivatives. In fact, the discontinuities of each field at its respective derivative are given by [27]

$$\left[\vec{E}\right]_\Lambda = 0, \quad \left[\vec{B}\right]_\Lambda = 0, \tag{1.16}$$

$$[\partial_t E_i]_\Lambda = \omega e_i, \quad [\partial_t B_i]_\Lambda = \omega b_i, \tag{1.17}$$

$$[\partial_i E_j]_\Lambda = -q_i e_j, \quad [\partial_j B_i]_\Lambda = -q_i b_j, \tag{1.18}$$

where e_i and b_i are the components of the wave polarization vectors, related to the derivatives of the fields in Λ, of the form $\vec{e} = \left[\partial \vec{E}/\partial \phi\right]_\Lambda$ and $\vec{b} = \left[\partial \vec{B}/\partial \phi\right]_\Lambda$, and the terms ω

and q_i are the angular frequency and the i-th component of the wave vector, respectively. Notice that this method is equivalent to considering the fields of a plane wave described by $E_i = \exp\left[-i\left(\vec{q}\cdot\vec{x}-\omega t\right)\right]e_i$ and $B_i = \exp\left[-i\left(\vec{q}\cdot\vec{x}-\omega t\right)\right]b_i$.

Let us now consider Maxwell's equations in the absence of sources using indices,

$$\partial_i D_i = 0, \qquad \partial_i B_i = 0, \tag{1.19}$$

$$\epsilon_{ijk}\partial_j E_k = -\partial_t B_i, \qquad \epsilon_{ijk}\partial_j H_k = \partial_t D_i. \tag{1.20}$$

Considering the discontinuity in the Ampère-Maxwell law, and using the constitutive relations Eq. (1.13) and Eq. (1.14), one obtains

$$\left[\epsilon_{ijk}\partial_j\left(\mu_{kl}^{(-1)}B_l + \tilde{\mu}_{kl}^{(-1)}E_l\right)\right]_\Lambda = \left[\partial_t\left(\varepsilon_{ij}E_j + \tilde{\varepsilon_{ij}}B_j\right)\right]_\Lambda, \tag{1.21}$$

and, through the chain rule,

$$\begin{aligned}
\varepsilon_{ijk}\Bigg\{&\frac{\partial\mu_{kl}^{(-1)}}{\partial E_m}\left[\partial_j E_m\right]_\Sigma B_l + \frac{\partial\mu_{kl}^{(-1)}}{\partial B_m}\left[\partial_j B_m\right]_\Sigma B_l + \mu_{kl}^{(-1)}\left[\partial_j B_l\right]_\Sigma + \\
&+\frac{\partial\tilde{\mu}_{kl}^{(-1)}}{\partial E_m}\left[\partial_j E_m\right]_\Sigma E_l + \frac{\partial\tilde{\mu}_{kl}^{(-1)}}{\partial B_m}\left[\partial_j B_m\right]_\Sigma E_l + \tilde{\mu}_{kl}^{(-1)}\left[\partial_j E_l\right]_\Sigma\Bigg\} = \\
&= \frac{\partial\epsilon_{ij}}{\partial E_k}\left[\partial_t E_m\right]_\Sigma E_j + \frac{\partial\epsilon_{ij}}{\partial B_k}\left[\partial_t B_k\right]_\Sigma E_j + \epsilon_{ij}\left[\partial_t E_j\right]_\Sigma + \\
&+\frac{\partial\tilde{\epsilon}_{ij}}{\partial E_k}\left[\partial_t E_m\right]_\Sigma B_j + \frac{\partial\tilde{\epsilon}_{ij}}{\partial B_k}\left[\partial_t B_k\right]_\Sigma B_j + \tilde{\epsilon}_{ij}\left[\partial_t B_j\right]_\Sigma .
\end{aligned} \tag{1.22}$$

Using Eqs. (1.16 - 1.18) for field discontinuities, the above equation can be rewritten as

$$\begin{aligned}
-\epsilon_{ijk}q_j\Bigg[&\left(\frac{\partial\mu_{kl}^{(-1)}}{\partial E_m}B_l + \frac{\partial\tilde{\mu}_{kl}^{(-1)}}{\partial E_m}E_l\right)e_m + \left(\frac{\partial\mu_{kl}^{(-1)}}{\partial B_m}B_l + \frac{\partial\tilde{\mu}_{kl}^{(-1)}}{\partial B_m}E_l\right)b_m + \\
&+ \mu_{kl}^{(-1)}b_l + \tilde{\mu}_{kl}^{(-1)}e_l\Bigg] \\
= \omega\Bigg[&\left(\frac{\partial\varepsilon_{ij}}{\partial E_k}E_j + \frac{\partial\tilde{\epsilon}_{ij}}{\partial E_k}B_j\right)e_k + \left(\frac{\partial\varepsilon_{ij}}{\partial B_k}E_j + \frac{\partial\tilde{\varepsilon}_{ij}}{\partial B_k}B_j\right)b_k + \\
&+ \varepsilon_{ij}e_j + \tilde{\varepsilon}_{ij}b_j\Bigg].
\end{aligned} \tag{1.23}$$

In the first two terms of the square brackets on the right side of the above equation, the index k is muted. Let us relabel it to m. We will do the same with the index j in the last two terms of the same square brackets and with the index l in the last two terms of the left-hand square brackets. Therefore, one obtains in the compact form,

$$\left(\frac{1}{\omega}\epsilon_{ijk}q_j\tilde{H}_{km} + C_{im}\right)e_m + \left(\frac{1}{\omega}\epsilon_{ijk}q_jH_{km} + \tilde{C}_{im}\right)b_m = 0, \tag{1.24}$$

where we defined the following tensors

$$\tilde{H}_{km} = \frac{\partial\mu_{kl}^{(-1)}}{\partial E_m}B_l + \frac{\partial\tilde{\mu}_{kl}^{(-1)}}{\partial E_m}E_l + \tilde{\mu}_{km}^{(-1)}, \tag{1.25}$$

$$H_{km} = \frac{\partial \mu_{kl}^{(-1)}}{\partial B_m} B_l + \frac{\partial \tilde{\mu}_{kl}^{(-1)}}{\partial B_m} E_l + \mu_{km}^{(-1)}, \tag{1.26}$$

$$C_{im} = \frac{\partial \varepsilon_{ij}}{\partial E_m} E_j + \frac{\partial \tilde{\varepsilon}_{ij}}{\partial E_m} B_j + \varepsilon_{im}, \tag{1.27}$$

$$\tilde{C}_{im} = \frac{\partial \varepsilon_{ij}}{\partial B_m} E_j + \frac{\partial \tilde{\varepsilon}_{ij}}{\partial B_m} B_j + \tilde{\varepsilon}_{im}. \tag{1.28}$$

Now, from Faraday's Law in Eq. (1.20) one gets

$$\epsilon_{ijk} \left[\partial_j E_k\right]_\Sigma = - \left[\partial_t B_i\right]_\Sigma , \tag{1.29}$$

then, from the field discontinuity definitions,

$$b_i = \frac{q_j}{\omega} \epsilon_{ijk} e_k. \tag{1.30}$$

Substituting the above term in the Eq. (1.24), one obtains, after rearranging some indexes

$$\left[v\epsilon_{ikl}\kappa_k \tilde{H}_{lj} + v^2 C_{ij} + \epsilon_{ikl}\epsilon_{npj}\kappa_p\kappa_k H_{ln} + v\epsilon_{lkj}\kappa_k \tilde{C}_{il}\right] e_j = 0, \tag{1.31}$$

where we use that $q_j = q\kappa_j$, in which κ_j is the unit j-th component of the wave vector, and $\omega/q = v$, where v is the phase velocity of the wave. The above equation defines an eigenvalue problem, which can be written in the form

$$Z_{ij} e_j = 0, \tag{1.32}$$

where Z_{ij} denotes the elements of the Fresnel matrix

$$Z_{ij} \doteq C_{ij} v^2 + \left(\epsilon_{ikl}\tilde{H}_{lj} + \epsilon_{lkj}\tilde{C}_{il}\right) \kappa_k v + \epsilon_{ikl}\epsilon_{npj} H_{ln}\kappa_k\kappa_p. \tag{1.33}$$

Non-trivial solutions of Eq. (1.32) are found via the generalized Fresnel equation, $\det |Z_{ij}| = 0$, where $|Z_{ij}|$ is the matrix whose elements are given by the above equation. One of the methods to solve this equation is the use of Cayley-Hamilton theorem [31, 32], as follows

$$det|Z_{ij}| = 2(Z_1)^3 - 3Z_1Z_2 + 2Z_3 = 0, \tag{1.34}$$

where $Z_1 \doteq Z_{ii}$, $Z_2 \doteq Z_{ij}Z_{ji}$ and $Z_3 \doteq Z_{ij}Z_{jk}Z_{ki}$. The dispersion relation derived from Eq. (1.34) describes the propagation of light rays in material media. In principle, one could get more than one effective metric associated with the optical medium, depending on the behavior of light propagating through it, for instance, in birefringent materials there are two different solutions for the phase velocities, called ordinary and extraordinary ray solutions, and there will be also different effective geometries related to each of them.

1.2.2 Linear magnetoeletric material

The main subject of this section is to investigate light propagation in a linear magnetoelectric material. Throughout this section 3-dimensional component notation is used, with the metric of the three-space in Galilean coordinates coinciding with the Kronecker delta δ_{ij}. So, without losing generality, we keep all indices at just one (lower) level, and Einstein's summation convention over repeated indices still applies.

When electric E_i and magnetic B_i fields are applied over an optical medium having magnetoelectric properties, polarization and magnetization phenomena may occur in such a way that both fields can contribute to both effects. If we restrict our analysis to the linear effects, the polarization (P_i) and the magnetization (M_i) vectors will be given by,

$$
\begin{aligned}
P_i &= \varepsilon_0 \chi^{(1)}_{ij} E_j + \alpha_{ij} H_j, \\
\mu_0 M_i &= \mu_0 \tilde{\chi}^{(1)}_{ij} H_j + \alpha_{ji} E_j,
\end{aligned}
$$

where spontaneous effects are not being considered. Here, α_{ij} represents the linear magnetoelectric coefficients, and it is assumed that the linear electric and magnetic susceptibility sectors are isotropic, in such a way that $\chi^{(1)}_{ij} = \chi\delta_{ij}$ and $\tilde{\chi}^{(1)}_{ij} = \tilde{\chi}\delta_{ij}$, respectively. In this case, the constitutive relations connecting the auxiliary fields to the fundamental electric and magnetic fields, can be conveniently written as

$$
D_i = \varepsilon E_i + \alpha_{ij} H_j, \tag{1.35}
$$

$$
B_i = \mu H_i + \alpha_{ji} E_j, \tag{1.36}
$$

where it was defined the isotropic electric permittivity, $\varepsilon = \varepsilon_0(1 + \chi)$, and magnetic permeability, $\mu = \mu_0(1 + \tilde{\chi})$, coefficients. Let us rewrite these relations for the induced fields, as

$$
D_i = (\varepsilon\delta_{ij} - \mu^{-1}\alpha_{ik}\alpha_{jk})E_j + \mu^{-1}\alpha_{ij}B_j, \tag{1.37}
$$

$$
H_i = \mu^{-1}B_i - \mu^{-1}\alpha_{ji}E_j. \tag{1.38}
$$

Comparing these equations with Eqs. (1.13) and (1.14), one obtains that the optical coefficients are given by

$$
\varepsilon_{ij} = \varepsilon\delta_{ij} - \mu^{-1}\alpha_{ik}\alpha_{jk}, \tag{1.39}
$$

$$
\mu^{(-1)}_{ij} = \mu^{-1}\delta_{ij}, \tag{1.40}
$$

$$
\tilde{\varepsilon}_{ij} = \mu^{-1}\alpha_{ij}, \tag{1.41}
$$

$$
\tilde{\mu}^{(-1)}_{ij} = -\mu^{-1}\alpha_{ji}. \tag{1.42}
$$

Plugging these parameters into Eq. (1.33), one obtains

$$
Z_{ij} = (\varepsilon\mu\delta_{ij} - \alpha_{ik}\alpha_{jk})\, v^2 + 2\epsilon_{rn(i}\alpha_{j)r}\kappa_n v - I_{ij}, \tag{1.43}
$$

where we defined the projector orthogonal to the wave vector, $I_{ij} \doteq \delta_{ij} - \kappa_i\kappa_j$. Solving $\det Z_{ij} = 0$ up to first order in α_{ij} one finds the solutions for the magnitude of the phase velocity,

$$v^{\pm} = \pm v_o - v_o^2 \sigma_i \kappa_i, \tag{1.44}$$

where we define $v_o^2 \doteq (\varepsilon\mu)^{-1}$, and

$$\sigma_i \doteq \frac{\epsilon_{ijk}\alpha_{jk}}{2}.$$

In this result we have discarded higher order contributions to σ_i. For later reference, it is worth writing this vector more explicitly in terms of its Cartesian components as

$$\vec{\sigma} = (\alpha_{[23]}, \alpha_{[31]}, \alpha_{[12]}). \tag{1.45}$$

Notice that only the antisymmetric part of the linear magnetoelectric coefficient α_{ij} contributes to the result in Eq. (1.44). Additionally, it should be mentioned that this solution describes an extraordinary light ray, in the sense that it depends on the direction of wave propagation. In the absence of magnetoelectric effect ($\sigma_i = 0$), the phase velocity reduces to an ordinary light-ray solution, whose velocity is v_o, as expected.

Furthermore, one can see that the system presents a nonreciprocal behavior, i.e., the two possible solutions describe waves that propagate in opposite directions with different velocities. Notice that this anisotropy is due to the coupling of α_{ij} to the direction of wave propagation. Additionally, one could consider the effect of assuming the magnetoelectric coefficients as functions of position, so that the magnitude of the light-ray velocity would change along its path through the medium. Such assumption requires a consistency analysis about the obtained wave solutions. For a moment, let us assume a general dependence of these coefficients on the coordinate z. In this case, it can be shown that considering $\alpha_{ij}(z)$ in Eqs. (1.35) and (1.38),

$$Z_{ij}e_j - \frac{iv}{\mu q}\epsilon_{ijk}\left[\partial_j \alpha_{lk}(z)\right] E_l e^{i(\vec{q}\cdot\vec{r} - \omega t)} = 0, \tag{1.46}$$

where E_l is the total field given by $E_p + E_0$ and Z_{ij} is given by Eq. (1.43). In order to maintain the previous results, one should guarantee that the second term in Eq. (1.46) does not contribute to the effect. If we consider a model in which α_{ij} vary sufficiently slowly with the coordinate z, these terms could be neglected and the consistency of the previous analysis is guaranteed. The behavior of the two phase velocities given by Eq. (1.44) are illustratively depicted in Fig. 1. Bottom (up) row of arrows symbolizes the behavior of the v^+ (v^-) mode, as indicated. Notice that the magnitude of the velocity changes as the light ray propagates through the material. In particular, for the chosen model, the mode v^+ is an increasing function of z, while mode v^- is a decreasing function of z.

Motivated by the previous analysis, we now investigate the idealised model for which the phase velocity given by Eq. (1.44) is assumed to be an exact solution, in the

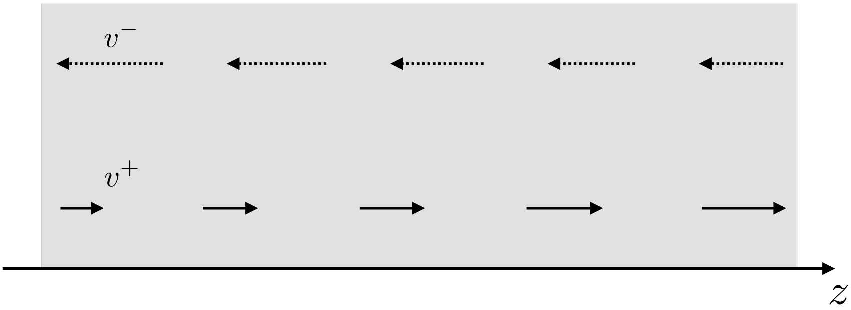

Fig. 1 – Propagation in z direction of monochromatic waves in an optical material whose magnetoelectric coefficient depends on the coordinate z. Solid and dashed arrows represent the positive and negative solutions, respectively. Their sizes are related to the magnitude of the corresponding phase velocity, given by Eq. (1.44). In this case we set α_{ij} as a decreasing functon of z.

sense that σ_i does not necessarily represent a small contribution, but could be as larger in magnitude as v_o. In such case, σ_i should be understood as a more general quantity which would not be especially related to the magnetoelectric effect, because other effects could also contribute to this result, including those depending on external fields. In fact, such a configuration can only be conceived in the realm of metamaterials. We will not be worried here about the derivation of these idealised constitutive relations behind such dispersion relation, but only about its physical consequences.

Let us set the propagation in the z direction, and study the behavior of light rays in such hypothetical material. In this particular case, the two possible solutions for the phase velocity described by Eq. (1.44) reduce to

$$v^+ = -\alpha + v_o, \tag{1.47}$$

$$v^- = -\alpha - v_o, \tag{1.48}$$

where $\alpha \doteq v_o^2\alpha_{[12]}$.

In order to exhibit a specific model presenting a transition in the behavior of light propagation, for which the coefficient α become bigger than v_0 at a certain point, we consider that this system can be constructed in such a way to allow α to be a function of z. Suppose that at some region, for instance $z < z_h$, $\alpha > v_0$. Therefore, for this region one has that $\alpha > v_o$ and both solutions for $v^\pm$ are negative, while in the region $z > z_h$ solutions propagating in both directions are allowed. These aspects are illustratively depicted in Fig. 2. Solutions of light rays propagating to the right cannot exist in the region given by $z < z_h$, since their velocities are negative due to the influence of this idealised optical effect. It is interesting to notice that in such region, birefringence occurs, as there will be two modes propagating in a same direction, but with different phase velocities. Furthermore, this region ($z < z_h$) is a sort of one-way system, as there will be no solution propagating to the right. On the other hand, there is no birefringence in the region given by $z > z_h$. In this region the two solutions correspond to rays propagating in opposite directions. However, these solutions correspond to quite different phase velocities, which makes the propagation

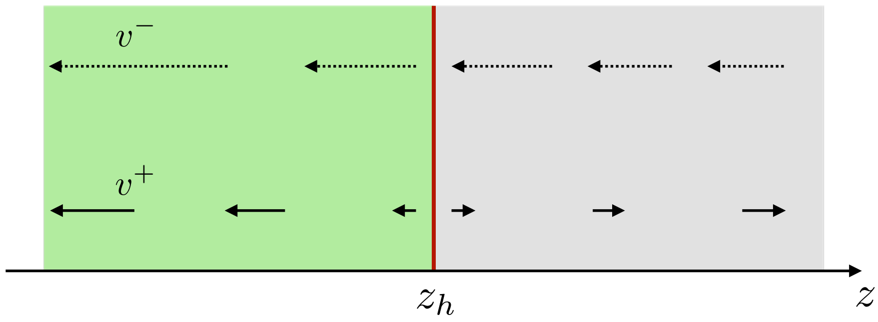

Fig. 2 – Propagation of monochromatic waves in a material whose optical coefficient depends on the coordinate z. Solid and dashed arrows represent the two possible solutions described by Eqs. (1.47) and (1.48), respectively. Their sizes are related to the magnitude of the corresponding phase velocities. The vertical line represents the transition, that is located at $z = z_h$.

in this region ($z > z_h$) non-symmetric under space reversion. Another interesting aspect is that the region nearby the transition ($z \approx z_h$) is a sort of slow light region for the mode described by Eq. (1.47). Its phase velocity is exactly zero at $z = z_h$ and increases as it moves away from this region. The very dependence of the magnitude of the velocity with the distance to the transition point is certainly dependent on the chosen model, such as the particular one set by Eq. (3.29), but the fact that its velocity must be zero at z_h and near zero in its immediate vicinity, is not dependent on the specific model, but it is a consequence of the presence of the transition.

It is straightforward to think of such optical behavior as an analog event horizon, in the sense that the point of transition splits the medium into two domains in which one of them only allows propagation in one direction. This correspondence will be investigated in the next section, where the effective metric produced by such a hypothetical optical system is examined.

1.3 Analogue models

In this section we obtain the analog model for both media analysed before. Firstly, we investigate the effective metric corresponding to the approximated linear magnetoelet-ric model. Finally we consider the idealised hypothetical model and through its dispersion relation we obtained the metric, which turns out to be related with light propagation near a Schwarzschild black hole.

1.3.1 Linear magnetoelectric model

Let us now examine the possible analog models based on a linear magnetoelectric medium. Hereafter we assume the background spacetime as described by the Minkowski metric, which, in Cartesian coordinates read $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The electric and magnetic four-vectors are defined, respectively, by $E^\mu = (0, \vec{E})$ and $B^\mu = (0, \vec{B})$. We introduce the 4-vector V^μ representing the velocity field of an observer relative to the optical material through which light is propagating. In the case of our interest the observer will always be at the rest frame of the optical material, i.e., $V^\mu = \delta^\mu_0$. In terms of such velocity field, $\omega = k_\mu V^\mu$. Furthermore, as $k^2 = k_\mu k_\nu \eta^{\mu\nu} = \omega^2 - q^2$, it follows that $q^2 = -k_\mu k_\nu h^{\mu\nu}$, where the projector in the three-dimensional space section was defined as

$$h^{\mu\nu} = \eta^{\mu\nu} - V^\mu V^\nu,$$

such that $h_{\mu\nu}V^\nu = 0$ and $h_{\mu\alpha}h^\alpha{}_\nu = h_{\mu\nu}$. We thus conveniently define the four-vector

$$q^\mu = h^{\mu\nu}k_\nu = k^\mu - \omega V^\mu = (0, \vec{q}).$$

Notice that $q_\mu q^\mu = -q^2$. Finally, we define the four-vector $\sigma^\mu = (0, \vec{\sigma})$, where $\vec{\sigma}$ is defined by Eq. (1.45).

Let us examine the possible models based on the solution given in Eq. (1.44). Squaring the phase velocity solution, and keeping only first order terms in magnetoelectric coefficients, one obtains that

$$\omega^2 - 2v_o^2\sigma_\nu q^\nu \omega = v_o^2 q^2.$$

Or yet, as $q^2 = -k^2 + \omega^2$, this equation can be recast in the convenient form

$$\left[V^u V^\nu + v_o^2 h^{\mu\nu} - 2v_o^2 \sigma^{(\mu}V^{\nu)}\right] k_\mu k_\nu = 0.$$

Now, one can identify, as in Eq. (1.1),

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - \left(1 - \frac{1}{v_o^2}\right) V^\mu V^\nu - 2\sigma^{(\mu}V^{\nu)},$$

whose inverse $g_{\mu\nu}$ is identified as the effective geometry,

$$g_{\mu\nu} = \eta_{\mu\nu} - \left(1 - v_o^2\right) V_\mu V_\nu + 2v_o^2\sigma_{(\mu}V_{\nu)}. \tag{1.49}$$

Separating this metric into components, for each sector, one has (up to a global factor v_o^2)

$$g_{00} = 1, \tag{1.50}$$

$$g_{0i} = -\frac{1}{2}\epsilon_{ijk}\alpha^{jk}, \tag{1.51}$$

$$g_{ij} = \frac{1}{v_o^2}\eta_{ij}. \tag{1.52}$$

The class of metrics described by this solution exhibits nonzero mixed terms, g_{0i}, as expected. As it can be seen, the linear magnetoelectric coefficient α^{ij} activates this term in the effective metric, and only its antisymmetric part contributes to it.

1.3.2 A toy model for an optical horizon

Let us now investigate the exact model based on the assumption that there might be a meta-medium for which the dispersion relation leading to Eq. (1.44) is exact. Notice that such a hypothetical system is not restricted to be a linear magnetoelectric material. In this case the dispersion relation is given by (using the four-dimensional notation introduced in Sec. 1.3.1),

$$\omega^2 - 2v_o^2\omega\sigma^\mu q_\mu + v_o^4\sigma^\mu\sigma^\nu q_\mu q_\nu - v_o^2 q^2 = 0.$$

Thus, one can conveniently rewrite this expression as $\tilde{g}^{\mu\nu}k_\mu k_\nu = 0$, where we define the effective metric

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - \left(1 - \frac{1}{v_o^2}\right)V^\mu V^\nu - 2\sigma^{(\mu}V^{\nu)} + v_o^2\sigma^\mu\sigma^\nu.$$

The optical geometry is then given by the inverse of this metric, namely

$$g_{\mu\nu} = \eta_{\mu\nu} - \left(1 - v_o^2 + v_o^4\sigma^2\right)V_\mu V_\nu + 2v_o^2\sigma_{(\mu}V_{\nu)}$$

where $\sigma^2 = -\sigma^\mu\sigma_\mu$.

The line element for a light ray propagating in z direction is

$$ds^2 = (v_o^2 - v_o^4\sigma^2)dt^2 - 2v_o^2\sigma_3 dzdt - dz^2 = 0. \tag{1.53}$$

This line element is similar to the one describing radial propagation in the Schwarzchild metric, when written in PG coordinates [25],

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - 2\sqrt{\frac{2m}{r}}drdt - dr^2.$$

In order to investigate this resemblance closer, let us choose a specific system for which $\sigma_i = \sigma\delta_{i3}$. Thus, Eq. (1.53) reduces to

$$ds^2 = (1 - v_o^2\sigma^2)(v_o dt)^2 - 2v_o\sigma dz(v_o dt) - dz^2.$$

In the above expression, $v_o = 1/\sqrt{\varepsilon\mu}$ plays the role of the speed of light in vacuum in PG line element ($c = 1/\sqrt{\varepsilon_0\mu_0} = 1$ in natural units here adopted), and it corresponds to the velocity of an ordinary light ray in the absence of the σ-coupling. Comparing this result with PG metric one sees that the term $v_o^2\sigma^2$, which encodes the hypothetical optical effect, plays the role of the term $2m/r$, which is related to the black hole mass. A similar behavior is also known to occur in acoustic black holes [21, 22, 23], where perturbations propagating in a moving fluid are trapped in a sort of acoustic horizon.

The above results suggest that natural or artificial optical systems provide possible scenarios to the investigation of kinematic aspects of solutions of general relativity. The above idealised model consists in an optical analog for Schwarzschild spacetime, but other solutions, including cosmological ones, could also be conceived in a similar way.

2 Acoustic analogue models

In this chapter we analyse the equivalence between acoustic waves propagating in a fluid and the propagation of a massless scalar field near a Schwarzschild black hole. It is shown that by considering linear perturbation in the functions describing the fluid, the EOM of the second order terms can be written in terms of an acoustic metric.

2.1 Fluid Action

Consider a fluid presenting an irrotational and homentropic flow[1], such that $\vec{v} \propto \nabla\psi$, where $\vec{v}$ is the fluid velocity and ψ is the velocity field, and $P = P(\rho)$, where P is the pressure and ρ the density. In the abscence of external forces the flow action is given by [33]

$$S = -\int d^4x \left[\rho\dot{\psi} + \frac{1}{2}\rho\left(\nabla\psi\right)^2 + u\left(\rho\right)\right], \tag{2.1}$$

where $u(\rho)$ is the internal energy density.

Let us then use the principle of least action and vary the action with respect to ψ and ρ. For ψ

$$\delta S = S[\psi + \delta\psi, \dot{\psi} + \dot{\delta\psi}, \rho] - S[\psi, \dot{\psi}, \rho] = 0. \tag{2.2}$$

Expanding and maintaining only first order terms in the variation $\delta\psi$, one obtains that

$$\int d^4x \left[\rho\frac{d}{dt}(\delta\psi) + \rho\nabla\psi\cdot\nabla(\delta\psi)\right] = 0. \tag{2.3}$$

Now, performing integration by parts in both terms of the above integral and recalling that $\delta\psi$ vanishes at the limits of integration, the above equation reduces to

$$\int d^4x \left[\dot{\rho} + \nabla\cdot(\rho\vec{v})\right]\delta\psi = 0, \tag{2.4}$$

and since $\delta\psi$ is an arbitrary variation, one finds the continuity equation

$$\dot{\rho} + \nabla\cdot(\rho\vec{v}) = 0. \tag{2.5}$$

Analogously, the variation with respect to ρ leads to the Bernoulli equation, namely

$$\dot{\psi} + \frac{1}{2}v^2 + \mu(\rho) = 0, \tag{2.6}$$

where $\mu(\rho) = (du/d\rho)$. Finally, let us take the gradient of the Bernoulli equation, as follows

$$\dot{\vec{V}} + (\vec{v}\cdot\nabla)\vec{v} + \nabla[\mu(\rho)] = 0, \tag{2.7}$$

[1] An homentropic flow has uniform and constant entropy, which makes it isentropic, but with the additional feature that every particle has the same level of entropy.

where we used that the spatial and time derivatives commute, and the vectorial calculus identity

$$\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}).$$

Furthermore, the last term in Eq. (2.7) can be written as

$$\nabla\mu = \frac{1}{\rho}\left(\rho\nabla\mu\right) = \frac{1}{\rho}\left[\nabla(\rho\mu) - \mu\nabla\rho\right].$$

thus, using integral by parts

$$\nabla\mu = \frac{1}{\rho}\left[\nabla\left(\int\rho d\mu + u\right) - \mu\nabla\rho\right] = \frac{1}{\rho}\nabla\left(\int\rho d\mu\right) = \frac{1}{\rho}\nabla P. \tag{2.8}$$

where the pressure is defined as $P = \int\rho d\mu$. Substituing this in Eq. (2.7), one finds the Euler equation given by

$$\dot{\vec{v}} + (\vec{v}\cdot\nabla)\vec{V} + \frac{1}{\rho}\nabla P = 0. \tag{2.9}$$

To finish this section it is worth analysing the symmetries of the action (2.1). The action is invariant under translations, such as

$$\begin{cases}\psi(x^i) \to \psi(x^i - a^i),\\ \rho(x^i) \to \rho(x^i - a^i),\end{cases}$$

and the conservation law related with this invariance is, by means of Noether's theorem, given by

$$\partial_t(\rho\partial_i\psi) + \partial_j\left[\rho(\partial_i\psi)({}_j\psi) - \delta_{ij}\left(\rho\dot{\psi} + \frac{1}{2}\rho(\nabla\psi)^2 + u(\rho)\right)\right] = 0. \tag{2.10}$$

One can still simplify this relation by using the definition of the pressure, as follows

$$P = \int\rho d\mu = \int d(\rho\mu) - \int\mu d\rho = \rho\mu - u,$$

thus with the Bernoulli equation, in Eq. (2.6), one obtains that [34]

$$-P = \rho\dot{\psi} + \frac{1}{2}\rho(\nabla\psi)^2 + u(\rho). \tag{2.11}$$

Hence, Eq. (2.10) reduces to the momentum conservation law

$$\partial_t(\rho\partial_i\psi) + \partial_j\Pi_{ij} = 0, \tag{2.12}$$

where $\Pi_{ij} = \rho(\partial_i\psi)({}_j\psi) + \delta_{ij}P$ is the momentum flux tensor.

2.2 Acoustic metric

In order to analyse the propagation of perturbations in the fluid, let us split the characteristic functions into two parts: the background (mean flow), described by ρ_0 and ψ_0, and the fluctuations, described by that ρ_1 and ψ_1, such that $\rho = \rho_0+\rho_1$ and $\psi = \psi_0+\psi_1$.

Up to second order contributions in the fluctuations, the action in Eq. (2.1) is written as

$$\begin{aligned} S = S_0 - \int d^4x \left(\rho_0 \dot{\psi}_1 + \rho_1 \dot{\psi}_0 + \rho_0 \nabla\psi_1 \cdot \nabla\psi_0 + \frac{1}{2}\rho_1 (\nabla\psi_0)^2 + \rho_1 \frac{\partial u}{\partial \rho}\bigg|_{\rho_0} \right) - \\ - \int d^4x \left(\rho_1 \dot{\psi}_1 + \rho_1 \nabla_0 \cdot \nabla\psi_1 + \frac{1}{2}\rho_0 (\nabla\psi_1)^2 + \frac{\rho_1^2}{2} \frac{\partial^2 u}{\partial \rho^2}\bigg|_{\rho_0} \right). \end{aligned} \tag{2.13}$$

Let us collect the linear terms in fluctuations. Using integration by parts, together with the fact that the fields vanish at the infinity, one obtains

$$S_1 = \int d^4x \left[\rho_1 \left(\dot{\psi}_0 + \frac{1}{2}(\nabla\psi_0)^2 + \frac{\partial u}{\partial \rho}\bigg|_0 \right) - \psi_1 \left(\dot{\rho}_0 + \nabla(\rho_0 \nabla\psi_0) \right) \right]. \tag{2.14}$$

The equation associated with ρ_1 is just the ρ_0 EOM for the unperturbed action and the one multiplied by ψ_1 is the continuity equation. Therefore, the linear contributions vanish entirely ($S_1 = 0$). The remaining action takes the form

$$S = S_0 + S_2, \tag{2.15}$$

where the action describing the propagation of linear perturbations in the fluid is given by

$$S_2 = - \int d^4x \left(\rho_1 \dot{\psi}_1 + \rho_1 \nabla_0 \cdot \nabla\psi_1 + \frac{1}{2}\rho_0 (\nabla\psi_1)^2 + \frac{v_s^2}{2\rho_0}\rho_1^2 \right), \tag{2.16}$$

in which v_s is the speed of sound defined by $v_s^2 = \rho \frac{\partial u}{\partial \rho}\Big|_{\rho_0}$.

Since there are time derivative terms of ρ_1 in the above equation, the EOM for this quantity is straightforward given by

$$\dot{\psi}_1 + \nabla_0 \cdot \nabla\psi_1 + \frac{v_s^2}{\rho_0}\rho_1 = 0. \tag{2.17}$$

From that, one can write ρ_1 in terms of ψ_1 and obtain that the action will reduce to

$$S_2 = - \int d^4x \left[\frac{1}{2}\rho_0^2 - \frac{\rho_0}{2v_s^2} \left(\dot{\psi}_1 + \vec{v_0} \cdot \nabla\psi_1 \right)^2 \right]. \tag{2.18}$$

Thus, the EOM for ψ_1 will be given, in its general form, by

$$-\partial_t \left[\frac{\rho_0}{2v_s^2} (\partial_t + \vec{v_0} \cdot \nabla\psi_1) \right] + \nabla \cdot \left\{ \vec{v_0} \left[-\frac{\rho_0}{v_s^2} (\partial_t \psi_1 + \vec{v_0} \cdot \nabla\psi_1) \right] + \rho_0 \nabla\psi_1 \right\} = 0. \tag{2.19}$$

Now, one can write the EOM for ψ_1 in a four-dimensional notation, as follows

$$\partial_\mu (f^{\mu\nu} \partial_\nu \psi_1) = 0, \tag{2.20}$$

where we have defined

$$f^{\mu\nu} \doteq \frac{\rho_0}{v_s^2} \begin{pmatrix} -1 & -\vec{v_0}^{\mathrm{T}} \\ -\vec{v_0} & v_s^2 \delta_{ij} - v_0^i v_0^j \end{pmatrix}. \tag{2.21}$$

One can also write this tensor as $f^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$, where

$$g^{\mu\nu} \doteq \frac{1}{\rho_0 v_s} \begin{pmatrix} -1 & -\vec{v_0}^{\mathrm{T}} \\ -\vec{v_0} & v_s^2\delta_{ij} - v_0^i v_0^j \end{pmatrix}, \tag{2.22}$$

and $\sqrt{-g} = \rho_0^2/v_s$. Then, let us introduce the "acoustic metric", given by the inverse of $g^{\mu\nu}$, as

$$g_{\mu\nu} \doteq \frac{\rho_0}{v_s} \begin{pmatrix} -(v_s^2 - v_0^2) & -\vec{v_0}^{\mathrm{T}} \\ -\vec{v_0} & I_{3\times 3} \end{pmatrix}, \tag{2.23}$$

where $I_{3\times 3}$ is the identity matrix. Henceforth, one can directly see that the equality after Eq. (2.21) holds for $g = \det(g)$, as we expected.

Therefore, Eq. (2.20) can be written as the d'Alembert equation in curved spacetime [21],

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi_1) = \Box_g\psi_1 = 0. \tag{2.24}$$

Thus, the action for the fluctuations reduces to

$$S_2 = -\frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\psi_1\partial_\nu\psi_1. \tag{2.25}$$

This is nothing but the action for the massless scalar field, minimally coupled with the gravitational field, propagating freely in a curved metric.

Finally, one can write the line element associated with the acoustic metric by means of Eq. (2.23) as follows

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\rho_0}{v_s}\left[-(v_s^2 - v_0^2)dt^2 - 2\delta_{ij}v_0^i dx^j dt + \delta_{ij}dx^i dx^j\right]. \tag{2.26}$$

For propagation in only one direction, this line element is very similar to the one associated with radial propagation in the Schwarzschild geometry, when written in the PG coordinates [25].

2.2.1 Geodesics

Let us now show that the field ψ_1 follows geodesics of the acoustic metric $g_{\mu\nu}$. We start by considering a plane-wave solution for this field, i.e.,

$$\psi_1 = \mathcal{R}\{ae^{i\phi}\}, \tag{2.27}$$

where a is a slowing varying amplitude and ϕ is a rapidly changing real phase. Let us introduce the gradient wave vector $K_\mu = \partial_\mu\phi$, which is orthogonal to the surface defined by ϕ. Using Eq. (2.24) one finds,

$$g^{\mu\nu}\partial_\mu(e^{i\phi}\partial_\nu\phi) = -e^{i\phi}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi - i\partial_\mu\partial_\nu\phi) = 0, \tag{2.28}$$

taking the real part of the last equation and using the definition of the wave vector, one obtains

$$K^{\nu}K_{\nu} = g_{\mu\nu}K^{\mu}K^{\nu} = 0, \tag{2.29}$$

where we defined $K^{\mu} = dx^{\mu}/d\lambda$, in which λ is a parameter. This defines a tangent vector along the curve defined by ϕ and parametrized by λ. Henceforth, K^{μ} is a null-like vector and sound waves will propagate along null geodesics of the acoustic metric.

For three dimensions, using that the parameter u is the usual Newtonian time, one gets

$$g_{\mu\nu}K^{\mu}K^{\nu} = g_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} = \frac{\rho_0}{v_s}\left[-(v_s^2 - v_0^2) - 2\delta_{ij}v_0^i\frac{dx^j}{dt} + \delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right] = 0, \tag{2.30}$$

thus

$$\left(\frac{d\vec{x}}{dt} - \vec{v_0}\right)^2 = v_s^2,$$

which implies that

$$\left|\frac{d\vec{x}}{dt} - \vec{v_0}\right| = v_s. \tag{2.31}$$

This is the wave equation describing propagation of sound with velocity $\vec{v_s}$ in a moving medium which moves with velocity $\vec{v_0}$.

2.2.2 Pseudo energy-momentum tensor

From this point it is possible to derive two different energy momentum-tensors, one using $\eta_{\mu\nu}$ (that is the Minkowski background in the Laboratory) and one using the acoustic metric. For the latter, we will use the name “pseudo energy-momentum tensor“. Calculations are done by means of the following equation

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{\sqrt{-g}}\int d^4x\left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}g^{\alpha\beta}\partial_{\alpha}\psi_1\partial_{\beta}\psi_1 + \sqrt{-g}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\partial_{\alpha}\psi_1\partial_{\beta}\psi_1\right]. \end{aligned} \tag{2.32}$$

Let us now use the Jacobi’s formula to calculate the derivative of the metric determinant. Consider a matrix A, thus for a general variation operation, the Jacobi’s formula is given by

$$\delta(\det A) = \det(A)\mathrm{Tr}(A^{-1}\delta A). \tag{2.33}$$

Hence,

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta g = \frac{\sqrt{-g}}{2}\mathrm{Tr}(g^{\mu\nu}\delta g_{\nu\alpha}) = -\frac{\sqrt{-g}}{2}(g_{\mu\nu}\delta g^{\mu\nu}), \tag{2.34}$$

where we have used that $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$ (see Appendix A). Hence, substituting this in Eq. (2.32), one obtains the pseudo energy-momentum tensor as

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\partial^{\gamma}\psi_1\partial_{\gamma}\psi_1 + \partial_{\mu}\psi_1\partial_{\nu}\psi_1. \tag{2.35}$$

In components we have

$$\mathrm{T}_{00} = \dot{\psi_1}^2 - \frac{(v_s^2 - v_0^2)}{2v_s^2}\left[(\dot{\psi}_2 + \nabla\psi_1 \cdot \vec{v_0})^2 - v_s^2(\nabla\psi_1)^2\right], \tag{2.36}$$

$$\mathrm{T}_{0i} = \dot{\psi}_1\partial_i\psi_1 - \frac{v_0^i}{2v_s^2}\left[(\dot{\psi}_2 + \nabla\psi_1 \cdot \vec{v_0})^2 - v_s^2(\nabla\psi_1)^2\right], \tag{2.37}$$

$$\mathrm{T}_{ij} = \partial_i\psi_1\partial_j\psi_1 + \frac{\delta_{ij}}{2v_s^2}\left[(\dot{\psi}_2 + \nabla\psi_1 \cdot \vec{v_0})^2 - v_s^2(\nabla\psi_1)^2\right]. \tag{2.38}$$

In Appendix A it is shown that this tensor satisfies the covariant form of the conservation law, given by

$$\mathrm{T}^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\mathrm{T}^{\mu\alpha}) + \Gamma^\alpha_{\mu\lambda}\mathrm{T}^\mu = 0, \tag{2.39}$$

where $\Gamma^\alpha_{\mu\lambda}$ are the connection coefficients defined, in terms of the acoustic metric, as

$$\Gamma^\alpha_{\mu\lambda} = \frac{1}{2}g^{\alpha\beta}(g_{\mu\beta},_\nu + g_{\nu\beta},_\mu - g_{\mu\nu},_\beta). \tag{2.40}$$

If the fluid and the background space are homogeneous, $\Gamma^\alpha_{\mu\lambda}$ vanish and the system is symmetric under translation in one specific direction. In this case, the covariant conservation law reduces to the conservation of the pseudo energy-momentum. However, since in general $\Gamma^\alpha_{\mu\lambda} \neq 0$, Eqs. (2.39) describe the energy and momentum exchanging between the sound waves and the mean flow [34].

2.3 Acoustic black holes

Let us now consider linear flows in, for instance, the z direction, i.e., $\vec{v_0} = (0, 0, v_0)$, and restrict our analysis to the propagation of sound waves parallel to the fluid flow, such that, in Eq. (2.30), $dx = dy = 0$. Hence, the propagation will be described by

$$\frac{\rho_0}{v_s}\left[-(v_s^2 - v_0^2) - 2v_0\frac{dz}{dt} + \left(\frac{dz}{dt}\right)^2\right] = 0. \tag{2.41}$$

Now, if we suppose that the fluid moves from right to left, i.e., $v_0 < 0$, there will be two possible solutions for the sound wave velocities, relative to the fluid movement. The downstream and upstream propagation are, respectively, given by

$$\frac{dz}{dt} = v_0 - v_s, \tag{2.42}$$

$$\frac{dz}{dt} = v_0 + v_s. \tag{2.43}$$

Notice that the downstream solution moves along with the flow, while the upstream solution propagates against it, from left to right. If there is a point from where the flow velocity becomes greater than the speed of sound, the surface defined by $|v_0| = v_s$ will split the space into two different regions. The first region is defined by $|v_0| < v_s$. In this case the solutions propagate in opposite directions, and the upstream waves, given by Eq. (2.43) are able to propagate to the right (they are positive). In the region where $|v_0| > v_s$ the solutions propagate in the same direction, together with the main flow. Notice that the upstream solutions become negative in this region, i.e., although they tend to propagate from left to right, in the supersonic region they are dragged to the left due to the fluid motion. The surface previously mentioned characterizes the "sonic horizon", in the sense that from that point no sound wave can escape out of the supersonic region. This is similar to the propagation of light near to a black hole event horizon, once the rays cross the horizon they can never return. Therefore, this acoustic system mimetizes the geometry of a black hole, and for that it is usually called "acoustic black hole" [21].

In terms of null coordinates, the upstream and downstream solutions will follow

$$x^- = c\left(t - \int \frac{dz}{c+v}\right), \tag{2.44}$$

$$x^+ = c\left(t + \int \frac{dz}{c-v}\right), \tag{2.45}$$

where the null geodesics are characterized by $x^- = constant$, for the upstream propagation, and $x^+ = constant$, for the downstream propagation. This is indeed the same behavior of light rays propagating along null geodesics of the Schwarzschild spacetime.

3 Hawking radiation

In this chapter we derived the results obtained by Stephen Hawking [1, 2] in a very simplified model, in which we consider only the collapse of an incoming radiation at some specific instant of time. We begin with the quantization of a massless scalar field in a curved background and then apply the results for a spacetime given by the metric of Vaidya [35]. Finally, we obtain the thermal spectrum and show that it corresponds to the *Planck distribution* for bosons.

3.1 Scalar field quantization

Consider the action describing the free propagation of a scalar field in Minkowski space given by

$$S = \frac{1}{2}\int d^4x(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2). \tag{3.1}$$

The EOM are given by

$$\frac{\partial L}{\partial\phi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right) = 0,$$

which result in

$$(\Box + m^2)\phi = 0, \tag{3.2}$$

where $\Box = \partial_\mu\partial^\mu$. Using a Legendre transformation, one obtains the correspondent Hamiltoninan of the scalar field,

$$H = \int d^3x\left(\frac{\partial L}{\partial(\partial_t\phi)}\partial_t\phi - L\right), \tag{3.3}$$

which results in

$$H = \frac{1}{2}\int d^3x(\pi^2 + (\nabla\phi)^2 + m^2), \tag{3.4}$$

where $\pi = \partial_t\phi$ is the canonical conjugate momentum.

Now, in order to quantize the field, let us promote the canonical functions to operators

$$\begin{aligned} \phi(t,\vec{x}) &\to \hat{\phi}(t,\vec{x}), \\ \pi(t,\vec{x}) &\to \hat{\pi}(t,\vec{x}), \end{aligned} \tag{3.5}$$

that satisfies the equal-time commutation relations

$$\begin{aligned} \left[\hat{\phi}(t,\vec{x}),\hat{\pi}(t,\vec{x'})\right] &= i\delta^3(\vec{x}-\vec{x'}), \\ \left[\hat{\phi}(t,\vec{x}),\hat{\phi}(t,\vec{x'})\right] &= \left[\hat{\pi}(t,\vec{x}),\hat{\pi}(t,\vec{x'})\right] = 0. \end{aligned} \tag{3.6}$$

In order to find solutions to the EOM, let us expand the field $\hat{\phi}$ in normal modes, as follows

$$\hat{\phi}(t,\vec{x}) = \int d^3\vec{k}\hat{\phi}_{\vec{k}}(t)f_{\vec{k}}(\vec{x}), \tag{3.7}$$

where the function $f_{\vec{k}}$ is given by

$$f_{\vec{k}}(\vec{x}) = N_k e^{i\vec{k}\cdot\vec{x}}, \tag{3.8}$$

which corresponds to the plane-wave solution. Plugging this into Eq. (3.2) one obtains that

$$\begin{aligned} \partial_t^2\hat{\phi}_{\vec{k}} &= -\omega^2\hat{\phi}_{\vec{k}}, \\ \nabla^2 f &= -k^2 f, \end{aligned} \tag{3.9}$$

where $\omega^2 = k^2 + m^2$. Solutions for the second equation are given by Eq. (3.8), while for the first one,

$$\hat{\phi}_{\vec{k}}(t) = \hat{a}^{(1)}_{\vec{k}}e^{-i\omega t} + \hat{a}^{(2)}_{\vec{k}}e^{i\omega t}. \tag{3.10}$$

Substituting Eqs. (3.8) and (3.10) in Eq. (3.7) one finds that, recalling the fact that $\hat{\phi}$ is a hermitian operator, $\hat{a^{\dagger}}^{(1)}_{-\vec{k}} = \hat{a}^{(2)}_{\vec{k}}$. With this, one can write the quantized field $\hat{\phi}(t,\vec{x})$ in the compact form

$$\hat{\phi} = \int d^3\vec{k}\left[\hat{a}_{\vec{k}}u(t,\vec{x}) + \hat{a}^{\dagger}_{\vec{k}}u^*(t,\vec{x})\right], \tag{3.11}$$

where we define

$$u(t,\vec{x}) = N_k e^{-i(\omega t - \vec{k}\cdot\vec{x})}, \tag{3.12}$$

in which N_k is an normalization constant. One can also write the canonical conjugate momentum as

$$\hat{\pi} = i\int d^3\vec{k}\omega\left[\hat{a}^{\dagger}_{\vec{k}}u^*(t,\vec{x}) - \hat{a}_{\vec{k}}u(t,\vec{x})\right]. \tag{3.13}$$

The definition in Eq. (3.11), in terms of the creation and annihilation operators, must hold for the commutation relations given by Eqs. (3.6), such that

$$\left[\hat{a}_{\vec{k}}, a^{\dagger}_{\vec{k}'}\right] = \delta(\vec{k} - \vec{k}'), \tag{3.14}$$

and all the other commutators vanish.

Now, in order to evaluate the normalization constant in Eq. (3.12), let us substitute Eqs. (3.11) and (3.13) in the first commutation relation in Eq. (3.6), as follows

$$\left[\hat{\phi}(t,\vec{x}), \hat{\pi}(\vec{x'},t)\right] = \int d^3\vec{k}\int d^3\vec{k'}(i\omega_k)\left\{u'u[\,\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k'}}] - u'^*u^*[\hat{a}^{\dagger}_{\vec{k'}}, \hat{a}_{\vec{k'}}]\right\}.$$

Thus, substituting the definition Eq. (3.12) in the above expression and using Eq. (3.14), one finds

$$\left[\hat{\phi}(t,\vec{x}), \hat{\pi}(\vec{x'},t)\right] = i\int d^3\vec{k}2\omega N_k^2 e^{i\vec{k}(\vec{x}-\vec{x'})} = i\delta^3(\vec{x} - \vec{x'}). \tag{3.15}$$

Hence, using that the Dirac delta function is given by

$$\delta^3(\vec{x}-\vec{x'}) = \int d^3\vec{k}\frac{1}{(2\pi)^3}e^{i\vec{k}(\vec{x}-\vec{x'})},$$

one obtains that the normalization constant must be given by $N_k = 1/\sqrt{16\pi^3\omega}$.

It is worth noticing that splitting the field into normal modes yields operators composed by positive and negative frequency solutions with respect to the Minkowskian time, such that

$$\frac{\partial}{\partial t}u_{\vec{k}}(t,\vec{x}) = -i\omega u_{\vec{k}}(t,\vec{x}). \tag{3.16}$$

Furthermore, the Klein-Gordon (KG) product is given by

$$(\phi,\psi) = -i\int d^3\vec{x}(\phi\partial_t\psi^* - \psi^*\partial_t\phi). \tag{3.17}$$

Therefore, the functions $u_{\vec{k}}(t,\vec{x})$ form a complete orthonormal basis with respect to this product.

Finally, the many-particle Fock space is built from the one-particle Hilbert space, which is constructed with the positive frequency solutions defined previously. First, one define the vacuum state $|0\rangle$, which is defined by means of the annihilation operator, as

$$\hat{a}_{\vec{k}}|0\rangle = 0. \tag{3.18}$$

The one-particle states are defined by acting the creation operator in this vacuum state, as follows

$$|1_{\vec{k}}\rangle = \hat{a^\dagger}_{\vec{k}}|0\rangle. \tag{3.19}$$

Therefore, the many particle states are given by

$$|n^{(1)}_{\vec{k}_1}, n^{(2)}_{\vec{k}_2}, ..., n^{(p)}_{\vec{k}_p}\rangle = \left(n^{(1)}!n^{(2)}!...n^{(p)}!\right)^{(-1/2)}(\hat{a^\dagger}_{\vec{k}_1})^{n^{(1)}}(\hat{a^\dagger}_{\vec{k}_2})^{n^{(2)}}...(\hat{a^\dagger}_{\vec{k}_p})^{n^{(p)}}|0\rangle, \tag{3.20}$$

where $\vec{k}_1, \vec{k}_2, ..., \vec{k}_p$ are different indices labeling the particle states.

3.2 Quantization in curved spacetimes

In this case, the lagrangian density is given by [36]

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - m^2\phi^2\right]. \tag{3.21}$$

The EOM are given by

$$(\Box_g + m^2)\phi(x) = 0, \tag{3.22}$$

where $\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu = \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)/\sqrt{-g}$ (see Appendix **??**).

In general, the Klein-Gordon product is written as follows

$$(\phi, \psi) = -i \int_{\Sigma} d\Sigma^{\mu} (\phi \partial_{\mu} \psi^{*} - \psi^{*} \partial_{\mu} \phi), \tag{3.23}$$

where the integral is performed in Σ, which is the Cauchy hypersurface defining the initial data.

Different from the Minkowski quantization, here it is difficult to establish positive and negative frequency solutions, as in general, the metric could change rapidly in time. However, in *stationary* spacetimes we have a timelike vector field ξ^{μ} that defines the isomorphism of the spacetime metric $\delta_{\xi^{\alpha}} g_{\mu\nu} = 0$, where $\delta_{\xi^{\alpha}}$ is an infinitesimal transformation generated by the vector ξ^{α} (as it appears in Appendix A). In this sense one could define the positive frequency solution given by

$$\xi^{\mu} \nabla_{\mu} u_{\vec{k}} = -i\omega u_{\vec{k}}, \tag{3.24}$$

and with that be able to extend the definition of Fock space to curved spacetimes.

In what follows, we will be interested in collapsing spacetimes, and although it seems that the previous analysis does not apply, one can still split the spacetime into two asymptotically stationary regions, one in the asymptotic past, labeled by "in", and one in the asymptotic future, after the collapse, labeled by "out". In order to analyse the quantum states in these regions one needs to make use of the *Bogoliubov transformations*.

3.2.1 Bogoliubov transformations

Here we are going to consider the discrete problem instead of the continuous space of possible normal modes. The only difference is that we replace the integrals by summation and the continuous $\vec{k}$ values by discrete indeces. Considering two asymptotically stationary regions, one has two possible expansions to the scalar field $\hat{\phi}$, one defined in the asymptotic past, characterized by the script "in", as follows

$$\hat{\phi} = \sum_{i} \left(\hat{a}_i^{in} u_i^{in} + \hat{a}_i^{in\dagger} u_i^{in*} \right), \tag{3.25}$$

and the other defined in the asymptotic future, characterized by the script "out",

$$\hat{\phi} = \sum_{i} \left(\hat{a}_i^{out} u_i^{out} + \hat{a}_i^{out\dagger} u_i^{out*} \right). \tag{3.26}$$

The orthonormal relations between the modes are given by

$$\begin{aligned} (u_i^{in}, u_j^{in}) &= \delta_{ij}, \\ (u_i^{in*}, u_j^{in*}) &= -\delta_{ij}, \\ (u_i^{in}, u_j^{in*}) &= 0. \end{aligned} \tag{3.27}$$

The same applies for the "out" modes.

Since each set of modes form a complete basis, one can be expanded in terms of the other. In general one has [37]

$$u_j^{out} = \sum_i (\alpha_{ij} u_i^{in} + \beta_{ij} u_i^{in*}). \tag{3.28}$$

These are called Bogoliubov transformations and α_{ij} and β_{ij} are the Bogoliubov coefficients. Taking this into account one can show, using the definitions in Eqs. (3.27), that

$$\alpha_{ij} = (u_i^{out}, u_j^{in}), \tag{3.29}$$

$$\beta_{ij} = -(u_i^{out}, u_j^{in*}). \tag{3.30}$$

In what follows, let us use the Einstein summation convention. Plugging Eq. (3.28) into the Klein-Gordon products one can find that

$$(u_i^{out}, u_j^{out}) = \delta_{ij} = (u_i^{out}, \alpha_{jk} u_k^{in} + \beta_{jk} u_k^{in*}),$$

so that

$$\alpha_{jk}^* \alpha_{ik} - \beta_{jk}^* \beta_{ik} = \delta_{ij}. \tag{3.31}$$

Moreover,

$$(u_i^{out}, u_j^{out*}) = 0 = (u_i^{out}, \alpha_{jk}^* u_k^{in*} + \beta_{jk}^* u_k^{in}).$$

Thus,

$$\beta_{jk} \alpha_{ik} - \alpha_{jk} \beta_{ik} = 0. \tag{3.32}$$

In the above results we have used the property of the Klein-Gordon product, as follows

$$(f_1, af_2 + bf_3) = -i \int d\Sigma^\mu (a^* f_1 \partial_\mu f_2^* + b^* f_1 \partial_\mu f_3^* - a^* f_2^* \partial_\mu f_1 - b^* f_3^* \partial_\mu f_1),$$

which results in

$$(f_1, af_2 + bf_3) = a^*(f_1, f_2) + b^*(f_1, f_3). \tag{3.33}$$

Now, one can invert the relation in Eq. (3.28) and finds that

$$u_i^{in} = \alpha_{ij}^* u_j^{out} - \beta_{ij} u_j^{out*}. \tag{3.34}$$

One can also calculate the creation and annihilation operators in terms of KG products, as follows

$$(\hat{\phi}, u_i^{in}) = \left(\hat{a}_j^{in} u_j^{in} + \hat{a}_j^{in\dagger} u_j^{in*}, u_i^{in} \right) = \hat{a}_j^{in} (u_j^{in}, u_i^{in}), \tag{3.35}$$

using Eq. (3.27) one obtains

$$\hat{a}_i^{in} = (\hat{\phi}, u_i^{in}). \tag{3.36}$$

Analogously

$$\hat{a}_i^{out} = (\hat{\phi}, u_i^{out}). \tag{3.37}$$

Using the definitions of $\hat{\phi}$, from Eqs. (3.25, 3.26), one can write these operators in terms of one another,

$$\begin{aligned} a_i^{in} &= \left(\hat{a}_j^{out} u_j^{out} + \hat{a^\dagger}_j^{out} u^{*out}_j, u_i^{in}\right) \\ &= \left[\alpha_{ji}\hat{a}_j^{out} + \hat{a}_j^{out\dagger}\left(u_j^{out*}, u_i^{in}\right)\right], \end{aligned} \tag{3.38}$$

thus,

$$\hat{a}_i^{in} = \left(\alpha_{ji}\hat{a}_j^{out} + \beta_{ji}^* \hat{a}_j^{out\dagger}\right). \tag{3.39}$$

Analogously,

$$\hat{a}_i^{out} = \left(\alpha_{ji}^*\hat{a}_j^{in} - \beta_{ji}^* \hat{a}_j^{in\dagger}\right). \tag{3.40}$$

In general, the vacuum states at each region are different, defined as

$$\hat{a}_i^{in}|in\rangle = 0, \tag{3.41}$$

$$\hat{a}_i^{out}|out\rangle = 0. \tag{3.42}$$

Therefore, the Fock spaces of u^{in} and u^{out} are not the same. Take for instance

$$\hat{a}_i^{in}|out\rangle = \beta_{ji}^*\hat{a}_j^{out\dagger}|out\rangle = \beta_{ji}^*|1_j^{out}\rangle \neq 0. \tag{3.43}$$

If the coefficients β_{ij} vanish, the vacuum state remains unchanged as the metric evolves.

Since, in general, the β_{ij} coefficients do not vanish, there will be a particle content related to the previous vacuum state $|in\rangle$, in the "out" region. In order to investigate that, let us calculate the number of modes u^{in} in the state $|out\rangle$, evaluated by means of the "in" particle number operator, $N_i^{in} = \hat{a}_i^{in\dagger}\hat{a}_i^{in}$,

$$\begin{aligned} \langle out|N_i^{in}|out\rangle &= \langle out|\hat{a}_i^{in\dagger}\hat{a}_i^{in}|out\rangle, \\ &= \langle out|(\beta_{ij}\hat{a}_j^{out})(\beta_{ki}^*\hat{a}_k^{out\dagger})|out\rangle = \beta_{ij}\beta_{ki}^*\langle 1_j^{out}|1_k^{out}\rangle. \end{aligned}$$

Henceforth, recalling that $\langle 1_j^{out}|1_k^{out}\rangle = \delta_{jk}$, one obtains

$$\langle out|N_i^{in}|out\rangle = \sum_j |\beta_{ij}|^2. \tag{3.44}$$

This demonstrates that there is a particle content of excited states of $|in\rangle$ in the final stationary metric, i.e., the difference between the asymptotic basis leads to nonzero expectation value of vacuum state after Bogoliubov transformation.

3.3 Vaidya spacetime

In this section, let us restrict ourselves to the simplest case of black hole formation, given by Vaidya's solution of Einstein's equations. The line element of such a solution is given by [35] [1]

$$ds^2 = -\left(1 - \frac{M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{3.46}$$

where Ω^2 is the solid angle and $M(v)$ is the total mass, depending only on the null coordinate v. When $M(v) = \textit{constant}$ one has the Schwarzschild spacetime, written in the Eddington-Finkelstein coordinates [38]. Here the collapse is driven by an incoming flux of radiation at a certain instant of time or an interval of the null coordinate v. Here we are going to consider that the flux of radiation occurs instantaneously, i.e., the collapse will be driven by an incoming shock wave located at, for instance, $v = v_0$. The two asymptotic regions in this case are: the Minskowski spacetime, that we will call "in", and the Schwarzschild region, called "out". This whole setup is depicted in Fig. (3). The Penrose diagram is depicted in terms of compact null variables, so that one can represent the whole spacetime (that goes to infinity) in a finite picture. The symbols in the graphic are characterized as follows: i^- is the past timelike infinity ($v \to -\infty$ and $u \to -\infty$), i^+ is the future timelike infinity ($v \to +\infty$ and $u \to +\infty$), i^0 is the spacelike infinity ($v \to +\infty$ and $u \to -\infty$), $\mathcal{I}^-$ is the past null infinity (v fixed and $u \to -\infty$) and finally, $\mathcal{I}^+$ is the future null infinity (u fixed and $v \to +\infty$) (for a review about Penrose diagrams check [37]).

Now, we consider the classic massless scalar field to be quantized in this scenario. The Klein-Gordon equation for this case is given by

$$\Box\phi = 0. \tag{3.47}$$

Recalling the fact that the background space is spherically symmetric, one can expand the scalar field into spherical harmonics solutions, as follows

$$\phi(t, \vec{x}) = \sum_{l,m} \frac{\phi_l(t,r)}{r} Y_{lm}(\theta, \varphi), \tag{3.48}$$

where Y_{lm} are the spherical harmonics.

Let us solve Eq. (3.47) using separation of variables. Substituting the above solution in the KG equation, for the Minkowski spacetime, $v < v_0$ (which corresponds to the "in" region), one obtains, suppressing the sum signal, that

$$-\frac{Y_{lm}}{r}\frac{\partial^2 \phi_l}{\partial t^2} + \frac{Y_{lm}}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{\phi_l}{r}\right) + \frac{\phi_l}{r^3 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} Y_{lm}\right) + \frac{\phi_l}{r^3 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2} Y_{lm} = 0.$$

Multiplyinig the above equation by $r^3/Y_{lm}\phi_l$ one finds that

$$-\frac{r^2}{\phi_l}\frac{\partial \phi_l}{\partial t^2} + \frac{r}{\phi_l}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{\phi_l}{r}\right) = -\left[\frac{1}{Y_{lm}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y_{lm}\right) + \frac{1}{Y_{lm}\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y_{lm}\right].$$

[1] The general Vaidya spacetime is given by

$$ds^2 = -\left(1 - \frac{M(v)}{r}\right) dv^2 + 2\varepsilon dvdr + r^2 d\Omega^2, \tag{3.45}$$

where the sign of ε indicates incoming (positive) or outgoing (negative) radiation.

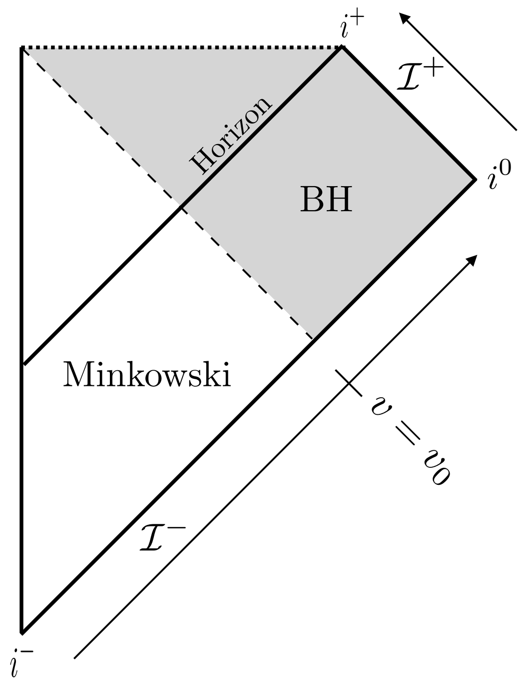

Fig. 3 – Penrose diagram for Vaidya spacetime. The dotted and dashed lines represents, respectively, the singularity and the shock wave. White and gray areas represent the flat spacetime and the black hole region, respectively. The collapse is driven by a shock wave located at $v = v_0$, after that a Schwarzschild black hole is formed.

Notice that the term on the left hand side depends only on angular coordinates, such that they must be a constant. This constant is of the form $l(l+1)$. This choice is related to the regularity of the spherical harmonics $Y_{lm}(\theta, \varphi)$ at the poles of the sphere, where $\theta = 0, \pi$. Henceforth, expanding the derivatives with respect to the radius r, one finally finds that the equation for the fields, in the "in" region, to be like

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right)\phi_l(r,t) = 0. \tag{3.49}$$

Analogously, for the Schwarzschild metric, $v > v_0$ (which corresponds to the "out" region), one obtains (see Appendix B)

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^{*2}} - V_l(r)\right)\phi_l(r,t) = 0, \tag{3.50}$$

where r^* is the Eddington-Finkelstein coordinate given by

$$r^* = r + 2M\ln|r/2M - 1|,$$

and $V(r)$ is the effective potential due to the curvature

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{3.51}$$

Noticed that at the horizon ($r = 2M$) the potential vanishes. Since we are interested only in the phenomena occurring at the horizon, where the potential vanishes, let us consider

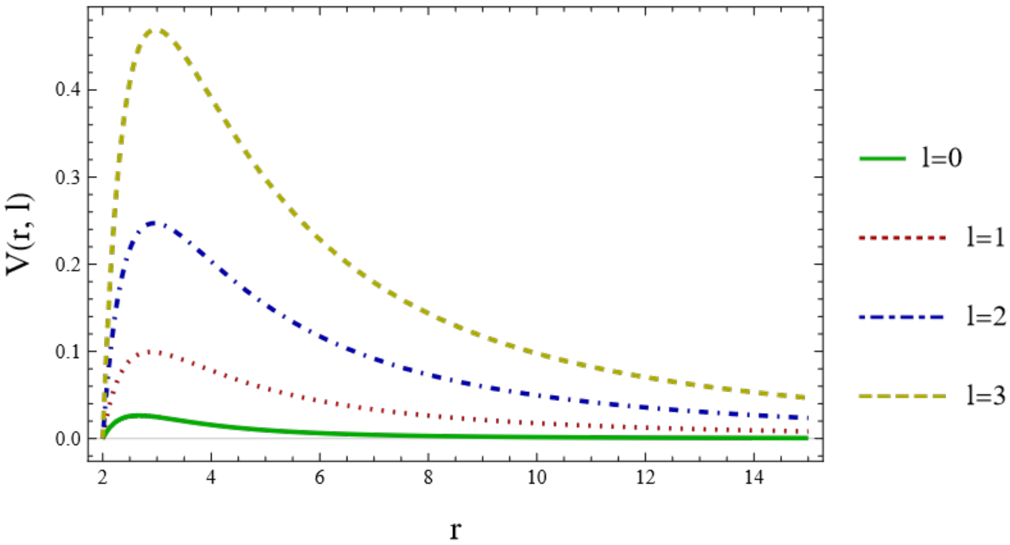

Fig. 4 – Potential defined by Eq. (4). Solid, dotted, dotdashed and dashed lines represents the behavior of the potential for $l = 0$, $l = 1$, $l = 2$, $l = 3$, respectively. Here we used that $M = 1$, such that the event horizon is located at $r = 2$.

an approximation in which we neglect the potential everywhere. One could also look at this approximation as setting the wave as the "s-wave component", that is the one less affected by the potential and is characterized by $l = 0$. The potential behavior is depicted in Fig. (4). Once this has been set, Eqs. (3.49) and (3.50) reduce to

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)\phi(r,t) = 0, \tag{3.52}$$

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^{*2}}\right)\phi(r,t) = 0. \tag{3.53}$$

Next, assuming the harmonic time dependence of the fields, one can write

$$\phi(r,t) = e^{-i\omega t}\phi(r). \tag{3.54}$$

With that, Eqs. (3.52) and (3.53) reduce to the radial dependent equations

$$\frac{d^2\phi(r)}{dr^2} + \omega^2\phi(r) = 0, \tag{3.55}$$

$$\frac{d^2\phi(r)}{dr^{*2}} + \omega^2\phi(r) = 0. \tag{3.56}$$

It is worth writing the line elements of each of the regions using null coordinates. The Minkowski line element is given, in spherical coordinates, by

$$ds_{in}^2 = -dt_{in}^2 + dr_{in}^2 + r_{in}^2 d\Omega^2,$$

where the subscript "in" indicates that this is the asymptotic past region of the asymptotically stationary spacetime. Choosing the null coordinates $u_{in} = t_{in} - r_{in}$ and $v = t_{in} + r_{in}$ one can rewrite the above expression as

$$ds_{in}^2 = -du_{in}dv + r_{in}^2 d\Omega^2. \tag{3.57}$$

Moreover, the Schwarzschild line element is given by

$$ds^2_{out} = -\left(1 - \frac{2M}{r_{out}}\right) dt^2_{out} + \left(1 - \frac{2M}{r_{out}}\right)^{-1} dr^2_{out} + r^2_{out} d\Omega^2, \tag{3.58}$$

where subscript "out" characterizes the asymptotic future region of the spacetime. Again, using the null coordinates, $u_{out} = t_{out} - r^*_{out}$ and $v = t_{out} + r^*_{out}$, one obtains

$$ds^2_{out} = -\left(1 - \frac{2M}{r_{out}}\right) du_{out} dv + r^2_{out} d\Omega^2. \tag{3.59}$$

The modes of the scalar field will be functions of these null coordinates, each one related to the possible incoming and outgoing solutions.

3.3.1 Quantized fields

Now we analyse the modes in each region using the treatment described in Sec. (3.2.1). First, since we restrict ourselves to the case in which $l = 0$, the solutions given by Eq. (3.48) can be written, generically, as

$$\hat{\phi}(x) = \frac{1}{2\sqrt{\pi}} \frac{\hat{\phi}(t, r)}{r} = \frac{1}{2\sqrt{\pi}} \frac{e^{-i\omega t}\hat{\phi}(r)}{r},$$

where we have used that $Y_{00} = 1/2\sqrt{\pi}$. Proceeding with the solutions of Eqs. (3.55) and (3.56) one finds, for the "in" region that

$$\hat{\phi}(x) = \frac{N_\omega}{2r\sqrt{\pi}} (\hat{a}^{(1)in}_\omega e^{-i\omega v} + \hat{a}^{(2)in}_\omega e^{-i\omega u_{in}}), \tag{3.60}$$

where N_ω is the normalization constant. For the "out" region one has

$$\hat{\phi}(x) = \frac{N_\omega}{2r\sqrt{\pi}} (\hat{a}^{(1)out}_\omega e^{-i\omega v} + \hat{a}^{(2)out}_\omega e^{-i\omega u_{out}}). \tag{3.61}$$

Now, in order to analyse the particle creation in the collapsing background we shall define the "in" Fock space associated with the asymptotic past region, defined at $\mathcal{I}^-$. The positive frequency modes are given by

$$u^{in}_\omega = \frac{N_\omega}{2r\sqrt{\pi}} e^{-i\omega v}.$$

In order to compute the normalization constant, let us substitute this in the KG product as follows

$$\begin{aligned}
(u^{in}_\omega, u^{in}_{\omega'}) &= -i \int_{\mathcal{I}^-} dv r^2 d\Omega (u^{in}_\omega \partial_v u^{in*}_{\omega'} - u^{in*}_{\omega'} \partial_v u^{in}_\omega) \\
&= N^2_\omega \int_{\mathcal{I}^-} [\omega' e^{iv(\omega - \omega')} + \omega e^{iv(\omega' - \omega)}] \\
&= 4\pi\omega N^2_\omega \delta(\omega - \omega') = \delta(\omega - \omega').
\end{aligned}$$

Therefore, the normalization constant is $N_\omega = 1/2\sqrt{\omega\pi}$ and the mode defined in the asymptotic past is given by

$$u_\omega^{in} = \frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega v}}{r}. \tag{3.62}$$

Analogously, the Fock space associated with the asymptotic future region, defined at $\mathcal{I}^+$, has its positive frequency modes given by

$$u_\omega^{out} = \frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega u_{out}}}{r}. \tag{3.63}$$

The normalization conditions for these modes are given by

$$(u_\omega^{out}, u_{\omega'}^{out}) = -i\int_{\mathcal{I}+} du_{out} r^2 d\Omega(u_\omega^{out}\partial_{u_{out}}u_{\omega'}^{out*} - u_{\omega'}^{out*}\partial_{u_{out}}u_\omega^{out}) = \delta(\omega-\omega'). \tag{3.64}$$

It is worth mentioning that although one could start with the surface $\mathcal{I}^+$ as a possible Cauchy surface, it will not correspond to the whole space of initial data, since one should add also the future horizon. To avoid any problem arising from this choice, in what follows we should set the asymptotic past surface as the Cauchy surface of initial data, with that one covers the whole spacetime configuration.

Let us determine the Bogoliubov coefficients related to the number of particles created by the collapse, thus

$$\beta_{\omega\omega'} = -(u_\omega^{out}, u_{\omega'}^{in*}) = i\int_{\mathcal{I}^-} dv r^2 d\Omega(u_\omega^{out}\partial_v u_{\omega'}^{in} - u_{\omega'}^{in}\partial_v u_\omega^{out}). \tag{3.65}$$

Notice that, in order to calculate this coefficient, one should know the behavior of the modes u_ω^{out} in the asymptotic past region, as we choose $\mathcal{I}^-$ as the Cauchy surface. In other words, one should obtain u_ω^{out} as a function of v. For that, we determine the form of these modes in the Minkowski region ($v < v_0$) using matching conditions. Before the shock wave $v < v_0$ we have that the modes are given by

$$u_\omega^{out} = \frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega u_{out}(u_{in})}}{r}. \tag{3.66}$$

The matching condition on $v = v_0$ implies that

$$r(v_0, u_{in}) = r(v_0, u_{out}), \tag{3.67}$$

but for Minkowski

$$r(v_0, u_{in}) = \frac{v_0 - u_{in}}{2},$$

and for Schwarzschild

$$r^*(v_0, u_{out}) = \frac{v_0 - u_{out}}{2} = r(v_0, u_{out}) + 2M\ln\left|\frac{r(v_0, u_{out})}{2M} - 1\right|. \tag{3.68}$$

Henceforth, substituting in Eq. (3.67), one obtains

$$u_{out} = u_{in} - 4M\ln\left|\frac{v_H - u_{in}}{4M}\right|, \tag{3.69}$$

where $v_H = v_0 - 4M$ is the the location of the null ray forming the horizon at $u_{out} = \infty$, characterized in the Fig. (3) by i^+.

Now, in the Minkowski region we need to guarantee the regularity at $r = 0$, so that the full mode "out" in such a region will be given by

$$u_\omega^{out} = \frac{1}{4\pi\sqrt{\omega}}\left[\frac{e^{-i\omega u_{out}(u_{in})}}{r} - \frac{e^{-i\omega u_{out}(v)}}{r}\Theta(v_H - v)\right], \tag{3.70}$$

where $u_{out}(v) = u_{out}(u_{in} \leftrightarrow v)$.

Since we are interested in the form of the "out" modes at $\mathcal{I}^-$, the first term of Eq. (3.70) will not contribute. At early times, $u_{out} \to -\infty$ and $v \to -\infty$, one has $u_{out}(v) \approx v$. So that, at the asymptotic past region

$$u_\omega^{out} \approx -\frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega v}}{r}. \tag{3.71}$$

Except for the negative sign, this is identical to the "in" mode, given by Eq. (3.62), so that $\beta_{\omega\omega'}$ vanish. Therefore in the asymptotic past region there is no particle emission, as expected.

At late times, $u_{out} \to +\infty$ and $v \to v_H$, one has

$$u_{out}(v) \approx v_H - 4M\ln\frac{v_H - v}{4M}. \tag{3.72}$$

Henceforth, in the regime near v_H, the modes are given by

$$u_\omega^{out} = -\frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega\left(v_H - 4M\ln\frac{v_H - v}{4M}\right)}}{r}. \tag{3.73}$$

This result shows that at late times, there is particle production, as the $\beta_{\omega\omega'}$ coefficients associated with these modes do not vanish [1, 2].

3.3.2 Thermal distribution

Let us then calculate the $\beta_{\omega\omega'}$ according to Eq. (3.65). First, let us use partial integration to neglect the boundary terms, so that

$$\beta_{\omega\omega'} = 2i\int_{\mathcal{I}^-} dv r^2 d\Omega u_\omega^{out}\partial_v u_{\omega'}^{in}. \tag{3.74}$$

Substituting Eqs. (3.62) and (3.73) in the equation above, one finds

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{v_H} dv e^{-i\omega\left[v_H - 4M\ln\frac{(v_H - v)}{4M}\right] + i\omega' v}.$$

Thus, introducing the new variable $x = v_H - v$, one obtains

$$\beta_{\omega\omega'} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}e^{-v_H(\omega' + \omega)}(4M)^{-4MiM}\int_0^\infty dx x^{4i\omega M}e^{i\omega' x}.$$

In principle, this quantity is not convergent, since we are not summing up for all possible values of the frequency. To avoid the divergence problem one can add an infinitesimal term $(-\epsilon)$ in the exponent of the exponential function in the integrand. Heceforth, using the following identity,

$$\int_0^\infty x^a e^{-bx} = b^{-1-a}\Gamma(1+a), \tag{3.75}$$

one obtains

$$\beta_{\omega\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{-v_H(\omega'+\omega)}(4M)^{-4MiM}(-i\omega'+\epsilon)^{-1-4i\omega M}\Gamma(1+4i\omega M). \tag{3.76}$$

Analogously, using the identity Eq. (3.29) one finds

$$\alpha_{\omega\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{-v_H(\omega'+\omega)}(4M)^{-4MiM}(i\omega'+\epsilon)^{-1-4i\omega M}\Gamma(1+4i\omega M). \tag{3.77}$$

Through the following identity

$$\ln(-\omega' - i\epsilon) = -i\pi + \ln\omega',$$

one obtains

$$(-i\omega'+\epsilon) = e^{i\pi}(i\omega'+\epsilon).$$

Therefore, comparing Eqs. (3.76) and (3.77) one gets

$$\alpha_{\omega\omega'} = -e^{4\pi\omega M}e^{2i\omega' v_H}\beta_{\omega\omega'},$$

and, finally,

$$|\alpha_{\omega\omega'}| = e^{4\pi\omega M}|\beta_{\omega\omega'}|. \tag{3.78}$$

Although one can use this relation to calculate the spectrum of particles emitted, it is better to analyse such aspect by means of wavepackets. The relation between $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ will remain unaltered, except for the fact that we consider wavepackets sharply peaked around a frequency ω_j.

A complete orthonormal set of wavepackets, described by discrete quantum numbers, is given by [37]

$$u_{jn}^{out} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} d\omega e^{2i\pi\omega n/\epsilon} u_\omega^{out}, \tag{3.79}$$

where $j \geq 0$ and n are integers. Now, let us substitute Eq. (3.63) in the above equation, so that one can rewrite it as

$$ru_{jn}^{out} = \frac{1}{4\pi\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} \frac{d\omega}{\sqrt{\omega}} e^{i\omega(2\pi n/\epsilon - u_{out})}.$$

The real part will be given by

$$u(u_{out}) \doteq \mathcal{R}(ru_{jn}^{out}) = \frac{1}{4\pi\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} \frac{d\omega}{\sqrt{\omega}} \cos\omega\left(\frac{2\pi n}{\epsilon} - u_{out}\right). \tag{3.80}$$

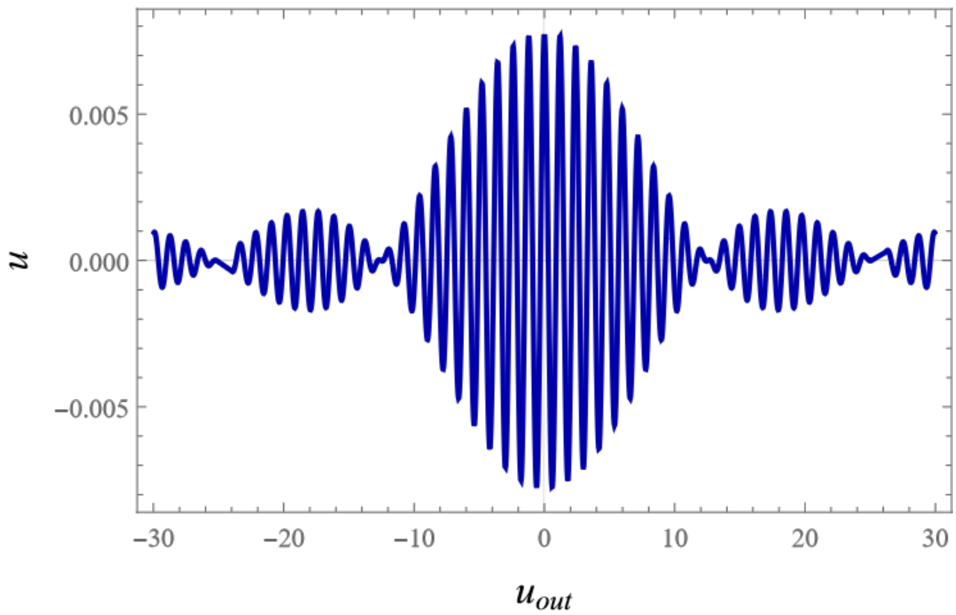

Fig. 5 – Real part of the wavepacket mode given by Eq. (3.79). Here we set $j = 10$, $\epsilon = 5$ and $n = 0$. The wavepackets are peaked around $u_{out} = 0$.

The wavepackets have a peak around $u_{out} = 2\pi n/\epsilon$ and, as we set ϵ to be an infinitesimal value, the modes are narrowly centered around $\omega_j = j\epsilon$. This behavior is depicted in Fig. (5).

It is worth noticing that the choice of treating wavepackets is not only mathematical convenient but also has a physical meaning. When we treat modes with definite frequency, the uncertainty of time measurements explodes, so that one could only analyse particle emission occurring in random instants of time. When we consider wavepackets we restrict ourselves to study those modes with frequency within ϵ of ω_j, emitted at $u_{out} = 2\pi n/\epsilon$.

Using this definition, after a rigorous derivation [37], one obtains the relation between $\alpha_{jn,\omega'}$ and $\beta_{jn,\omega'}$ to be

$$|\alpha_{jn,\omega'}| = e^{4\pi\omega_j M}|\beta_{jn,\omega'}|, \tag{3.81}$$

which is almost unchanged, compared with Eq. (3.78).

Let us now use Eq. (3.31), which in this case is given by

$$\int_0^\infty d\omega'(\alpha_{jn,\omega'}\alpha^*_{j'n',\omega'} - \beta_{jn,\omega'}\beta^*_{j'n',\omega'}) = \delta_{jj'}\delta_{nn'}.$$

For $j = j'$ and $n = n'$, the equation above reduces to

$$\int_0^\infty d\omega' \left(|\alpha_{jn,\omega'}|^2 - |\beta_{jn,\omega'}|^2\right) = 1. \tag{3.82}$$

Therefore, substituting Eq. (3.81), one obtains that the expectation value of number of particles emitted at late times is given by

$$\langle in|N^{out}_{jn}|in\rangle = \int_0^\infty d\omega'|\beta_{jn,\omega'}|^2 = \frac{1}{e^{8\pi M\omega_j} - 1}. \tag{3.83}$$

Finally, comparing this with the *Planck distribution* of radiation for bosons, namely

$$\frac{1}{e^{\hbar\omega_j/k_BT}-1},$$

where k_B is the Boltzman's constant, one obtains the temperature of the particles distribution

$$T_H = \frac{\hbar}{8\pi k_B M}. \tag{3.84}$$

This is called *Hawking temperature* of the black hole [2]. Notice that it is inversely proportional to the black hole mass. In terms of the mass of the sun, one can write

$$T_H \approx 10^{-7}\frac{M_s}{M}\mathrm{K}.$$

Notice that this quantity is very small when we consider astrophysical black holes, with mass many times larger than the mass of the sun.

Final Remarks

In the first part of this thesis we analysed analog models obtaind from linear magnetoelectric materials. There are different ways of producing such analogies between light propagation in optical media and in curved spacetimes. The way explored in this project is based on the description of light propagation in an optical medium through an effective geometric interpretation, as formally discussed in Sec. 1.1. In such scenario, it is possible to relate the optical coefficients of the medium in consideration with the metric components of a gravitational analog. Another way is to start with a metric solution of general relativity and relate the modification of the electromagnetic fields in such curved spacetime with the constitutive relations of a hypothetical optical medium. More specifically, in the presence of gravity, the electromagnetic field in the empty space has its properties affected by the curvature of the spacetime. As already known since Einstein's early publications on GR, the contravariant and covariant forms of the electromagnetic tensor, which are associated by the metric, are related by means of an expression that mimics the constitutive relations of a material medium. The metric components describing a curved spacetime can thus be compared with the susceptibility coefficients that describe an effective optical medium, so that a formal analogy is possible [5, 39]. One immediate conclusion is that such an analogy requires a linear magnetoelectric medium. In this context, it was recently noticed that the term that plays the role of the magnetoelectric coefficient is antisymmetric [40]. On the other hand, as shown in Sec. 1.3, if we start by analysing the propagation of light in a material medium that exhibits a non-symmetric magnetoelectric coefficient, one finds that such a system mimics a curved spacetime presenting nonzero time-space mixed terms in the metric which depend only on the antisymmetric part of α_{ij}, as described by Eq. (1.51). It is important to mention that, in the context of Sec. 1.3, the analog models are constructed by means of the solutions for plane-wave propagation. Some approximations are usually implemented because in natural systems the optical coefficients are originated in the expansion of the free-energy density of the optical material in terms of the electromagnetic fields. For instance, in the case of the magnetoelectric effect, second order contributions in α_{ij} are usually suppressed, as these terms are expected to be corrections when compared to the other linear terms.

As specifically examined in Sec. 1.3.2, the magnetoelectric effect motivates the conception of an idealised analog model containing an event horizon. Light propagation in the neighborhood of such analog horizon exhibits a non-symmetric spatial behavior. At one side of it, there will be only one direction in which both wave solutions can propagate. This aspect is similar to the behavior of light propagation in the interior of the Schwarzschild event horizon. In fact, one can notice that the solutions given by

Eqs. (1.47) and (1.48) are very similar to the solutions that describe a radial propagation in the Schwarzschild metric, written in the Painlevé-Gullstrand coordinates. Another point that should be mentioned is that the interior of the analog event horizon ($z < z_h$ in the toy model) becomes a birefringent system, because the two distinct solutions propagate in the same direction (given by the wave vector) with different phase velocities. Moreover, it is noteworthy that near the horizon one of the solutions behaves as a slow-light mode. It propagates with a phase velocity that gets an arbitrarily small value as it gets close to the horizon, on each side of it. Additionally, even outside the horizon ($z > z_h$), reversing the propagation direction leads to different behaviors of the light rays. This space inversion symmetry breaking is known to occur in the presence of magnetoelectric couplings.

The main aspect behind the event horizon solution was the assumption of an optical coefficient that depends on position. This behavior can be naively imagined to be artificially produced by using a material whose optical properties are conveniently sensitive to temperature differences, or even by joining parallel layers of materials, specially conceived to guarantee that in each layer the optical effect would occur with a different magnitude.

Furthermore, analog models for gravity can also be studied in the context of nonlinear couplings. Generally, if we consider the expansion of the polarization and magnetization vectors in terms of the applied fields, several effects in different orders of magnitude are bond to appear in a optical medium, depending on its physical properties. In such nonlinear systems the applied fields may explicitly appear in the metric components of the corresponding analog model, which can lead to richer scenarios to investigate GR metric solutions. Additionally, it should be mentioned that when natural materials are being considered, the implemented approximations to first order effects in the linear magnetoelectric coupling can be partially justified by means of the possible existence of nonlinear effects. For instance, if we had kept second order contributions to the linear coupling, the magnitude of corresponding effects would be of the same order, or even smaller, than those associated with higher-order couplings, and its presence would require further analysis.

In the second part of this project, we studied analog models obtained from acoustic black holes and the Hawking's radiation phenomenon. When Hawking first described the process of black hole evaporation, one of the main experimental consequences was that the temperature of the radiation, emitted in the process of collapse, is very small when we consider astrophysical scenarios, as it is inversely proportional to the black hole mass. This result arose an experimental problem, as the measurement tools available are not able to detect such small temperatures. Although it can be argued that in primordial black holes (before inflation) the temperature of the radiation would be larger, since their mass are relatively small, there is still no evidence of the existence of such a structure in our universe. In fact, even if at early stages of the universe they could have been produced [41, 42], it is argued that the inflation period could have accelerate their process of evaporation so that they do not exist anymore [43]. Therefore, the only source of this kind of radiation would be the astrophysical black holes as, for instance, the one at the center of our galaxy, called M87. Such black holes are extremely massive, henceforth the temperature of their radiation would be many times smaller than the cosmic microwave background temperature.

Therefore, one possible way of studying such a phenomenon would be by constructing analog systems in terrestrial laboratories, those that mimics the kinematic aspects of this GR solution. Here we made a review on acoustic black holes and showed that the propagation of acoustic perturbations inside a moving medium behaves exactly like the propagation of a massless scalar field near a black hole given by the Schwarzschild solution. The soundwaves are trapped by a sort of "sonic horizon" that, in our case, drag the perturbations to the direction of the flow propagation.

Moreover, we also studied the quantization of fields in curved spacetimes. We restrict ourselves to the investigation of the quantized modes in a simple model of spacetime presenting a gravitational collapse, the Vaidya spacetime. The metric transition, from flat to curved, occurs due the collapse of incoming radiation, a shockwave concentrated in a particular instant of time. Such a metric is obviously non stationary however, we could split the whole spacetime configuration into two asymptotically stationary regions, called the "asymptotic past", characterized by the Minkowski metric, and "asymptotic future" given by the Schwarzschild metric. Henceforth, by means of the Bogolubiov transformations we showed that the vacuum states of the quantum fields at each region are different and this yields to the production of particles during the transition. Finally, by studying the modes at each stationary stage of the metric, we recover the result first analysed by Hawking [2], in which he showed that the thermal spectrum of the emission is equal to the Planck distribution for bosons and identified the "Hawking temperature", given by Eq. (3.84).

Appendix

APPENDIX A – Energy-momentum tensor in curved spacetime

In curved spacetime, a field action will, generally, be given by

$$S = \frac{1}{c}\int \mathcal{L}\sqrt{-g}d\Omega. \tag{A.1}$$

Now, consider the following transformation

$$x^{\mu} \to x^{\mu\prime} = x^{\mu} + \xi^{\mu}, \tag{A.2}$$

where $\xi^{\mu} \to 0$. Under this transformation, the metric transforms as follows

$$g^{\mu\nu}(x^{\alpha}) \to g^{\mu\nu\prime}(x^{\alpha\prime}) = \frac{\partial x^{\mu\prime}}{\partial x^{\alpha}}\frac{\partial x^{\nu\prime}}{\partial x^{\beta}}g^{\alpha\beta}. \tag{A.3}$$

Thus, substituting the transformations in Eqs. (A.2) one finds, up to first order in ξ^{μ}, the metric transformation

$$g^{\mu\nu\prime}(x') = g^{\alpha\beta}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + \delta^{\mu}_{\alpha}\xi^{\nu}{}_{,\beta} + \delta^{\nu}_{\beta} + \delta^{\nu}\beta\xi^{\mu}{}_{,\alpha}) = g^{\mu\nu}(x) + g^{\mu\beta}\xi^{\nu}{}_{,\beta} + g^{\alpha\nu}\xi^{\mu}{}_{,\alpha}\,. \tag{A.4}$$

In order to maintain the dependence on the old coordinates, let us expand

$$g^{\mu\nu\prime}(x^{\alpha\prime}) = g^{\mu\nu\prime}(x^{\alpha} + \xi^{\alpha}) = g^{\mu\nu\prime}(x) + \frac{\partial g^{\mu\nu}}{x^{\alpha}}\xi^{\alpha}. \tag{A.5}$$

Henceforth, substituting this on Eq. (A.4) one obtains

$$g^{\mu\nu\prime}(x) = g^{\mu\nu}(x) - \frac{\partial g^{\mu\nu}}{\partial x^{\alpha}}\xi^{\alpha} + g^{\mu\beta}\xi^{\nu}{}_{,\beta} + g^{\alpha\nu}\xi^{\mu}{}_{,\alpha}\,. \tag{A.6}$$

Now, using, that the covariant derivative of ξ^{μ} is given by

$$\xi^{\nu}{}_{;\alpha} = \xi^{\nu}{}_{,\alpha} + \Gamma^{\nu}_{\beta\alpha}\xi^{\beta}, \tag{A.7}$$

one finds that the linear terms in ξ^{μ} in Eq. (A.6) are written as

$$g^{\mu\beta}\xi^{\nu}{}_{,\beta} + g^{\alpha\nu}\xi^{\mu}{}_{,\alpha} - \frac{\partial g^{\mu\nu}}{\partial x^{\alpha}}\xi^{\alpha} = \left(\xi^{\mu;\nu} + \xi^{\nu;\mu}\right) + \xi^{\alpha}\left(g^{\mu\nu}{}_{;\beta}\right). \tag{A.8}$$

Since the covariant derivative of the metric tensor vanishes one finally obtains that the metric transforms as

$$g^{\mu\nu\prime} = g^{\mu\nu} + \delta g^{\mu\nu}, \tag{A.9}$$

where $\delta g^{\mu\nu} = \xi^{\mu;\nu} + \xi^{\nu;\mu}$. This is a diffeomorphism of the metric $g^{\mu\nu}$. Analogously,

$$g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \tag{A.10}$$

where $\delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu}$.

Let us then consider the variation principle in the action of Eq. (A.1) with respect to the metric, so that

$$\begin{aligned}\delta S &= S[g^{\mu\nu} + \delta g^{\mu\nu}, \partial_\alpha(g^{\mu\nu} + \delta g^{\mu\nu})] - S[g^{\mu\nu}, \partial_\alpha g^{\mu\nu}] \\ &= \frac{1}{c}\int \left[\frac{\partial\sqrt{-g}L}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\sqrt{-g}L}{\partial g^{\mu\nu}{}_{,\alpha}}\delta\partial_\alpha g^{\mu\nu}\right] d\Omega.\end{aligned} \tag{A.11}$$

Now, using integration by parts one finds

$$\delta S = \frac{1}{2c}\int \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} d\Omega, \tag{A.12}$$

where we define

$$\frac{1}{2}\sqrt{-g}T_{\mu\nu} = \frac{\partial\sqrt{-g}L}{\partial g^{\mu\nu}} - \partial_\alpha\frac{\partial\sqrt{-g}L}{\partial(g^{\mu\nu}{}_{,\alpha})}. \tag{A.13}$$

Using that,

$$g^{\mu\nu}\delta_{\alpha\nu} = -(\xi_\alpha;_\nu + \xi_\nu;_\alpha)g^{\mu\nu} = -(\xi_\alpha^{;\mu} + \xi^\mu;_\alpha) = -g_{\nu\alpha}\delta g^{\mu\nu},$$

one obtains that $T_{\mu\nu}\delta g^{\mu\nu} = -T^{\mu\nu}\delta g_{\mu\nu}$. Substituting this relation in Eq. (A.12), it can be rewritten as

$$\delta S = -\frac{1}{2c}\int \sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} d\Omega. \tag{A.14}$$

Let us introduce the diffeomorphism definition in Eq. (A.12) and use the fact that the energy-momentum tensor is symmetric, such that

$$\delta S = \frac{1}{c}\int \sqrt{-g}T_{\mu\nu}\xi^{\mu;\nu} d\Omega. \tag{A.15}$$

Integrating by parts and recalling that the fields ξ^μ vanish at infinity, one has, imposing invariance under diffeomorphism

$$\delta S = -\frac{1}{c}\int \sqrt{-g}T^\nu_\mu;_\nu \xi^\mu d\Omega = 0. \tag{A.16}$$

Since ξ^μ are small but arbitrary quantities, one finally obtains that

$$T^\nu_\mu;_\nu = 0. \tag{A.17}$$

This is the covariant form of the energy-momentum conservation equation.

APPENDIX B – Klein-Gordon equation for Schwarzschild background

Consider the general d'Alembertian operator for a curved spacetime, applied in a field $\phi(t,\vec{x})$, as follows

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\nabla_\mu(\partial_\nu\phi) = g^{\mu\nu}[(\partial_\nu\phi)_{,\mu} - \Gamma^\alpha_{\mu\nu}(\partial_\alpha\phi)], \tag{B.1}$$

where we used that the covariant derivative is given by [44]

$$\nabla_\mu T_\nu = (T_\nu)_{,\mu} - \Gamma^\alpha_{\mu\nu}T_\alpha. \tag{B.2}$$

Substituting Eq. (2.40) in Eq. (B.1), and recalling that the metric is a symmetric tensor, one obtains that

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{g^{\alpha\gamma}}{2}g^{\mu\nu}(g_{\mu\nu,\gamma})\partial_\alpha\phi - g^{\alpha\gamma}(g_{\mu\alpha,\nu})\partial_\alpha\phi. \tag{B.3}$$

Now, using the identity (see Appendix A)

$$\frac{1}{\sqrt{-g}}\partial_\gamma\sqrt{-g} = \frac{1}{2}g^{\mu\nu}(g_{\mu\nu,\gamma})$$

and the Leibniz rule in the last term of Eq. (B.3), one finally obtains

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\nabla_\nu\phi &= g^{\mu\nu}\partial_\mu\partial_\nu\phi + g^{\alpha\gamma}\frac{1}{\sqrt{-g}}\partial_\gamma\sqrt{-g}\partial_\alpha\phi + g^{\gamma\alpha}{}_{,\gamma}\,\partial_\alpha\phi \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi). \end{aligned} \tag{B.4}$$

For a massless scalar field, the KG equation reduces to

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0. \tag{B.5}$$

The Schwarzschild contravariant metric, in spherical coordinates, is given by

$$g^{\mu\nu} = \mathrm{diag}\left(-\frac{1}{A(r)}, A(r), \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right), \tag{B.6}$$

where $A(r) = 1 - 2M/r$ and $g = \det g_{\mu\nu} = r^2\sin^2\theta$. Plugging this into Eq. (B.5) one gets

$$-\frac{1}{A}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(Ar^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0,$$

thus, using the spherical harmonics expansion Eq. (3.48), one obtains, after some algebra

$$-\frac{\partial^2 f_l}{\partial t^2} + \frac{A}{r}\frac{\partial}{\partial r}\left(Ar^2\frac{\partial f_l}{\partial r}\right) - A\frac{l(l+1)}{r^2}f_l = 0, \tag{B.7}$$

where we used the same argument of Sec. 3.3 for the spherical harmonics differential equation. Expanding the second term, one obtains

$$-\frac{\partial^2 f_l}{\partial t^2} + A^2 \frac{\partial^2 f_l}{\partial r^2} + AA' \frac{\partial f_l}{\partial r} - A \left[\frac{2M}{r^3} + \frac{l(l+1)}{r^2} \right] f_l = 0, \tag{B.8}$$

where $A' = dA/dr$. Recalling that the Eddington-Finkelstein coordinate r^* can also be written as [38],

$$r^* = \int \frac{dr}{A(r)},$$

one has,

$$\frac{\partial^2 f_l}{\partial r^{*2}} = A^2 \frac{\partial^2 f_l}{\partial r^2} + AA' \frac{\partial f_l}{\partial r}.$$

Henceforth, substituting this result in Eq. (B.8) one finally obtains the d'Alembert equation for Schwarzschild background, given by Eq. (3.50).

References

[1] Hawking, S. W. Black hole explosions? *Nature* **247**, 30 (1974). 5, 10, 31, 42

[2] Hawking, S. W. Particle creation by black holes. *Communications in Mathematical Physics* **43**, 199 (1975). 5, 10, 31, 42, 45, 48

[3] Gordon, W. Zur lichtfortplanzung nach der relativitätstheorie. *Communications in Mathematical Physics* **43**, 199 (1975). 9

[4] Tamm, T. E. The electrodynamics of anisotropic media in the special theory of relativity. *Zhurnal Russkogo Fiziko-Khimicheskogo Obshchestva, Otdel Fizicheskii* **56**, 248 (1924). 9

[5] Plebanski, J. Electromagnetic waves in gravitational fields. *Physical Review* **118**, 1396 (1960). 9, 46

[6] Gutiérrez, S. A., Dudley, A. L. & Plebanski, J. F. Signals and discontinuities in general relativistic nonlinear electrodynamics. *Journal of Mathematical Physics* **22**, 2835 (1981). 9

[7] Novello, M., De Lorenci, V. A., Salim, J. M. & Klippert, R. Geometrical aspects of light propagation in nonlinear electrodynamics. *Physical Review D* **61**, 045001 (2000). 9

[8] Barceló, C., Liberati, S. & Visser, M. Analogue gravity from field theory normal modes? *Classical and Quantum Gravity* **18**, 3595 (2001). 9

[9] Carusotto, I., Fagnocchi, S., Recati, A., Balbinot, R. & Fabbri, A. Numerical observation of hawking radiation from acoustic black holes in atomic bose–einstein condensates. *New Journal of Physics* **10**, 103001 (2008). 9

[10] Barceló, C., Liberati, S. & Visser, M. Analog gravity from bose-einstein condensates. *Classical and Quantum Gravity* **18**, 1137 (2001). 9

[11] Balbinot, R. & Fabbri, A. Quantum correlations across the horizon in acoustic and gravitational black holes. *Physical Review D* **105**, 045010 (20022). 9

[12] Mackay, T. G. & Lakhtakia, A. Towards a metamaterial simulation of a spinning cosmic string. *Physics Letters A* **374**, 2305 (2010). 9

[13] Barceló, C., andG. G. Moreno, J. E. S. & Jannes, G. Chronology protection implementation in analogue gravity. *The European Physical Journal C* **82**, 299 (2022). 9

[14] Hawking, S. W. Chronology protection conjecture. *Physics Review D* **46**, 603 (1992). 9

[15] De Lorenci, V. A. & Jr., E. S. M. Spinning strings, cosmic dislocations, and chronology protection. *Physical Review D* **70**, 047502 (2004). 9

[16] Fiebig, M. Revival of the magnetoelectric effect. *Journal of Physics D: Applied Physics* **38**, 123 (2005). 9

[17] Rivera, J. P. A short review of the magnetoelectric effect and related experimental techniques on single phase (multi-) ferroics. *European Physical Journal B* **71**, 299 (2009). 9

[18] Schmid, H. On a magnetoelectric classification of materials. *International Journal of Magnetism* **4**, 337 (1973). 9

[19] Tabares-Muñoz, C., Rivera, J. P., Bezinges, A., Monnier, A. & Schmid, H. Measurement of the quadratic magnetoelectric effect on single crystalline bifeo3. *Japanese Journal of Applied Physics* **24**, 1051 (1985). 9

[20] Lapine, M., Shadrivov, I. V. & Kivshar, Y. S. Colloquium: Nonlinear metamaterials. *Reviews of Modern Physics* **86**, 1093 (2014). 9

[21] Unruh, W. G. Experimental black-hole evaporation? *Physical Review Letters* **46**, 1351 (1981). 10, 23, 27, 30

[22] Balbinot, R., Fabbri, A., Fagnocchi, S. & Parentani, R. Hawking radiation from acoustic black holes, short distance and backreaction effects. *La Rivista del Nuovo Cimento* **28**, 1 (2005). 10, 23

[23] Barceló, C., Liberati, S. & Visser, M. Analogue gravity. *Living Reviews in Relativity* **14**, 3 (2011). 10, 23

[24] Weinfurtner, S. & *et al.* Measurement of stimulated hawking emission in an analogue system. *Physical Review Letters* **106**, 021302 (2011). 10

[25] Painlevé, P. La mecanique classique et la theorie de la relativite. *Comptes Rendue de l'Academie de Sciences* **173**, 677 (1921). 10, 23, 27

[26] Gullstrand, A. Allegemeine l orper-problems in der einsteinshen osung des statischen eink gravitations theorie. *Arkiv for Matematik, Astronomi och Fysik* **76**, 1 (1922). 10

[27] Hadamard, J. *Leçons sur la Propagation des Ondes et les Equations de Hydrodynamique* (Librairie Scientifique A. Hermann, Paris, France, 1903). 12, 15

[28] Novello, M., Salim, J. M., De Lorenci, V. A. & Elbaz, E. Nonlinear electrodynamics can generate a closed spacelike path for photons. *Physical Review D* **63**, 103516 (2001). 13

[29] De Lorenci, V. A. & Klippert, R. Analogue gravity from electrodynamics in nonlinear media. *Physical Review D* **65**, 064027 (2002). 13

[30] De Lorenci, V. A. Effective geometry for light traveling in material media. *Physical Review E* **65**, 026612 (2002). 13

[31] Lang, S. *Algebra* (Addison-Wesley, London, England, 1971). 17

[32] Lane, S. M. & Birkhof, G. *Algebra* (MacMillan, London, England, 1970). 17

[33] Landau, L. & Lifshitz, E. *Fluid Mechanics* (Pergamon, London, England, 1959). 24

[34] Stone, M. Acoustic energy and momentum in a moving medium. *Physical Review E* **62**, 1 (1999). 25, 29

[35] Vaidya, P. C. The gravitational field of a radiating star. *Proceedings of the Indian Academy of Sciences - Section A* **33**, 264 (1951). 31, 37

[36] Birrel, N. & Davies, P. *Quantum fields in curved spaces* (Cambridge University Press, Melbourne, Australia, 1982). 33

[37] Fabbri, A. & Salas, J. N. *Modeling black hole evaporation* (Imperial College Press, London, England, 2005). 35, 37, 43, 44

[38] Finkelstein, D. Past-future asymmetry of the gravitational field of a point particle. *Physical Review* **110**, 965 (1958). 37, 53

[39] Post, E. J. *Formal structure of the electromagnetics* (North-Holland, Amsterdam, Netherlands, 1962). 46

[40] Gibbons, G. & Werner, M. The gravitational magnetoelectric effect. *Universe* **5**, 88 (2019). 46

[41] Carr, B. The primordial black hole mass spectrum. *Astrophysical Journal* **201**, 1 (1975). 48

[42] Musco, I., Miller, J. C. & Rezzolla, L. Computations of primordial black-hole formation. *Classical and Quantum Gravity* **22**, 1405 (2005). 48

[43] Peiris, H. & *et al.* First-year wilkinson microwave anisotropy probe (wmap)* observations: Implications for inflation. *Astrophysical Journal Supplement* **148**, 213 (2003). 48

[44] Foster, J. & Nightingale, J. *A short course in General Relativity* (Longman, New York, United States, 1979). 52