**UNIVERSIDADE FEDERAL DE ITAJUBÁ - UNIFEI**
**PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA**

# Asymptotic symmetries and infrared phenomena in gauge theories and gravity

**Luis Fernando Temoche Hurtado**

Itajubá, 17 de abril de 2023

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Dissertação submetida ao Programa de Pós-Graduação em Física como parte dos requisitos para obtenção do Título de Mestre em Ciências Físicas.

**Área de Concentração: Física**

**Orientador: Dr. Alexis Roa Aguirre**

**17 de abril de 2023**
**Itajubá**

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**Defesa Pública de Dissertação de Mestrado**

Às quinze horas do dia vinte e três de fevereiro de dois mil e vinte e três, na Sala Virtual, foram abertos os Trabalhos de Defesa Pública de Dissertação de Mestrado intitulada **"Asymptotic symmetries and infrared phenomena in gauge theories and gravity"**, do candidato **LUIS FERNANDO TEMOCHE HURTADO**, como parte das exigências para obtenção do Título de Mestre em Física, na Área de Concentração: **Física.** A Comissão Examinadora foi composta pelos doutores: Bruce Lehmann Sánchez Vega da UFMG, 1º Examinador; Eduardo Henrique Silva Bittencourt da UNIFEI, 2º Examinador; Alexis Roa Aguirre da UNIFEI, 3º Examinador - Orientador e Presidente da Sessão. Conforme Art. 54 da Norma de Programas de Pós-Graduação da UNIFEI, instalados os trabalhos de Defesa Pública de Dissertação, seguiram-se as seguintes fases: Apresentação dos demais membros da banca pelo presidente; chamada do candidato e leitura do título do trabalho a ser defendido; exposição oral do conteúdo do trabalho pelo candidato; e arguição do candidato por cada examinador. Feito isso, foi suspensa a Sessão de Defesa para início da Sessão de Julgamento. Durante a mesma, os examinadores atribuíram ao candidato os seguintes conceitos: 1º Examinador "A"; 2º Examinador "A"; 3º Examinador "A", sendo o candidato considerado **"APROVADO"**. Voltando à Sessão de Defesa Pública de Dissertação, o presidente convocou o candidato e proclamou o resultado final, à vista do Quadro Demonstrativo de Apuração, entregando-lhe uma cópia do mesmo; apresentou seus agradecimentos e encerrou a sessão. Os trabalhos foram encerrados às dezoito horas do dia vinte e três de fevereiro de dois mil e vinte e três e, para que tudo constasse, foi lavrada a presente ata, a qual foi lida, aprovada e assinada pelos membros da Comissão Examinadora. O examinador externo à UNIFEI autoriza o presidente da banca a assinar esta ata em seu nome.

Itajubá, 23 de fevereiro de 2023.

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# Resumo

O objetivo da seguinte dissertação é apresentar a conexão existente no comportamento do regime infravermelho em ambas as teorias de calibre e gravidade. O primeiro elemento desta análise é o estudo dos os teoremas soft , originalmente desenvolvidos por Weinberg [1][2]. Em um sentido geral, os processos de espalhamento são governados por meio de restrições que controlam a forma como as partículas soft são produzidas. O segundo elemento são as simetrias espaço-temporais assintóticas desenvolvidas por Bondi, van der Burg, Metzner e Sachs [3] [4] dos espaço-tempos assintoticamente planos. Eles fornecem o surgimento de cargas conservadas associadas ao comportamento dos campos através nos infinitos nulos. Terceiro, mas não menos importante, a existência de efeitos de memória, como os estudados por Christodolou [5][6] e Thorne [7] no limite infravermelho. Novamente, em termos simples, eles se referem ao surgimento de perturbações no tecido do espaço-tempo devido à sua propagação, levando a deslocamentos de campo. Nós exploramos a conexão em termos da aplicação de transformadas de Fourier e identidades de Ward. Por simplicidade, vamos-nos focar no caso da Eletrodinâmica Quântica e da Gravidade Quântica.

**Palavras-chaves**: Espaço-tempo assintoticamente plano. Identidade de Ward. Teorema "Soft".

# Abstract

The objective of the following dissertation is to present the existent connection in the infrared regime behavior of both gauge theories and gravity. The first element in this picture analysis is the study of soft theorems, originally developed by Weinberg [1][2]. In a general sense, scattering processes are governed through constraints that control the way soft particles are produced. The second one shall be the asymptotic spacetime symmetries developed by Bondi, van der Burg, Metzner and Sachs [3] [4] of asymptotically flat spacetimes. They lead to the appearance of conserved charges associated to the behavior of the fields at null infinities. The third but not least, the existence of memory effects, like the ones studied by Christodolou [5][6] and Thorne [7] in the infrared limit. Again, in simple terms, they refer to the surgence of perturbations in the fabric of spacetime due to its propagation, leading to field shifts. We explore the connection in terms of the application of Fourier transforms and Ward identities. For simplicity we focus on the case of Quantum Electrodynamics and Quantum Gravity.

**Key-words**: Asymptotically flat spacetime. Ward identity. Soft theorem.

# Contents

# 1 Introduction

The asymptotic symmetries of spacetime where originally studied as symmetries of the bulk when considering the propagation of gravitational waves and their behavior towards infinity. In the seminal works by Bondi et. al. [3] and Sachs [4], the authors deduced the existence of a group of isometries correspondent to the case of asymptotically flat spacetimes.

The existence of the *soft limit* of S-matrix preceded the derivation of asymptotic symmetries. We understand the *soft limit* as scattering processes in quantum field theory ($QFT$) where there is production of soft particles, whose energy $E$ satisfies the condition $E \longrightarrow 0$. Then we talk about infrared phenomena due to the production of these particles. The so called soft theorems of scattering amplitudes factorized into an amplitude with soft factor. Original developments were made by Bloch and Nordseick [8], Low [9], Yennie et. al. [10] and Weinberg [2]. The case of memory effects on the other hand referred to the appearance of a finite variation in the average value of some quantity after a certain process occurs. This was originally studied by Zeldovich [11], Christodolou [5], Wiseman [12] and Thorne [7] in the context of propagation of gravitational waves.

The existence of a connection between all these infrared phenomena was introduced in the literature by different authors. On the side of soft theorems and asymptotic symmetries by Barnich [13], Kapec et. al [14][15], Campiglia and Laddha [16], Seraj [17] and Gabai [18], and in the case of memory effect and those symmetries, originally introduced by Strominger & Zhiboedov [19]. Based on different approaches, all of them claim the possible interpretation of the symmetry group of asymptotically flat spacetime as the sources of conserved quantities which lead to the existence of the so-called soft theorems for scattering amplitudes. Similarly an interpretation is made for the memory effect, where a vacuum transition in those types of spacetimes are generated by the application of certain transformations which are elements of its symmetry group.

As mentioned in the title, the study of infrared phenomena on gauge and gravity theories is the main focus of our dissertation. In particular, the issue of infrared divergences on field theory is a common subject developed on standard texts on the field, such as Weinberg [20] or Schwartz [21] recently. Generally speaking, infrared divergences appear on field theory when questions arise about the calculation of cross sections for different scattering processes, like $e^+e^- \longrightarrow \mu^+\mu^-$. The question which arises, is how this process differs from a similar one involving the production of soft particles, either on the initial or final state, or from virtual particles. This leads to a proper treatment involving regularization procedures beyond the scope of the present work. However, it gives us some motivation on the subject on how does infrared divergence of scattering processes can be dealt properly through the understanding of soft particle production due to symmetry considerations. Besides that, opens the possibility of analysis of particle jet production and its interpretation.

This dissertation is organized as follows: on *Chapter 2* we aim to describe the case of the infrared triangle for the case of electromagnetism. Both the classical and quantum mechanical cases of asymptotic symmetries are discussed and its link to conservation laws through the interpretation of Ward identities on null infinities as the so-call Soft Photon Theorem. *Chapter 3* discusses a similar analysis, this time applied to Quantum Gravity. In the context of this dissertation, Quantum Gravity refers to a Quantum Field theory describing the physics of relativistic particles and quantum mechanical nature with spin 2, which interact with any system with certain amount of mass and energy. These particles of spin 2 can be portrait as the excitations of the gravitational field at quantum scales. Being our study focused on the infrared spectrum, problems as the non-renormalizability of the gravitational field described by the metric are set aside. On the other hand, through the introduction of the Bondi-van der Burg-Metzner-Sachs symmetries (BMS) we derive the set of spacetime isometries known as BMS group, and then link its existence to the known Soft Graviton Theorem. On third place, *Chapter 4* looks into the definition of the memory effect and its interpretation as vacuum transition on an asymptotically flat spacetime. Finally, conclusions about the link between the soft theorems, asymptotic symmetries and memory effect are presented, as a picture of the current state of research derived from these developments.

# 2 Infrared divergences for electromagnetism

Analysis of infrared divergences for electromagnetism can be started through the study of the Lienard-Wiechert potentials [22]. The main objective of this section is to show that these solutions are discontinuous at the boundary of the Minkowski spacetime.

## 2.1 Classical divergences-Lienard Wiechert potential

Given the position $\mathbf{x}'(\tau)$ of a particle with charge $e$, we want to compute the field potential generated due to its movement. Let us start from the wave equation for the potential

$$\Box A^{\alpha} = -4\pi J^{\alpha}, \tag{2.1}$$

where the potential $A^{\alpha}$ and 4-current $J^{\alpha}$, are given as

$$A^{\alpha}(x) = \int d^4x' D_{+}(x-x')J^{\alpha}(x'), \tag{2.2}$$

$$J^{\alpha}(x) = e^2 \int d\tau u^{\alpha}\delta^4(x-x'). \tag{2.3}$$

Here, the retarded Green function $D_{+}(x-x')$ is defined as

$$D_{+}(x-x') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\Theta(t-\tau)\delta(t-\tau-|\mathbf{x}-\mathbf{x}'|), \tag{2.4}$$

where $|\mathbf{x}(t)-\mathbf{x}'(\tau)|$ is the the relative position to the source of the field, and

$$t-\tau = |\mathbf{x}-\mathbf{x}'|. \tag{2.5}$$

Replacing both expressions in the wave equation leads to

$$A^{\alpha}(x) = \frac{e^2}{2\pi}\int d\tau \Theta(y_0(\tau))\delta(y^{\beta}y_{\beta}(\tau))u^{\alpha}, \tag{2.6}$$

where $y^{\alpha}(\tau) = x^{\alpha} - x'^{\alpha}$, $u^{\alpha} = \gamma(1,\boldsymbol{\beta})$, with $\gamma = (1-\beta^2)^{-1/2}$. Then, by deriving the 4-potential, we get

$$\partial_{\alpha}A_{\beta} = \frac{e^2}{2\pi}\Big[\int d\tau\delta(y^0)\partial_{\alpha}(x^0 - x'^0)\delta(y_{\mu}y^{\mu})u_{\beta} + \int d\tau\Theta(y^0)\partial_{\alpha}\delta(y_{\mu}y^{\mu})u_{\beta}\Big], \tag{2.7}$$

where the first integral is evaluated in crossing light trajectories, so it cancels out. Then the above expression is reduced to

$$\partial_{\alpha}A_{\beta} = \frac{e^2}{2\pi}\int d\tau\Theta(y^0)\partial_{\alpha}\delta(y_{\mu}y^{\mu})u_{\beta}. \tag{2.8}$$

Consider both Dirac delta and derivative identities,

$$\partial_\alpha \delta(f) = \frac{\partial f}{\partial x^\alpha}\frac{d\tau}{df}\frac{d}{d\tau}\delta(f), \tag{2.9}$$

$$\partial_\alpha f = \partial_\alpha(y_\mu y^\mu) = 2(x_\alpha - x'_\alpha), \tag{2.10}$$

$$df = -2(x^\mu - x'^\mu)u_\mu(\tau)d\tau, \tag{2.11}$$

where $f = y_\mu y^\mu$. Now, by substituting in (2.8), one gets

$$\partial_\alpha A_\beta = \frac{e^2}{4\pi|y_\nu u^\nu|}\frac{d}{d\tau}\left[\frac{x_\alpha - x'_\alpha}{|x^\mu - x'^\mu|u_\mu}\right]u_\beta\Big|_\tau. \tag{2.12}$$

The Faraday tensor $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, will be given by:

$$F_{\alpha\beta} = \frac{e^2}{4\pi|y_\nu u^\nu|}\frac{d}{d\tau}\left[\frac{y_\alpha u_\beta - y_\beta u_\alpha}{y^\mu u_\mu}\right]\Big|_\tau. \tag{2.13}$$

For the special case of constant $u_\alpha$,

$$F_{\alpha\beta} = \frac{-e^2}{4\pi|y_\nu u^\nu|^3}\left[y_\alpha u_\beta - y_\beta u_\alpha\right]. \tag{2.14}$$

Now, considering $n$ moving sources with charges $Q_k$ and the 4-velocities $u_k^\alpha$ , we obtain the radial component for electric field

$$F_{rt}(r,t) = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k(r - t\hat{x}\cdot\boldsymbol{\beta}_k)}{|\gamma_k^2(t - r\hat{x}\cdot\boldsymbol{\beta}_k)^2 - t^2 + r^2|^{3/2}}, \tag{2.15}$$

where $r^2 = \mathbf{x}\cdot\mathbf{x}$ and $\mathbf{x} = r\hat{x}$. Expression (2.15) is discontinuous at $r \longrightarrow \infty$, requiring the necessity of analyzing it towards the conformal infinities of Minkowski spacetime, whose causal structure is developed in its correspondent Penrose diagram, which is presented on Appendix A, where is established that $\mathcal{I}^+$ and $\mathcal{I}^-$ are the future and past null infinities and $i^0$ is the spacelike infinity. In fact, the expression (2.15) can be rewritten in terms of retarded coordinates $u = t - r$, as

$$F_{rt} = F_{ru} = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k(r - (u+r)\hat{x}\cdot\boldsymbol{\beta}_k)}{|\gamma_k^2(u + r - r\hat{x}\cdot\boldsymbol{\beta}_k)^2 - (u+r)^2 + r^2|^{3/2}}. \tag{2.16}$$

Taking $u$ fixed, and $r \to \infty$ ($r >> u$), we have

$$F_{rt}\Big|_{\mathcal{I}^+} = F_{ru}\Big|_{\mathcal{I}^+} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1 - \hat{x}\cdot\boldsymbol{\beta}_k)^2}. \tag{2.17}$$

Similarly, for the advanced coordinates $v = t + r$, we obtain

$$F_{rt} = F_{rv} = \frac{e^2}{4\pi}\sum_{k=1}^{n}\frac{Q_k\gamma_k(r - (v-r)\hat{x}\cdot\boldsymbol{\beta}_k)}{|\gamma_k^2(v - r - r\hat{x}\cdot\boldsymbol{\beta}_k)^2 - (v-r)^2 + r^2|^{3/2}}, \tag{2.18}$$

$$F_{rt}\Big|_{\mathcal{I}^-} = F_{rv}\Big|_{\mathcal{I}^-} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1 + \hat{x}\cdot\boldsymbol{\beta}_k)^2}. \tag{2.19}$$

For obtaining the $F_{rt}$ value at $\mathcal{I}^+_-$, we take the limit at $u \to -\infty$. Similar analysis is applied for evaluating $F_{rt}$ at $\mathcal{I}^-_+$, this time at $v \to \infty$. The field tensor exhibits a singular behavior at $i^0$, since $F_{ru}\Big|_{\mathcal{I}^+} \neq F_{rv}\Big|_{\mathcal{I}^-}$. From the asymptotic expressions (2.17) and (2.19), we can see that the leading contribution of the Taylor expansion for the field tensor will be $F^{(2)}_{rt}$, i.e. the $\frac{1}{r^2}$ order term. However, it is possible to identify an antipodal matching condition when $\hat{x} \to -\hat{x}$, as follows

$$\lim_{r\to\infty} r^2 F_{ru}\Big|_{\mathcal{I}^+_-}(\hat{x}) = \lim_{r\to\infty} r^2 F_{rv}\Big|_{\mathcal{I}^-_+}(-\hat{x}). \tag{2.20}$$

Therefore, the leading term in the radial electric field for a collection of n particles at any point $\mathcal{I}^+_-$ will be equal to the value of the field at the antipodal point on $\mathcal{I}^-_+$. As a consequence of equation (2.20), we are going to deduce that the infrared behavior of electromagnetic radiation at infinity leads to the identification of a set of conserved quantities which come to existence because of the residual large gauge symmetry of the system. In particular we are interested in the zero energy limit, the so called soft modes. From here, we will use a combination of both retarded or advanced coordinates together with the complex stereographic coordinates for the angular sector.

Let us start from the Minkowski line element expressed in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{2.21}$$

Then replacing $u = t - r$, leads to

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{2.22}$$

Note that for $u$ fixed, this element reduces to the $S^2$ unit sphere line element. Here we can introduce stereographic coordinates expressed as $X = \cot\frac{\theta}{2}\cos\phi$ and $Y = \cot\frac{\theta}{2}\sin\phi$. However, for our purposes will be useful to express it as complex coordinates, $z = X + iY$. Then, the line element becomes equivalent to

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \tag{2.23}$$

where

$$\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}. \tag{2.24}$$

Analogously, this procedure can be applied for $v = t + r$, with $X = -\cot\frac{\theta}{2}\cos\phi$ and $Y = -\cot\frac{\theta}{2}\sin\phi$, leading to

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{2.25}$$

Note that the point $z$ mapped with the retarded coordinates at (2.23), is related to its antipodal point on the sphere mapped with the advanced coordinates (2.25) by the transformation $z \to -1/\bar{z}$. This refers to the fact that the $\{z, \bar{z}\}$ for the retarded Bondi coordinates defined as

$$x^1 + ix^2 = \frac{2rz}{1+z\bar{z}}, \tag{2.26}$$

$$x^3 = r\frac{1-z\bar{z}}{1+z\bar{z}}, \tag{2.27}$$

are mapped to the $\{z, \bar{z}\}$ pair for the advanced Bondi coordinates defined as

$$x^1 + ix^2 = -\frac{2rz}{1+z\bar{z}}, \tag{2.28}$$

$$x^3 = -r\frac{1-z\bar{z}}{1+z\bar{z}}, \tag{2.29}$$

through the application of the transformation referred to. In terms of these coordinates, we can rewrite the matching condition in the following way:

$$F^{(2)}_{ur}(z,\bar{z})\Big|_{\mathcal{I}^+_-} = F^{(2)}_{vr}(z,\bar{z})\Big|_{\mathcal{I}^-_+}. \tag{2.30}$$

The above result will imply the existence of an infinite number of conserved charges for the electromagnetic theory in Minkowski spacetime. This can be implemented by considering an arbitrary function $\epsilon(z,\bar{z})$ which satisfies the following boundary condition

$$\epsilon(z,\bar{z})\Big|_{\mathcal{I}^+_-} = \epsilon(z,\bar{z})\Big|_{\mathcal{I}^-_+}. \tag{2.31}$$

Depending on the explicit analytic expression for $\epsilon(z,\bar{z})$, this will lead to a different set of symmetries and subsequently, conservation laws. From Noether's theorem, see Appendix B, the conserved current (B.10)is determined by

$$\begin{aligned} f_\mu &= \Big[ -\frac{1}{4}\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial^\mu A^\nu)} + j_\sigma \frac{\partial A^\sigma}{\partial(\partial^\mu A^\nu)}\Big]\delta A^\nu \\ &= -\frac{1}{4}(4F_{\mu\nu})\delta A^\nu \\ &= -F_{\mu\nu}\delta A^\nu. \end{aligned} \tag{2.32}$$

Given the local gauge transformation

$$A'_z = A_z + \partial_z \epsilon(z,\bar{z}) \tag{2.33}$$

Then, the conserved charge (B.11) at $\mathcal{I}^\pm$, being this space a 3-dimensional spacetime, leads to volumetric integral for the charge

$$\begin{aligned} Q^+_\epsilon &= -\int d^2z\gamma_{z\bar{z}}F_{0\nu}\delta A^\nu \\ &= -\int d^2z\gamma_{z\bar{z}}F_{uz}\delta A^z \\ &= -\int d^2z\gamma_{z\bar{z}}F_{uz}\partial^z\epsilon \\ &= \int_{\mathcal{I}^+} dud^2z\Big[\partial_u(\partial_z A_{\bar{z}} + \partial_{\bar{z}}A_z) + \gamma_{z\bar{z}}j_u\Big] \end{aligned} \tag{2.34}$$

Now, by introducing the retarded radial gauge fixing [23]

$$\mathcal{A}_r = 0, \tag{2.35}$$

$$\mathcal{A}_u\Big|_{\mathcal{I}^+} = 0, \tag{2.36}$$

the field tensor components become

$$\mathcal{F}_{z\bar{z}} = \partial_z \mathcal{A}_{\bar{z}} - \partial_{\bar{z}} \mathcal{A}_z, \tag{2.37}$$

$$\mathcal{F}_{uz} = \partial_u \mathcal{A}_z - \partial_z \mathcal{A}_u, \tag{2.38}$$

$$\mathcal{F}_{rz} = \partial_r \mathcal{A}_z - \partial_z \mathcal{A}_r, \tag{2.39}$$

$$\mathcal{F}_{ur} = \partial_u \mathcal{A}_r - \partial_r \mathcal{A}_u, \tag{2.40}$$

where we consider the following expansion of the potentials

$$\mathcal{A}_z(u,r,z,\bar{z}) = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u,z,\bar{z})}{r^n}, \tag{2.41}$$

$$\mathcal{A}_u(u,r,z,\bar{z}) = \frac{A_u(u,z,\bar{z})}{r} + \sum_{n=1}^{\infty} \frac{A_u^{(n)}(u,z,\bar{z})}{r^{n+1}}. \tag{2.42}$$

From Maxwell's equations we know that

$$\partial^r \mathcal{F}_{ru} + \partial^z \mathcal{F}_{zu} + \partial^{\bar{z}} \mathcal{F}_{\bar{z}u} = e^2 j_u. \tag{2.43}$$

Preserving just the leading order terms of (2.43), it reduces to

$$\partial^u F_{ur}^{(2)} + D^z F_{uz}^{(0)} + D^{\bar{z}} F_{u\bar{z}}^{(0)} = e^2 j_u^{(2)}, \tag{2.44}$$

where the covariant derivative $D^z$ is with respect to the sphere $S^2$, leading to

$$Q_\epsilon^+ = \frac{1}{e^2} \int_{\mathcal{I}^+} dud^2z\epsilon\Big[\partial_u(\partial_z A_{\bar{z}}^{(0)} + \partial_{\bar{z}} A_z^{(0)}) + e^2 \gamma_{z\bar{z}} j_u\Big]. \tag{2.45}$$

Different kinds of gauge symmetries lead to different conservation laws. For instance $\epsilon = 1$ leads to the total initial electric charge

$$Q_1^+ = \int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}} j_u^{(2)}. \tag{2.46}$$

Other type of conservation law, shall be the total conservation of outgoing electric charge at fixed angle $(w, \bar{w})$, from the generator $\epsilon = \delta^2(z-w)$, where $(w,\bar{w})$ is a fixed angle on the $S^2$ sphere. The conservation charge is given by

$$Q_{w\bar{w}}^+ = \frac{1}{e^2} \int_{-\infty}^{\infty} du\Big[\partial_u(\partial_w A_{\bar{w}} + \partial_{\bar{w}} A_w) + e^2 \gamma_{w\bar{w}} j_u\Big]. \tag{2.47}$$

Now we define the conserved charges and conservation law at null infinities $\mathcal{I}^+$ as

$$\begin{aligned} Q_\epsilon^+ &= \frac{1}{e^2} \int_{\mathcal{I}^+_-} d^2z dr \gamma_{z\bar{z}} j_u, \\ &= \frac{1}{e^2} \int_{\mathcal{I}^+_-} d^2z dr \gamma_{z\bar{z}} \partial^r F_{ru}, \\ &= \frac{1}{e^2} \int_{\mathcal{I}^+_-} d^2z \gamma_{z\bar{z}} \epsilon F_{ru}^{(2)}, \end{aligned} \tag{2.48}$$

and at $\mathcal{I}^-$, as

$$
\begin{aligned}
Q_\epsilon^- &= \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z dr \gamma_{z\bar{z}} j_v, \\
&= \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z dr \gamma_{z\bar{z}} \partial^r F_{rv}, \\
&= \frac{1}{e^2}\int_{\mathcal{I}^-_+} d^2z \gamma_{z\bar{z}} \epsilon F^{(2)}_{rv},
\end{aligned} \tag{2.49}
$$

where the antipodal matching condition leads to

$$Q^+_\epsilon = Q^-_\epsilon, \tag{2.50}$$

being $\gamma_{z\bar{z}}$ the metric of the $S^2$ sphere living on every point at $\mathcal{I}^\pm$ and where the $F^{(2)}_{ru}$ come from the asymptotic expansion towards infinity made for the potentials (2.41).

Our understanding of the asymptotic symmetries in-built on the electrodynamics at null infinity is subject to right interpretation of these new set of gauge transformations. As explained by Strominger [23], the concept is rooted on the idea of *asymptotic symmetry group (ASG)*. A symmetry group can be understood as a set of transformations which leave invariant a lagrangian, composed both of trivial transformations and non-trivial ones. Then the *ASG* corresponds to the subset of transformations which act non-trivially on the system. Examples are for instance the *BMS group* [3] for the case of gravitational radiation, or the two copies of Virasoro algebra *SL(2,R)* used to describe the asymptotic symmetry of an $AdS_3$ spacetime, originally developed by Brown & Henneaux [24]. Then, for the case of QED, this asymptotic symmetry corresponds to the large gauge transformations presented previously, which generate phase transitions on particle states at fixed angle on the null boundary.

The large gauge symmetry of the system comes from the factorization of the charges in two contributions, called soft and hard terms, being the latter the one with non-zero energy. For instance charge at future null infinity is given by

$$Q^+_\epsilon = Q^+_S + Q^+_H, \tag{2.51}$$

$$Q^+_S = -\frac{1}{e^2}\int_{\mathcal{I}^+} dud^2z(\partial_z \epsilon F^{(0)}_{u\bar{z}} + \partial_{\bar{z}}\epsilon F^{(0)}_{uz}), \tag{2.52}$$

$$Q^+_H = \int_{\mathcal{I}^+} dud^2z \epsilon \gamma_{z\bar{z}} j^{(2)}_u. \tag{2.53}$$

From this ansatz we can deduce the commutators algebra for the large gauge symmetries. We follow the protocol introduced by Frolov[25] and Ashtekar[26], were the quantization relations at any conformal hypersurface $\Sigma$ in Minkowski spacetime is defined as

$$[\Phi_{\mathcal{I}^+}(u,z,\bar{z}), \Phi_{\mathcal{I}^+}(u',z',\bar{z}')] = \frac{i}{2}\partial_u \delta(u-u')\delta^2(z-z'), \tag{2.54}$$

$$[\Phi_{\mathcal{I}^-}(v,z,\bar{z}), \Phi_{\mathcal{I}^-}(v',z',\bar{z}')] = \frac{i}{2}\partial_u \delta(v-v')\delta^2(z-z'), \tag{2.55}$$

where $\mathcal{I}^+$ and $\mathcal{I}^-$ are the hypersurfaces $\Sigma$ where the commutation relations are evaluated. Considering $\Phi \to F_{\mu\nu}$, then

$$[F_{uz}(u,z,\bar{z}), F_{uw}(u',w,\bar{w})] = \frac{i}{2}\partial_u \delta(u-u')\delta^2(z-w). \tag{2.56}$$

Integrating with respect to $u$ and fixing integration constants to 0, this leads to

$$[A_z(u,z,\bar{z}), A_w(u,w,\bar{w})] = \frac{i}{4}\Theta(u-u')\delta^2(z-w), \tag{2.57}$$

$$\left[Q_\epsilon^+, A_z^{(0)}(u,z,\bar{z})\right] = i\partial_z \epsilon(z,\bar{z}), \tag{2.58}$$

$$\left[Q_\epsilon^-, A_z^{(0)}(v,z,\bar{z})\right] = i\partial_z \epsilon(z,\bar{z}). \tag{2.59}$$

From these results, we can see that the infinite number of symmetries generated by the conserved charges $Q_\epsilon^+$ in a canonical formalism are just gauge transformations with parameter $\epsilon$ [23].

## 2.2 Quantum divergences-Ward identities for QED

Our presentation of the Ward identities in Quantum Field Theory and its interpretation as conserved quantities from symmetries of quantum mechanical systems are based on the references [21] and [27]. The generating functional for QED is defined as follows:

$$Z[j,\eta,\bar{\eta}] = \int [D\bar{\psi}][D\psi][DA] \exp\Big(i\int d^4x(\mathcal{L} - \frac{1}{2\lambda}(\partial\cdot A)^2 + \bar{\eta}\psi + \bar{\psi}\eta + j^\mu A_\mu)\Big), \tag{2.60}$$

being the lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{2.61}$$

where $j^\mu = e\bar{\psi}\gamma^\mu\psi$ is the charged current, $\{\psi,\bar{\psi}\}$ are the fermionic fields, $\gamma^\mu$ the gamma matrix and $\{\eta,\bar{\eta}\}$ Grasmann fields. Under a local gauge transformation the fermion fields transform as:

$$\psi \longrightarrow \psi'(x) = \exp(ie\alpha(x))\psi(x), \tag{2.62}$$

$$\bar{\psi} \longrightarrow \bar{\psi}'(x) = \exp(-ie\alpha(x))\bar{\psi}(x). \tag{2.63}$$

Considering the infinitesimal transformation we obtain

$$\mathcal{L}' = \mathcal{L} - e(\partial_\mu\alpha(x))\bar{\psi}\gamma^\mu\psi(x). \tag{2.64}$$

Now let us apply the gauge transformation for a generic correlation function. We introduce the concept of time-ordering for a field product. Mathematically speaking it is defined as

$$T(\hat{\phi}(x)\hat{\phi}(y)) = \Theta(x^0-y^0)\hat{\phi}(x)\hat{\phi}(y) + \Theta(y^0-x^0)\hat{\phi}(x)\hat{\phi}(y). \tag{2.65}$$

where $\Theta(z)$ is the Heaviside function. Its purpose would as stated, to order relatively the action of the operators depending on their time location. Then, we can evaluate the vacuum expectation value of the action of two fields as

$$\begin{aligned}&\langle T(\psi'(x_1)\bar{\psi}'(x_2))\rangle\\ &\quad= \frac{1}{Z[0]}\int[D\bar{\psi}][D\psi][DA]\psi'(x_1)\bar{\psi}'(x_2)\exp(i\int d^4x(\mathcal{L}' - \frac{1}{2\lambda}(\partial\cdot A)^2))\\ &\quad= \frac{1}{Z[0]}\int[D\bar{\psi}][D\psi][DA]\psi(x_1)\bar{\psi}(x_2)(1+ie\alpha(x_1))(1-ie\alpha(x_2))\\ &\qquad\exp(i\int d^4x(\mathcal{L} - \frac{1}{2\lambda}(\partial\cdot A)^2 - e(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi)),\end{aligned}\tag{2.66}$$

Introducing first order expansion in $\alpha(x)$:

$$\begin{aligned}&= \frac{1}{Z[0]}\int[D\bar{\psi}][D\psi][DA]\psi(x_1)\bar{\psi}(x_2)\Big(1+ie\alpha(x_1) - ie\alpha(x_2) - \int d^4x ie(\partial_\mu\alpha)j^\mu\Big)\\ &\quad\times\exp\Big(i\int d^4x(\mathcal{L} - \frac{1}{2\lambda}(\partial\cdot A)^2)\Big),\end{aligned}\tag{2.67}$$

leads to:

$$\begin{aligned}&= \frac{1}{Z[0]}\int[D\bar{\psi}][D\psi][DA]\psi(x_1)\bar{\psi}(x_2)\exp(i\int d^4x(\mathcal{L} - \frac{1}{2\lambda}(\partial\cdot A)^2))+\\ &\quad\frac{1}{Z[0]}\int d^4x\alpha(x)\int[D\bar{\psi}][D\psi][DA]\psi(x_1)\bar{\psi}(x_2)(ie\delta(x-x_1) - ie\delta(x-x_2) + ie\partial_\mu j^\mu)\\ &\quad\exp\Big(i\int d^4x(\mathcal{L} - \frac{1}{2\lambda}(\partial\cdot A)^2)\Big).\end{aligned}\tag{2.68}$$

Considering the invariance of the correlation function under gauge transformations we obtain

$$0 = ie\delta(x-x_1)\langle T(\psi(x_1)\bar{\psi}(x_2))\rangle - ie\delta(x-x_2)\langle T(\psi(x_1)\bar{\psi}(x_2))\rangle + ie\partial_\mu\langle T(j^\mu(x)\psi(x_1)\bar{\psi}(x_2))\rangle.\tag{2.69}$$

Finally, we obtain the Schwinger-Dyson equation for QED at leading order [21],

$$i\partial_\mu\langle T(j^\mu(x)\psi(x_1)\bar{\psi}(x_2))\rangle = -ie\delta(x-x_1)\langle T(\psi(x_1)\bar{\psi}(x_2))\rangle + ie\delta(x-x_2)\langle T(\psi(x_1)\bar{\psi}(x_2))\rangle.\tag{2.70}$$

This result can be interpreted as the generalization of the conservation law $\partial_\mu J^\mu = 0$. By defining the Fourier transform of the 4 current correlation function as

$$M^\mu(p,q_1,q_2) = \int d^4x d^4x_1 d^4x_2 e^{ipx}e^{iq_1x_1}e^{-iq_2x_2}\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle,\tag{2.71}$$

and

$$M_0(q_1,q_2) = \int d^4x_1 d^4x_2 e^{iq_1x_1}e^{-iq_2x_2}\langle\psi(x_1)\bar{\psi}(x_2)\rangle,\tag{2.72}$$

$$M_0(q_1+p,q_2) = \int d^4x d^4x_1 d^4x_2 e^{ipx}e^{iq_1x_1}e^{-iq_2x_2}\delta^4(x-x_1)\langle\psi(x_1)\bar{\psi}(x_2)\rangle,\tag{2.73}$$

we obtain finally the Ward-Takahashi identity as

$$ip_\mu M^\mu(p, q_1, q_2) = M_0(q_1 + p, q_2) - M_0(q_1, q_2 - p). \tag{2.74}$$

To express this result in proper coordinates, we wil identify the scattering amplitude due to charge conservation as $\langle T(j^\mu(x)\psi(x_1)\bar{\psi}(x_2))\rangle \rightarrow \langle(\mathcal{Q}^+_\epsilon \mathcal{S} - \mathcal{S}\mathcal{Q}^-_\epsilon)\rangle$. If we identify the action of the asymptotic conserved charge

$$\begin{aligned}
\mathcal{Q}^-_\epsilon|in\rangle &= -2\int d^2z\partial_{\bar{z}}\epsilon\partial_z N^-(z,\bar{z})|in\rangle + \sum_{k=1}^{m} \mathcal{Q}^{in}_k \epsilon(z^{in}_k, \bar{z}^{in}_k)|in\rangle, &(2.75)\\
\langle out|\mathcal{Q}^+_\epsilon &= 2\int d^2z\partial_z\epsilon\partial_{\bar{z}}\langle out|N(z,\bar{z}) + \sum_{k=1}^{n} \mathcal{Q}^{out}_k \epsilon(z^{in}_k, \bar{z}^{in}_k)\langle out|, &(2.76)
\end{aligned}$$

where $N^\pm$ and $N_z$ are defined for the electromagnetic field as

$$\int\limits_{-\infty}^{\infty} du\, F^{(0)}_{uz} \equiv N_z = e^2\partial_z N, \tag{2.77}$$

and $N^\pm$ is $N$ evaluated at $\mathcal{I}^\pm$, then, the correlation function becomes

$$\begin{aligned}
\langle(\mathcal{Q}^+_\epsilon \mathcal{S} - \mathcal{S}\mathcal{Q}^-_\epsilon)\rangle &= 2\int d^2z\,\partial_z\epsilon\,\partial_{\bar{z}}\langle out|N(z,\bar{z})\mathcal{S} + 2\mathcal{S}\int d^2z\partial_{\bar{z}}\epsilon\,\partial_z N^-(z,\bar{z})|in\rangle\\
&\quad +\langle out|\sum_{k=1}^{n} \mathcal{Q}^{out}_k \epsilon(z^{in}_k, \bar{z}^{in}_k)\mathcal{S} - \mathcal{S}\sum_{k=1}^{m} \mathcal{Q}^{in}_k \epsilon(z^{in}_k, \bar{z}^{in}_k)|in\rangle. &(2.78)
\end{aligned}$$

Due to integration by parts, we see that

$$\begin{aligned}
\int d^2z\,\partial_{\bar{z}}\epsilon\,\partial_z N^-(z,\bar{z}) &= \int dz\,\partial_z N^- \int d\bar{z}\,\partial_{\bar{z}}\epsilon\\
&= \int d\bar{z}\,\partial_{\bar{z}}\epsilon\Big|^{\mathcal{I}^+}_{\mathcal{I}^-} N^- - \int d^2z\,\partial_z\partial_{\bar{z}}\epsilon N^-\\
&= -\int d^2z\,\partial_z\partial_{\bar{z}}\epsilon N^-. &(2.79)
\end{aligned}$$

As a consequence our Ward identity is expressed as

$$2\int d^2z\partial_z\partial_{\bar{z}}\langle out|(\partial_z N\mathcal{S} - \mathcal{S}\partial_z N^-)|in\rangle = \left[\sum_{k=1}^{m} \frac{Q^{in}_k}{z - z^{in}_k} - \sum_{k=1}^{n} \frac{Q^{out}_k}{z - z^{out}_k}\right]\langle out|\mathcal{S}|in\rangle. \tag{2.80}$$

The identification with the photon soft theorem requires an expansion on plane wave modes. Starting from the potential we have

$$A_\mu(x) = e\sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega}\Big[\epsilon^{*\alpha}_\mu(q)a^{out}_\alpha(q)e^{iqx} + \epsilon^\alpha_\mu(q)a^{out}_\alpha(q)^\dagger e^{-iqx}\Big]. \tag{2.81}$$

Given $u = t - r$, $\bar{q} = q\hat{q}$, $\bar{x} = r\hat{x}$ and $q^2 = 0 \rightarrow \omega_q = q$, the expansion becomes

$$\begin{aligned}
A_\mu(x) &= e\sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega}\Big[\epsilon^{*\alpha}_\mu(q)a^{out}_\alpha(q)e^{-i\omega_q u - i\omega_q r(1-\hat{q}\hat{x})} + \epsilon^\alpha_\mu(q)a^{out}_\alpha(q)^\dagger e^{i\omega_q u + i\omega_q r(1-\hat{q}\hat{r})}\Big]\\
&= \frac{e}{8\pi^2}\sum_{\alpha=\pm}\int_0^\infty d\omega_q\,\omega_q\int_0^\pi \sin\theta\Big[\epsilon^{*\alpha}_\mu(q)a^{out}_\alpha(q)e^{-i\omega_q u - i\omega_q r(1-\cos\theta)}\\
&\quad +\epsilon^\alpha_\mu(q)a^{out}_\alpha(q)^\dagger e^{i\omega_q u + i\omega_q r(1-\cos\theta)}\Big], &(2.82)
\end{aligned}$$

where the integral in spherical coordinates is given by

$$\int d^3q = \int_0^\infty d\omega_q \int_0^\pi d\theta \int_0^{2\pi} d\phi\, \omega_q^2 \sin\theta. \tag{2.83}$$

Given the saddle point at $\theta = 0$, we use the Taylor expansion at leading order

$$\begin{aligned} A_\mu(x) &= \frac{e}{8\pi^2}\sum\nolimits_{\alpha=\pm} \int_0^\infty d\omega_q \omega_q \epsilon_\mu^{*\alpha}(q) a_\alpha^{out}(q) e^{-i\omega_q u} \int_0^\pi d\theta\theta e^{-i\omega_q r\frac{\theta^2}{2}} \\ &\quad + \int_0^\infty d\omega_q \omega_q \epsilon_\mu^{\alpha}(q) a_\alpha^{out\dagger}(q) e^{i\omega_q u} \int_0^\pi d\theta\theta e^{i\omega_q r\frac{\theta^2}{2}} \\ &= -\frac{ie}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty \frac{d\omega_q}{\omega_q} \Big[\epsilon_\mu^{*\alpha}(q) a_\alpha^{out}(q) e^{-i\omega_q u} - \epsilon_\mu^{\alpha}(q) a_\alpha^{out\dagger}(q) e^{i\omega_q u}\Big]. \end{aligned} \tag{2.84}$$

This leads to the calculation of the $A_z = \partial_z x^\mu A_\mu$, then

$$A_z(x) = -\frac{ie}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty \frac{d\omega_q}{\omega_q} \Big[\partial_z x^\mu \epsilon_\mu^{*\alpha}(q) a_\alpha^{out}(q) e^{-i\omega_q u} - \partial_z x^\mu \epsilon_\mu^{\alpha}(q) a_\alpha^{out\dagger}(q) e^{i\omega_q u}\Big] \tag{2.85}$$

Considering the identity $\partial_z x^\mu \epsilon_\mu^{*\alpha}(q) = \frac{\sqrt{2}}{1+z\bar{z}}$, we obtain

$$A_z(x) = \frac{-i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}} \sum_{\alpha=\pm} \int_0^\infty \frac{d\omega_q}{\omega_q} \Big[a_\alpha^{out}(q) e^{-i\omega_q u} - a_\alpha^{out\dagger}(q) e^{i\omega_q u}\Big] + \mathcal{O}(r^{-1}). \tag{2.86}$$

For the large $r$ approximation, we recover the leading order

$$A_z^{(0)}(x) = \frac{-i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}} \sum_{\alpha=\pm} \int_0^\infty \frac{d\omega_q}{\omega_q} \Big[a_\alpha^{out}(q) e^{-i\omega_q u} - a_\alpha^{out\dagger}(q) e^{i\omega_q u}\Big]. \tag{2.87}$$

We are finally able to connect our results with the Ward-Takahashi identity due to the identity

$$e^2 \partial_z N = \lim_{\omega\to 0} \int_{-\infty}^\infty du (e^{i\omega u} + e^{-i\omega u}). \tag{2.88}$$

For both $\partial_z N$ and $\partial_z N^-$:

$$\partial_z N = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} [\omega a_+^{out}(\omega\hat{x}) + \omega a_-^{out}(\omega\hat{x})^\dagger], \tag{2.89}$$

$$\partial_z N^- = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} [\omega a_+^{in}(\omega\hat{x}) + \omega a_-^{in}(\omega\hat{x})^\dagger]. \tag{2.90}$$

Then, our Ward identity becomes

$$\lim_{\omega\to 0} [\omega\langle out|(a_+^{out}(\omega\hat{x})\mathcal{S} - \mathcal{S}a_-^{in}(\omega\hat{x})^\dagger|in\rangle] = \sqrt{2}e(1+z\bar{z})\Big[\sum_{k=1}^m \frac{Q_k^{in}}{z-z_k^{in}} - \sum_{k=1}^n \frac{Q_k^{out}}{z-z_k^{out}}\Big]\langle out|\mathcal{S}|in\rangle. \tag{2.91}$$

## 2.3 Soft theorem for photons

The upcoming discussion is based on the arguments of Weinberg shown in [1], [2], and [20]. We start by stating the photon soft theorem

$$\langle \hat{a}_+^{out}(q) S\rangle = e\Big[\sum_{k=1}^m \frac{Q_k^{out} p_k^{out}\cdot\epsilon^+}{p_k^{out}\cdot q} - \sum_{k=1}^n \frac{Q_k^{in} p_k^{in}\cdot\epsilon^+}{p_k^{in}\cdot q}\Big]\langle S\rangle. \tag{2.92}$$

The modification of the S matrix is subject to the appearance of soft photons in both in and out states. Its coupling is introduced according to Feynman diagrams by a vertex and propagator factors. First, the vertex is given by

$$\mathcal{L}_{int} = -A^\mu j_\mu, \tag{2.93}$$

$$V \equiv ie\epsilon^\mu 2Qp_\mu, \tag{2.94}$$

and the propagator:

$$S_F(p,q) = \frac{-i}{(p+q)^2 + m^2} = \frac{-i}{2p\cdot q}. \tag{2.95}$$

As a consequence

$$VS_F = (ie\epsilon^\mu 2Qp_\mu)[\frac{-i}{2p\cdot q}] = \frac{eQ\epsilon\cdot p}{p\cdot q}. \tag{2.96}$$

The soft photons can be produced in both in and out states undistinguisably, leading to the so called *Soft Factor*,

$$\sum_{k=1}^{m} \frac{eQ_k^{out} p_k^{out}\cdot\epsilon}{p_k^{out}\cdot q} - \sum_{k=1}^{n} \frac{eQ_k^{in} p_k^{in}\cdot\epsilon}{p_k^{in}\cdot q}. \tag{2.97}$$

Having obtained this result, we analyze the Lorentz invariance and associated global charge conservation for this system. Due to ortogonality of polarization and momentum vectors, we have that:

$$\epsilon^\mu q_\mu = 0, \tag{2.98}$$

and then

$$\epsilon^{\mu\prime} = \epsilon^\mu + q^\mu. \tag{2.99}$$

Then, we consider the following transformations:

$$q^\mu = \frac{\omega}{1+z\bar{z}}(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}), \tag{2.100}$$

$$\epsilon^{+\mu} = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \tag{2.101}$$

$$\epsilon^{-\mu} = \frac{1}{\sqrt{2}}(z, 1, i, \bar{z}), \tag{2.102}$$

$$(p_k'^{\,in})^\mu = E_k^{in}\Big(1, \frac{z_k^{in}+\bar{z}_k^{in}}{1+z_k^{in}\bar{z}_k^{in}}, \frac{-i(z_k^{in}-\bar{z}_k^{in})}{1+z_k^{in}\bar{z}_k^{in}}, \frac{1-z_k^{in}\bar{z}_k^{in}}{1+z_k^{in}\bar{z}_k^{in}}\Big). \tag{2.103}$$

By using the above expression, we finally find the Ward identity in the following form,

$$\lim_{\omega\to 0}[\omega\langle(\hat{a}_+^{out}S - S\hat{a}_-^{in\dagger})\rangle] = \sqrt{2}e(1+z\bar{z})\Bigg[\sum_k \frac{Q^{out}}{z-z_k^{out}} - \frac{Q_k^{in}}{z-z_k^{in}}\Bigg]\langle S\rangle. \tag{2.104}$$

Finally we obtain that the Ward identity obtained previously can be reexpressed in terms of a connection with the soft photon theorem in the Bondi coordinates introduced in the present chapter.

# 3 Infrared Divergences for Gravity

## 3.1 Spacetime symmetries

To build up spacetime symmetry, we start by considering spacetime isotropy. Mathematically this condition is equivalent to the invariance of the infinitesimal interval under Lorentz transformations

$$ds'^2 = ds^2. \tag{3.1}$$

By considering the Minkowski metric $g_{\mu\nu} = (-1, 1, 1, 1)$, we have

$$\begin{aligned} ds'^2 &= g'_{\mu\nu} dx'^{\mu} dx'^{\nu} \\ &= g_{\alpha\beta}(\Lambda^{\alpha}_{\mu} dx^{\mu})(\Lambda^{\beta}_{\nu} dx^{\nu}) \\ &= (g_{\alpha\beta}\Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu})(dx^{\mu} dx^{\nu}) \\ &= ds^2. \end{aligned} \tag{3.2}$$

The previous result, leads to the following condition:

$$g_{\mu\nu} = g_{\alpha\beta}\Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu} = (\Lambda^{T})^{\alpha}_{\mu} g_{\alpha\beta}\Lambda^{\beta}_{\nu} \longrightarrow g = \Lambda^{T} g \Lambda, \tag{3.3}$$

where $\Lambda \in SO(1, 3)$ is an element of the Lorentz group satisfying the condition (3.3), i.e. a 4-dimensional matrix with 6 independent elements.

Spacetime symmetries are associated to the geometry of spacetime. Lorentz invariance, for example, is characteristic of flat spacetimes, while diffeomorphism invariance corresponds to generalized curved spacetimes. Now, in our present context, we are examining the case of asymptotically flat spacetimes. The natural question that arises is what would be the symmetry group associated to that geometry. As first developed by Bondi, van der Burg, Metzner and Sachs [3], these set of isometries would be denominated the BMS group. An additional comment on the symmetry structure behind both flat and asymptotically flat spacetimes. From the group theory perspective, Lorentz group can be considered a subgroup of BMS. Beyond, both supertranslations and superrotations can be interpreted as extensions of translations and boosts on the celestial sphere, $S^2$ at null infinities $\mathcal{I}^{\pm}$ .

## 3.2 BMS group

The key idea behind the development of the Bondi, van der Burg, Metzner & Sachs (BMS) group of transformations [3] is the study of isometries for the case of asymptotically flat spacetimes. According to Wald [28], this can be accomplished by defining it as an isolated system, with an appropriate boundary representing points at infinity.

### 3.2.1 Minkowski spacetime and conformal infinity

In spherical coordinates, the line element is given by

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{3.4}$$

With the objective in mind of studying gravitational radiation, whose propagation speed is the speed of light, we consider the application of null advanced $v$, and retarded $u$ coordinates such as:

$$v = t + r, \tag{3.5}$$

$$u = t - r, \tag{3.6}$$

leading to:

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{3.7}$$

Now we manipulate the infinity through conformal transformations. First, for example, study the case of outgoing radiation at $u = \textit{constant}$ and $v = \infty$. To introduce the compactification of this point, we make one further transformation as $V = 1/v$. Then:

$$ds^2 = \frac{1}{V^2}dudV + \frac{1}{4}(\frac{1}{V} - u)^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{3.8}$$

As a second step, we need to deal with the singularity at $V = 0$. To skip it, we introduce the factorization of the so called conformal factor $\Omega$, as follows

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{3.9}$$

$$\Omega^2 = \frac{4}{(1+v^2)(1+u^2)}, \tag{3.10}$$

which can be accomplished through the conformal transformation

$$T = \arctan v + \arctan u, \tag{3.11}$$

$$R = \arctan v - \arctan u. \tag{3.12}$$

In the end, the equivalent non-physical metric is given by

$$d\hat{s}^2 = -dT^2 + dR^2 + \sin R^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{3.13}$$

Now, we can say that there exists a conformal isometry of Minkowski spacetime constrained to the region [28]

$$-\pi < T + R < \pi, \tag{3.14}$$

$$-\pi < T - R < \pi. \tag{3.15}$$

In terms of the new variables, the conformal infinities of Minkowski spacetime are

- $i^-$ $\rightarrow$ past timelike infinity $R = 0, T = -\pi$
- $i^+$ $\rightarrow$ future timelike infinity $R = 0, T = \pi$
- $\mathcal{I}^-$ $\rightarrow$ past timelike infinity $R \in (0, \pi), T = -\pi + R$
- $\mathcal{I}^+$ $\rightarrow$ future timelike infinity $R \in (0, \pi), T = \pi - R$
- $i^0$ $\rightarrow$ spatial infinity $R = \pi, T = 0$

All these locations gives a proper prescription for the describing the asymptotically flat infinity structure.

### 3.2.2 Construction of asymptotic flatness at curved spacetimes

With the previous results in mind, our objective becomes to define asymptotic flatness on curved spacetimes. The key role is subject to the introduction of conformal infinity on this arena as well. On our former concept, this included past and future timelike infinity $i^\pm$ on it, but not anymore. Besides, it would be necessary to introduce smoothness and differentiability at spatial infinity $i^0$ without excluding physically relevant systems from our setup like isolated bodies at early and late times, that is at $i^\pm$.

Then an asymptotically flat spacetime set on a manifold $M$ and with a metric $g_{\mu\nu}$ is defined by Ashtekar & Hansen [29] by the following conditions

- $\exists(\tilde{M}, \tilde{g}_{\mu\nu})$, infinitely differentiable where $(M, g_{\mu\nu})$ can be mapped.
- Shall exist a conformal isometry $\psi : M \rightarrow \psi(M) \subset \tilde{M}$ with conformal factor $\Omega$

with the subsequent conditions:

1. The metric $\tilde{g}_{\mu\nu}$ shall be continuous at $i^0$, with directional limits well defined on it.
2. The conformal factor $\Omega$ shall be continuous and two times differentiable at $i^0$ on $\tilde{M}$.
3. Well defined non-physical metric both in the null $\mathcal{I}^\pm$ and spacelike $i^0$ infinities.
4. Both the metric $g_{\mu\nu}$ and its non-physical equivalent $\tilde{g}_{\mu\nu}$ can only be related through an infinite stretching.

As it is known, Lorentz invariance in flat spacetimes can be pictured as dynamical equations to be invariant under that set of transformations [30]. If we extend this to the case of small gravitational effects from General Relativity, the natural question will be: what kind of symmetries are in place under these conditions?. In other words, given physical systems embedded in an asymptotically flat spacetime, what kind of symmetries shall they obey? With that problem in mind, further developed in [4], the problem can be addressed equivalently as what are the boundary conditions that field equations subject to extremely small gravitational fields to have a uniform an smooth behavior at $r \to \infty$. Mathematically speaking, is found that a field $\Phi$ with behavior $\Phi = O(r^{-N})$ as it approaches $r \to \infty$, is called "uniform" and "radially smooth" if

$$\frac{\partial \Phi}{\partial u} = O(r^{-N-1}), \tag{3.16}$$
$$\frac{\partial \Phi}{\partial r} = O(r^{-N}), \tag{3.17}$$
$$\frac{\partial \Phi}{\partial \theta} = O(r^{-N}), \tag{3.18}$$
$$\frac{\partial \Phi}{\partial \phi} = O(r^{-N}). \tag{3.19}$$

If these conditions are violated, the radiative modes of the field $\Phi$ are claimed to be carrying infinite amount of energy, giving rise to an infinite mass term for the field. A remarkable feature of this construction, is that it is completely independent from a particular gravitational field profile. In other words, one is able to separate the kinematics of spacetime from the dynamics of the gravitational field at spatial infinity [4]. Keep in mind that, all this construction is perfectly compatible to the conditions introduced in [26] previously.

Our aim is to find compatibility with the principles introduced by Sachs, as well as with the consideration of having an asymptotically Lorentz covariant spacetime. We start from a general Lorentzian metric, expressed in Bondi coordinates [3],

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\frac{m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + U_z dudz + U_{\bar{z}}dud\bar{z} + \cdots, \tag{3.20}$$

where the functions $m_B, C_{zz}, C_{\bar{z}\bar{z}}, U_z, U_{\bar{z}}$ shall be also uniform and smooth at $r \to \infty$, so we can recover Minkowski metric in the flat limit,

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{3.21}$$

The idea behind of expressing Minkowski in retarded/advanced coordinates is to recover the BMS symmetry group as a product of the rigid translations and a six-parametric group isomorphic to the homogeneous orthochronous (improper) Lorentz group, that is the conformal group for the $S^2$ sphere.

Asymptotic Flatness criteria is induced through the falloff conditions of the Weyl tensor computed from (3.20). Being defined as:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}(g_{\nu\rho}R_{\sigma\mu} + g_{\mu\sigma}R_{\rho\nu} - g_{\nu\sigma}R_{\rho\mu} - g_{\mu\rho}R_{\sigma\nu}) + \frac{1}{6}R(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}), \tag{3.22}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor, $R_{\sigma\mu}$ the Ricci tensor and $R$ the Ricci scalar. It encodes the curvature in free space, and governs the propagation of gravitational waves. Then, it is the appropriate observable for measuring the effects of curvature for the asymptotically flat spacetime. From the calculation of the Weyl tensor associated to the metric (3.20), we extract the following components

$$\begin{aligned} C_{rzrz} &= -\frac{C_{\bar{z}\bar{z}}C_{zz}^2\gamma_{z\bar{z}}}{2(C_{zz}C_{\bar{z}\bar{z}}-r^2\gamma_{z\bar{z}}^2)^2} && (3.23) \\ C_{rurz} &= \frac{1}{8(rC_{zz}C_{\bar{z}\bar{z}}-r^3\gamma_{z\bar{z}}^2)^2}\Big\{C_{zz}^2[C_{\bar{z}\bar{z}}^2U_z+rC_{\bar{z}\bar{z}}+rC_{z\bar{z}}(U_{\bar{z}}\gamma_{z\bar{z}}+2rD_{\bar{z}}\gamma_{z\bar{z}}) \\ &\quad -r^2\gamma_{z\bar{z}}D_{\bar{z}}C_{\bar{z}\bar{z}}]+r^3\gamma_{z\bar{z}}^2[2r\gamma_{z\bar{z}}(U_z\gamma_{z\bar{z}}-D_{\bar{z}}C_{zz})-C_{\bar{z}\bar{z}}D_zC_{zz}] \\ &\quad +r^2C_{zz}\gamma_{z\bar{z}}[C_{\bar{z}\bar{z}}(-2U_z\gamma_{z\bar{z}}+2rD_z\gamma_{z\bar{z}}+D_{\bar{z}}C_{zz})-r\gamma_{z\bar{z}}D_zC_{\bar{z}\bar{z}}]\Big\} && (3.24) \end{aligned}$$

Its behavior towards $r\to\infty$ is the one which defines the condition of asymptotic flatness. Specifically, the criteria is that

$$C_{rzrz} \sim \mathcal{O}(r^{-3}), \tag{3.25}$$

$$C_{rurz}, C_{rur\bar{z}} \sim \mathcal{O}(r^{-3}). \tag{3.26}$$

By the introduction of Taylor expansion on both expressions at (3.23), we recover

$$\begin{aligned} C_{rzrz} &\sim -\frac{C^{\bar{z}\bar{z}}C_{zz}^2}{2r^3}+\mathcal{O}(r^{-5}) && (3.27) \\ C_{rurz} &\sim \frac{1}{4r^2}(U_z-D^zC_{zz})+\frac{1}{8r^3}\{2C^{zz}C_{\bar{z}\bar{z}}D^z\gamma_{z\bar{z}}-C_{\bar{z}\bar{z}}D_zC^{zz}-C_{zz}D_zC^{\bar{z}\bar{z}}\} \\ &\quad +\mathcal{O}(r^{-4}). && (3.28) \end{aligned}$$

The first constraint is satisfied, given $C_{rzrz}\sim\mathcal{O}(r^{-3})$ and beyond. About the second constraint, the first non-zero contribution comes at $\mathcal{O}(r^{-2})$, with

$$\frac{1}{4r^2}(U_z-D^zC_{zz})=0. \tag{3.29}$$

where $C_{zz}$ is the initial conditions tensor. So to make vanish this term, we consider

$$U_z=D^zC_{zz}. \tag{3.30}$$

With all that in mind, there is a third constraint due to the interpretation of the radial coordinate $r$ as luminosity distance as defined by Sachs in [4][30]. Mathematically speaking, we get that

$$\partial_r \det\left[\frac{g_{z\bar{z}}}{r^2}\right]=0. \tag{3.31}$$

Given that $g_{z\bar{z}}=r^2\gamma_{z\bar{z}}+rC_{z\bar{z}}+\mathcal{O}(r^{-1})$, this condition generates as consequence that the $C_{z\bar{z}}$ is traceless

$$C_z^z=0. \tag{3.32}$$

As result of the falloff conditions, the line element for this kind of spacetimes is given as

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_{B}}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + \\ &\quad + D^z C_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} + \frac{1}{r}\left[\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right]dudz \ldots, \end{aligned} \tag{3.33}$$

where $m_B$ is the Bondi mass and $N_z = \partial_z C$, being $C$ the fully contracted expresion of $C_{zz}$. $N_z$ is called the angular momentum aspect of the metric, due to its connection through integration to the angular momentum. Originally, this was deduced by Bondi et. al in [3, 30].

## 3.3 BMS algebra

The symmetry group associated to these geometrical conditions shall preserve both local and global boundary conditions at the null infinity [26], defined in the previous section. These are equivalent to the conservation of the fall-off conditions for the 4D Lorentzian spacetime (3.20). For the Killing equations to be solved for this general case, we are able to determine the elements of the BMS algebra. Even though our computations are expressed in the $\{u, r, A, B\}$ coordinates, this is also valid for $\{u, r, z, \bar{z}\}$ via the identification $\{A, B\} \longrightarrow \{z, \bar{z}\}$. Then, we have

$$U = 1 - \frac{2m_B}{r} + \mathcal{O}(r^{-2}), \tag{3.34}$$

$$\beta = \mathcal{O}(r^{-2}), \tag{3.35}$$

$$U_A = \frac{1}{r^2}D^A C_{BA} + \mathcal{O}(r^{-3}), \tag{3.36}$$

$$g_{AB} = r^2\gamma_{AB} + rC_{AB} + \mathcal{O}(1). \tag{3.37}$$

By expanding (3.20), we obtain the following metric elements

$$g_{uu} = -U + \frac{1}{4}g_{z\bar{z}}U^z U^{\bar{z}}, \tag{3.38}$$

$$g_{ur} = -e^{2\beta}, \tag{3.39}$$

$$g_{uA} = \frac{1}{2}D^B C_{BA} + \mathcal{O}(r^{-1}), \tag{3.40}$$

$$g_{rr} = g_{rA} = 0, \tag{3.41}$$

$$g_{AB} = r^2\gamma_{AB} + rC_{AB} + \mathcal{O}(1). \tag{3.42}$$

Our objective is to determine the set of transformations which leave the metric invariant up to the fall-off conditions imposed. To determine this, we perform the calculation of the Killing vectors for the system

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu\xi^\rho + g_{\nu\rho}\partial_\mu\xi^\rho. \tag{3.43}$$

We analyze each element as follows

$$\mathcal{L}_\xi g_{rr} = 2g_{ur}\partial_r\xi^u = 0 \longrightarrow \partial_r\xi^u = 0 \longrightarrow \xi^u = \xi^u(u, x^A). \tag{3.44}$$

Then

$$\mathcal{L}_\xi g_{ur} = \partial_u \xi^u g_{ur} + \mathcal{O}(r^{-1}) = 0 \longrightarrow \xi^u = \xi^u(x^A) = f(x^A). \tag{3.45}$$

So far, from the previous constraints and fall-off conditions the Killing vector $\xi$ shall have the following decomposition

$$\xi = f\partial_u + \sum_{n=0}^{\infty} \frac{\xi^{r(n)}}{r^n}\partial_r + \sum_{n=1}^{\infty} \frac{\xi^{A(n)}}{r^n}\partial_A. \tag{3.46}$$

Now our Killing equation for $g_{uu}, g_{rA}$ and $g_{AB}$ become

$$\begin{aligned}
\mathcal{L}_\xi g_{uu} &= f\partial_u g_{uu} + \xi^r \partial_r g_{uu} + \xi^A \partial_A g_{uu} + 2\partial_u \xi^r g_{ru} + 2\partial_u \xi^A g_{Au} \\
&= -2\partial_u \xi^{r(0)} + \frac{-2\partial_u \xi^{r(1)} + 2f\partial_u m_B + \partial_u \xi^{A(1)} D^B C_{BA}}{r} + \mathcal{O}(r^{-2}), \qquad (3.47)\\
\mathcal{L}_\xi g_{rA} &= \partial_r \xi^C g_{CA} + g_{ur}\partial_A f \\
&= -\gamma_{AB}(\xi^{B(1)} + D^B f) - \frac{2\gamma_{AB}\xi^{B(2)} + C_{AB}\xi^{B(1)}}{r} + \mathcal{O}(r^{-2}), \qquad (3.48)\\
\mathcal{L}_\xi g_{AB} &= f\partial_u g_{AB} + \xi^r \partial_r g_{AB} + \xi^C \partial_C g_{AB} + g_{uB}\partial_A f + \partial_A \xi^C g_{CB} + g_{uA}\partial_B f + \partial_B \xi^C g_{CA} \\
&= r\Big[f\partial_u C_{AB} + 2\gamma_{AB}\xi^{r(0)} + D_A \xi^{(1)}_B + D_B \xi^{(1)}_A\Big] + f\partial_u h^{(0)}_{AB} + C_{AB}\xi^{r(0)} \\
&\quad +2\gamma_{AB}\xi^{r(1)} + \frac{1}{2}D^C C_{CA} D_B f + \frac{1}{2} D^C C_{CB} D_A f + \xi^{C(1)} D_C C_{AB} + D_A \xi^{(2)}_B \\
&\quad +C_{BC} D_A \xi^{C(1)} + D_B \xi^{C(2)}_A + C_{AC} D_B \xi^{C(1)} + \mathcal{O}(r^{-1}). \qquad (3.49)
\end{aligned}$$

Now to cancel out the terms violating the fall-off conditions in (3.48), we obtain for the angular components of the $\xi$:

$$\begin{aligned}
\xi^{A(1)} &= -D^A f, \qquad (3.50)\\
\xi^{B(2)} &= \frac{1}{2} C^{BC} D_C f. \qquad (3.51)
\end{aligned}$$

Similarly for its radial components, both the Killing equations

$$f\partial_u C_{AB} + 2\gamma_{AB}\xi^{r(0)} + D_A \xi^{(1)}_B + D_B \xi^{(1)}_A = 0 \longrightarrow \xi^{r(0)} = \frac{1}{2} D_A D^A f, \tag{3.52}$$

and luminosity distance condition

$$\begin{aligned}
\det\left[\frac{g_{AB}}{r^2}\right] &= \det\left[\gamma_{AB} + \frac{C_{AB}}{r} + \mathcal{O}(r^{-2})\right] \\
&= \det\gamma \exp tr\Big[\log\Big(\delta^B_A + \frac{1}{r}C^B_A + \frac{1}{r^2}h^{(0)B}_A + \mathcal{O}(r^{-3})\Big)\Big] \\
&= \det\gamma\Big[1 + \frac{1}{r}C^A_A + \frac{1}{r^2}(h^{(0)A}_A - \frac{1}{2}C^B_A C^A_B + \frac{1}{2}C^A_A C^B_B) + \mathcal{O}(r^{-3})\Big]. \qquad (3.53)
\end{aligned}$$

Given that $C^A_A = 0$ due to Weyl tensor properties, the only non-trivial term to be cancelled is

$$h^{(0)A}_A - \frac{1}{2}C^B_A C^A_B = 0 \longrightarrow h^{(0)A}_A = \frac{1}{2} C^B_A C^A_B. \tag{3.54}$$

Then its variation is given by

$$\begin{aligned}
\delta h_A^{(0)A} &= C_{AB}\delta C^{AB} \\
&= f\partial_u h_A^{(0)A} + 4\xi^{r(1)} + 2D_A f D_B C^{AB} - C^{AB} D_A D_B f \\
&= \frac{f}{2}\partial_u (C^{AB} C_{AB}) - 2C^{AB} D_A D_B f \\
&= 0,
\end{aligned} \tag{3.55}$$

and then,

$$\xi^{r(1)} = -\frac{1}{2} D_A f D_B C^{AB} D_A D_B f. \tag{3.56}$$

This result leads us to obtain the Killing vector associated to the preservation of asymptotically flat spacetimes, as

$$\begin{aligned}
\xi(f) &= f\partial_u + \Big[ - \frac{D^A f}{r} + \frac{\frac{1}{2} C^{AB} D_B f}{r^2} + \mathcal{O}(r^{-3}) \Big] \partial_A \\
&\quad + \Big[ \frac{1}{2} D^2 f + \frac{-\frac{1}{2} D_A f D_B C^{AB} - \frac{1}{4} C^{AB} D_A D_B f}{r} + \mathcal{O}(r^{-2}) \Big] \partial_r .
\end{aligned} \tag{3.57}$$

Being $f = f(x^A)$ the parameter for our group of transformations, then its Lie algebra is built up from the Lie brackets of (3.57). Defined as:

$$\begin{aligned}
[V, W] &= [V^\mu \partial_\mu, W^\nu \partial_\nu] \\
&= [V^\nu \frac{\partial W^\mu}{\partial x^\nu} - W^\nu \frac{\partial V^\mu}{\partial x^\nu}] \partial_\mu .
\end{aligned} \tag{3.58}$$

Then our respective brackets will be obtained from the expansion

$$\begin{aligned}
[\xi(f_1), \xi(f_2)] &= [\xi^u(f_1)\partial_u \xi^\mu(f_2) - \xi^u(f_2)\partial_u \xi^\mu(f_1)] \\
&\quad + [\xi^r(f_1)\partial_r \xi^\mu(f_2) - \xi^r(f_2)\partial_r \xi^\mu(f_1)] \\
&\quad + [\xi^A(f_1)\partial_A \xi^\mu(f_2) - \xi^A(f_2)\partial_A \xi^\mu(f_1)].
\end{aligned} \tag{3.59}$$

and as a result, we get that

$$[\xi(f_1), \xi(f_2)]^u = 0, \tag{3.60}$$

$$[\xi(f_1), \xi(f_2)]^r = \mathcal{O}(r^{-1}), \tag{3.61}$$

$$[\xi(f_1), \xi(f_2)]^A = \mathcal{O}(r^{-2}). \tag{3.62}$$

Our symmetry vector $\xi(f)$ shall be identified from now on as the *Super-translation* generator. To interpret and connect the action of super-translations to charge conservation and subsequent Ward Identities and Soft theorems, we shall consider the action of (3.57) on the initial data of the system, encoded in $m_B, C_{AB}, N_{AB}$:

$$\mathcal{L}_{\xi(f)} C_{AB} = f\partial_u C_{AB} + \gamma_{AB} D^2 f - 2D_A D_B f, \tag{3.63}$$

$$\mathcal{L}_{\xi(f)} N_{AB} = f\partial_u N_{AB}, \tag{3.64}$$

$$\mathcal{L}_{\xi(f)} m_B = f\partial_u m_B + \frac{1}{4}(N^{AB} D_A D_B f + 2D_A f D_B M N^{AB}). \tag{3.65}$$

where the Bondi news tensor is defined as

$$N_{AB} = \frac{1}{2}\partial_u C_{AB}, \tag{3.66}$$

and particularly in the retarded Bondi coordinates is expressed as

$$N_{zz} = \partial_u C_{zz}. \tag{3.67}$$

Making explicit identification with the Bondi retarded coordinates (2.23), then

$$\mathcal{L}_{\xi(f)} C_{AB} \longrightarrow \mathcal{L}_{\xi(f)} C_{zz} = f\partial_u C_{zz} + \partial_u C_{zz} + \gamma_{zz} D^2 f - 2D_z^2 f, \tag{3.68}$$

and considering the vanishing of the curvature it is obtained that

$$C_{zz} = -2D_z^2 C. \tag{3.69}$$

However, under supertranslations that condition is not fullfilled, because under its action

$$\mathcal{L}_\xi C = f, \tag{3.70}$$

generating vacuum degeneracy up to $f$.

Further development of the new concept of supertranslations is required. First, from a group theory point of view, supertranslations are a generalization of the spacetime translations in the Poincare group. In Bondi retarded coordinates, they can be pictured as translations along every null generator of $\mathcal{I}^+$. In addition, about the infrared behavior of gravity theories, these transformations preserve metric gauge choice and fall-off conditions for asymptotically flat spacetimes. Then, the BMS group is defined separately on null infinities where we identify both BMS$^+$ and BMS$^-$ invariance groups.

At both classical and quantum level, the BMS invariance has dynamical consequences. The former can be analyzed through the lens of Einstein's field equations. To have a full BMS invariance, the equations of motion are constrained by continuity conditions at the spatial infinity $i^0$, consequence of the initial value problem in General Relativity. In practical terms, it will imply the introduction of the definition of Christodolou and Klainerman (CK) spaces, as explained by He et. al. [31] and in last instance, the matching conditions for the bulk fields of the metric

$$C(z,\bar{z})\Big|_{\mathcal{I}^\pm} = C(z,\bar{z})\Big|_{\mathcal{I}^\mp}, \tag{3.71}$$

$$m_B(z,\bar{z})\Big|_{\mathcal{I}^\pm} = m_B(z,\bar{z})\Big|_{\mathcal{I}^\mp}. \tag{3.72}$$

Expressed in terms of the BMS group parameter $f(z,\bar{z})$, we get

$$f(z,\bar{z})\Big|_{\mathcal{I}^\pm} = f(z,\bar{z})\Big|_{\mathcal{I}^\mp}, \tag{3.73}$$

the same matching condition introduced for gauge fields.

For scattering amplitudes, as will be seen in the next section, spontaneous symmetry breaking of the BMS invariance leads to the appearance of soft gravitons as Goldstone modes. The preservation of the symmetry becomes a Ward Identity and Soft Graviton Theorem.

## 3.4 Quantum Gravity Ward Identity

Scattering amplitudes invariant under BMS is subject to the continuity of Cauchy initial data from $\mathcal{I}^-$ to $\mathcal{I}^-$ through $i^0$. Spacetimes subject to this type of expansion are called *Christodoulou & Klainerman spaces (CK)*[5][6]. In our application context, we can picture them as described in [32], spacetimes which subject to the evolution any physical interaction, like black hole mergers or particle scattering, they return to a vacuum configuration. Having in consideration all interactions are allowed, except those whose energy surpasses the threshold for the formation of black holes. Having now fixed a gauge for the metric (Bondi gauge), introduce the fall-off conditions and CK space configuration, the bulk and boundary fields in the metric become constrained like

$$N_{zz}(u)\Big|_{u\to\pm\infty} \sim |u|^{-3/2}, \tag{3.74}$$

and the Weyl tensor

$$\Psi_2^0(u,z,\bar{z}) = -\lim_{r\to\infty}(rC_{uzr\bar{z}}\gamma^{z\bar{z}}) = -m_B + \frac{1}{4}C^{zz}N_{zz} - \frac{1}{2}\gamma^{z\bar{z}}(\partial_{\bar{z}}U_z - \partial_z U_{\bar{z}}), \tag{3.75}$$

where $m_B$ is the Bondi mass, and $asconsequenceto$ $m_B\Big|_{\mathcal{I}^+_+} = 0,$

$m_B\Big|_{\mathcal{I}^+_-} = -M,$ (3.76)where $M$ can be identified as the ADM mass. Besides

$$\Big[\partial_{\bar{z}}U_z - \partial_z U_{\bar{z}}\Big]_{\mathcal{I}^+_\pm} = 0. \tag{3.77}$$

Now we proceed to discuss the scattering problem in the context of BMS invariance. Given the continuity conditions for the initial data, we introduce the supertranslation generator at $\mathcal{I}^+$, and its action on the news tensor $N_{zz}$:

$$\begin{aligned} T^+(f) &= \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\gamma_z f m_B \\ &= \frac{1}{16\pi G}\int dud^2zf\Big[\gamma_{z\bar{z}}T_{uu} + \frac{1}{2}\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}}U_z)\Big], \\ \{T^+(f), N_{zz}\} &= f_u N_{zz}. \end{aligned} \tag{3.78}$$

The action of $T^+(f)$ as seen in the expansion, is composed of two terms. The first corresponds to the introduction of supertranslation on the radiative modes and the second is a boundary term which properly manipulated is equivalent to

$$\frac{1}{32\pi G}\int_{\mathcal{I}^+} dud^2z f\partial_u\Big[\partial_z U_{\bar{z}} + \partial_{\bar{z}} U_z\Big] = \lim_{\omega\to 0}\frac{1}{32\pi G}\int_{\mathcal{I}^+} dud^2z(e^{i\omega u} + e^{-i\omega u})\Big[N^{z}_{\bar{z}} D^2_z f + N^{\bar{z}}_{z} D^2_{\bar{z}} f\Big], \tag{3.79}$$

the soft graviton term. The latter will be responsible for the spontaneous symmetry breaking in the scattering process. As a consequence then

$$\{T^+(f), T^+(f')\} = 0, \tag{3.80}$$

and the S-matrix invariance

$$T^+(f)\mathcal{S} - \mathcal{S}T^-(f) = 0. \tag{3.81}$$

Due to this symmetry, the action/coupling of soft graviton current generated by those fields lead to the so called quantum gravity Ward identity, which associates to this insertion of the well-known soft factor, given as:

$$\langle z_1^{\text{out}}, \cdots | \colon P_z\mathcal{S}\colon | z_1^{\text{in}}, \cdots\rangle = \langle z_1^{\text{out}}, \cdots |\mathcal{S}| z_1^{\text{in}}, \cdots\rangle \left[\sum_{k=1}^{m}\frac{E_k^{\text{out}}}{z - z_k^{\text{out}}} - \sum_{k=1}^{n}\frac{E_k^{\text{in}}}{z - z_k^{\text{in}}}\right], \tag{3.82}$$

where the soft graviton current defined as:

$$P_z \equiv \frac{1}{2G}\left(\int_{-\infty}^{\infty} dv\partial_v V_z - \int_{-\infty}^{\infty} du\partial_u U_z\right). \tag{3.83}$$

## 3.5 Soft graviton theorem

A simplified version of this identity is deduced from an Einstein-scalar gravity theory. Take in consideration the weak field approximation given by $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, so the action becomes

$$S = -\int d^4x\sqrt{-g}\left[\mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{s}}\right], \tag{3.84}$$

where

$$\begin{aligned}\mathcal{L}_{\text{grav}} &= -\frac{2}{\kappa^2}R = -\frac{1}{2}\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h + \partial^\mu h_{\mu\nu}\partial_\rho h^{\nu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \cdots, \\ \mathcal{L}_{\text{s}} &= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}\kappa h^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\sigma\phi\partial_\sigma\phi\right] + \cdots,\end{aligned} \tag{3.85}$$

and $h = det(h_{\mu\nu})$. We introduce for this case one soft graviton in the scattering amplitude with the two topologically inequivalent Feynman diagrams (see Figure 1). In analogy to the case of the photon, is known that regardless of the nature of the quantum field theory studied, the *Lehmann-Symanzik-Zimmerman (LSZ) reduction formula* [21] is applicalbe and the calculation reduces to the product of the expectation value of the time ordered product of fields with the Fourier transform of $(\Box + m^2)$-like factors associated to the dynamical coupling of the gauge/gravity field with its source, as

$$\begin{aligned}\langle p_1 \dots p_n | S | p'_1 \dots p'_m \rangle &= \left[ i \int d^4 x'_1 e^{-ip'_1 x'_1} (\Box'_1 + m^2) \right] \dots \left[ i \int d^4 x_n e^{ip_n x_n} (\Box_n + m^2) \right] \\ &\quad \times \langle \Omega | T\{ \mathcal{O}'_1(x'_1) \dots \mathcal{O}'_m(x'_m) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \} | \Omega \rangle. \end{aligned} \tag{3.86}$$

When the fields are treated as free for both the initial and final states, then on the asymptotic case we find that $(\Box + m^2) \longrightarrow 0$. However those not lead to a zero scattering amplitude, due to the fact that the time ordering expectation value contains factors like $\frac{1}{(\Box + m^2)}$ which cancel the former $(\Box + m^2)$. So we see cancelation of infinities due to zeros. As a result, the *LSZ formula* cancels all but relevant terms [21], as vertex and interaction terms. For the case of our interacting scalar field theory, the Fourier transformed interacting term come from both the vertex interaction of the soft graviton as well as its propagator, for the case of coupling with external legs of the collision. First the vertex is given as

$$\frac{-i}{(p+q)^2 - i\epsilon} = \frac{-i}{2p \cdot q}, \tag{3.87}$$

due to the on-shell condition. Since the interaction term goes proportional to $2p_\mu p_\nu$, we find the coupling term

$$\sqrt{8\pi G} \frac{\epsilon^{\mu\nu} p_\mu p_\nu}{p \cdot q}. \tag{3.88}$$

As consequence we have that, as developed originally by Weinberg [2][20] and analyzed by Strominger [23], the amplitude $\mathcal{M}_{\mu\nu}$ for an scattering process of $m$ incoming and $n$

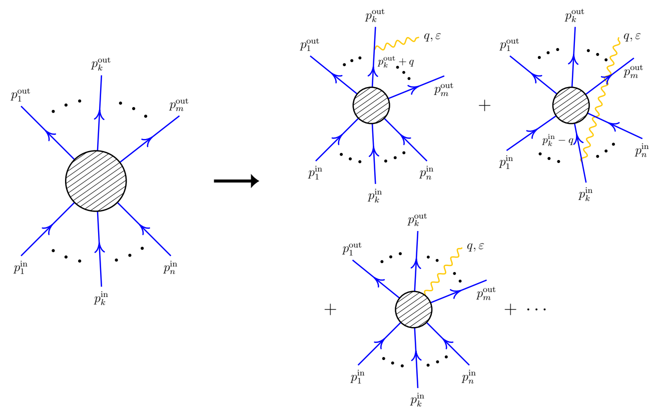


Figure 1 – Representation of an scattering process for m incoming and n outgoing scalars particles. The first two diagrams represent the insertion of the soft graviton on the external particles, while the third one considers this for an internal particle. Figure extracted from Strominger [23]

outgoing particle states is determined by

$$\begin{aligned}\mathcal{M}_{\mu\nu}(q, p'_1, \cdots, p'_m, p_1, \cdots, p_n) = &\sum_{k=1}^{m} \mathcal{M}(p'_1, \cdots, p'_k + q, \cdots, p'_m, p_1, \cdots, p_n) \frac{-i}{(p'_k + q)^2 - i\epsilon} \\ &\times \left[\frac{i\kappa}{2}\left(p'_{k\mu}(p'_k + q)_\nu + p'_{k\nu}(p'_k + q)_\mu - \eta_{\mu\nu} p'_k \cdot (p'_k + q)\right)\right] \\ &+ \sum_{k=1}^{n} \mathcal{M}(p'_1, \cdots, p'_m, p_1, \cdots, p_k + q, \cdots, p_n) \frac{-i}{(p_k - q)^2 - i\epsilon} \\ &\times \left[\frac{i\kappa}{2}\left(p_{k\mu}(p_k - q)_\nu + p_{k\nu}(p_k - q)_\mu - \eta_{\mu\nu} p_k \cdot (p_k - q)\right)\right].\end{aligned} \tag{3.89}$$

The soft graviton theorem is the leading term in $q$-expansion, which comes from generalizing (3.88) for the case of m-incoming and n-outgoing particles, such as

$$\mathcal{M}_{\mu\nu}(q, p'_1, \cdots, p'_m, p_1, \cdots, p_n) = \frac{\kappa}{2}\left[\sum_{k=1}^{m} \frac{p'_{k\mu} p'_{k\nu}}{p'_k \cdot q} - \sum_{k=1}^{n} \frac{p_{k\mu} p_{k\nu}}{p_k \cdot q}\right] \mathcal{M}(p'_1, \cdots, p'_m, p_1, \cdots, p_n), \tag{3.90}$$

where $q \to 0$. The contraction of the helicity vector with the scattering amplitude, enables to obtain the scattering ratio. Through the application of Lorentz invariance $e^\mu(\bar{q}, \pm 2)\mathcal{M}^{\alpha\beta}_{\mu\nu} \to e'^\mu(\bar{q}, \pm 2)\mathcal{M}'^{\alpha\beta}_{\mu\nu}$, leading to

$$e'^\mu(\bar{q}, \pm 2)\mathcal{M}'^{\alpha\beta}_{\mu\nu} = (e^\mu + q^\mu)\mathcal{M}^{\alpha\beta}_{\mu\nu}, \tag{3.91}$$

where $\{\alpha, \beta\}$ are matrix indices. Implying energy-momentum conservation as

$$\sum_n \eta_n f_n p^\nu_n = \sqrt{8\pi G_N} \sum_n \eta_n p^\nu_n, \tag{3.92}$$

where $f_n$ is the coupling constant for this quantum gravity model, Lorentz invariance guarantees $p^\mu$ conservation if, as mentioned by Weinberg [20]. *low energy massless particles of spin 2 couple in the same way to all forms of energy and momentum.* In addition, Weinberg's analysis gives us an important conclusion connecting asymptotically flat spacetime physics with curved spacetime cases as *Einstein's principle of equivalence is a necessary consequence of Lorentz invariance as applied to massless particles of spin 2.*

# 4 Memory Effect and the Infrared Triangle

In the present chapter we discuss the concept of memory effect as an infrared phenomena in physics. Our objective is to identify its connection with the soft factors derived from the soft theorems analyzed previously.

## 4.1 Electromagnetic memory effect

As main references for the electromagnetic memory effect we will consider [33] and [34]. The electromagnetic memory effect can be understood as the velocity kick received by a set of radiation detectors, placed to profile the E-M waves propagating on their surroundings. From a theoretical point of view, this is a result of the waves zero modes interacting with the in and out states, that will produce a net change $\Delta A_u$ on the leading order terms of the $A_\mu$ potential. The net change will be proportional to the soft factor.

Being $r$ the regulator, we take a Taylor expansion of the potential for $r \longrightarrow 0$ as follows

$$A_\mu(u, z, \bar{z}) = \lim_{r \to 0} A_\mu(r, u, z, \bar{z}) = \frac{1}{r} A_u + A_z + O(r) + \dots, \tag{4.1}$$

Then Maxwell's equations (2.43)(2.44) for the $O(r^{-1})$ term are given by

$$\partial_u A_u = \partial_u (D^z A_z + D^{\bar{z}} A_{\bar{z}}) + e^2 j_u. \tag{4.2}$$

The choice of these terms is essential for the IR dynamics of the field after the E-M radiation has propagated away from the source, corresponds as usual to the constraints imposed via the asymptotically flat spacetime fall-off conditions presented in (2.41). From that expansion of the potential, we find at leading order, the terms which shall contribute to the Faraday tensor will be $F_{ur} = F_{rz} = \mathcal{O}(r^{-2})$, $F_{z\bar{z}} = \mathcal{O}(1)$ and $F_{uz} = -A_z^{(1)}$. As a consequence, this leads to the correspondent expansion of the Faraday tensor components like

$$\begin{aligned} F_{ur} &= A_u, \\ F_{z\bar{z}} &= \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z, \\ F_{uz} &= \partial_u A_z, \\ F_{rz} &= -A_z^{(1)}. \end{aligned} \tag{4.3}$$

The net variation is measured in the space-time region surrounding the sources, which encloses $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$. Given that along $u$ and using $F_z = 0$ at the boundaries of $\mathcal{I}^+$, then we integrate (4.2)

$$\Delta A_u \Big|_{\mathcal{I}^-_+}^{\mathcal{I}^+_-} = 2 D^z \Delta A_z \Big|_{\mathcal{I}^-_+}^{\mathcal{I}^+_-} + e^2 \int du j_u. \tag{4.4}$$

The electromagnetic radiation generated through this velocity kick can be identified as a soft factor contribution through the computation of $\Delta A_z$ as

$$\Delta A_z = -\frac{e}{4\pi}\epsilon_z^{*+}\omega S_p^{(0)+}, \tag{4.5}$$

where

$$S_p^{(0)\pm} = eQ\frac{p\cdot\epsilon^{\pm}}{p\cdot q}. \tag{4.6}$$

Some additional comments can be made about electromagnetic memory. First, memory effect can be seen even in slow motion and weak field scenarios. In those cases, it corresponds to the projection of the dipole moment responsible for the velocity kick. From Newton's second law

$$m\frac{d^2\bar{x}}{dt^2} = q\bar{E}, \tag{4.7}$$

so then the velocity kick is obtained through integration by the identification

$$\bar{E} = \frac{1}{r}P\Big[\frac{d^2\bar{p}}{dt^2}\Big], \tag{4.8}$$

$$\delta\bar{v} = \frac{q}{mr}P\Big[\frac{d}{dt}\bar{p}(t=\infty) - \frac{d}{dt}\bar{p}(t=-\infty)\Big], \tag{4.9}$$

where $P$ is the projection of the dipole moment. Second, as will be mentioned in the upcoming section, memory effects can be understood in terms of two contributions. For the electromagnetic memory, these two are categorized as the *ordinary* and *null* velocity kicks. The latter is found to be the proportional to the charge radiated to infinity per unit solid angle. For further information, refer to [34].

## 4.2 Gravitational memory effect

Our approach to the gravitational memory effect concept is based on the developments made by Strominger on [19]. As in the previous section, we can picture the gravitational memory effect as a kick on the gravitational initial data due to the propagation of radiation through spacetime. As explained by Bieri [34][35], original Christodolou memory effect [5] can be understood as the force exerted by the propagating gravitational waves on the motion of the detector, leading to the relative acceleration between its constituents. The integration of this acceleration effect over all time, leads to the residual velocity of the detector. In principle this shall be forbidden due to the consideration of the energy conservation law, leading to non-existent radiating modes either on the $|\mathrm{in}\rangle$ or $|\mathrm{out}\rangle$ states in the scattering process. This leads to the classification of two different gravitational memory effects, the linear and non-linear, focusing ourselves only on the former.

From a formal point of view, everything starts with the classical modes of the asymptotically flat metric, Bondi mass $m_B$ and initial data tensor $C_{z\bar{z}}$. They are constrained by Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{4.10}$$

through

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu}, \tag{4.11}$$

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T^M_{uu}], \tag{4.12}$$

where $N_{zz} = \partial_u C_{zz}$, and $T_{uu}$ corresponds to the full stress-energy tensor, while $T^M_{uu}$ contains only the contribution of the matter content in spacetime. The action of supertranslations over the gravitational field degrees of freedom then are given by

$$\begin{aligned} \mathcal{L}_f m_B &= f\partial_u m_B, \\ \mathcal{L}_f C_{zz} &= f N_{zz} - 2D_z^2 f, \end{aligned} \tag{4.13}$$

whose algebra generators are given by

$$\zeta_f = f\partial_u + D^z D_z f \partial_r - \frac{1}{r}(D^{\bar{z}} f \partial_{\bar{z}} + D^z f \partial_z). \tag{4.14}$$

The transformation that our system has been subject to is obtained through the calculation of the difference between initial and final data after a super translation transformation is applied. Defining the change as

$$\Delta C_{zz} \equiv C_{zz}(u_f) - C_{zz}(u_i), \tag{4.15}$$

$$\Delta m_B \equiv M_f - M_i. \tag{4.16}$$

where $M_i$ and $M_f$ are the Bondi mass $m_B$ measured either on the initial or final state. Integrating the Einstein equations (4.11) with respect to $u$, we obtain:

$$D_z^2 \Delta C^{zz} = 2\int_{u_i}^{u_f} du\; T_{uu} + 2\Delta m_B. \tag{4.17}$$

Finally, it leads to the net variation

$$\Delta C^{zz}(z,\bar{z}) = 2\int d^2 z' \gamma_{z'\bar{z}'} G(z,\bar{z};z',\bar{z}')\left(\int_{u_i}^{u_f} du\; T_{uu}(z',\bar{z}') + \Delta m_B\right). \tag{4.18}$$

The Green function $G(z,\bar{z};z',\bar{z}')$ encodes [19] the effect of the impulse of radiation carried over the detectors, and is expressed as

$$G(z,\bar{z};z',\bar{z}') = -\tfrac{1}{\pi}\sin^2\tfrac{\Theta}{2}\log\left[\sin^2\tfrac{\Theta}{2}\right], \qquad \text{with} \qquad \sin^2\tfrac{\Theta(z,z')}{2} \equiv \tfrac{|z-z'|^2}{(1+z\bar{z}')(1+z'\bar{z})}. \tag{4.19}$$

This effect introduces the continuity conditions of CK space for action of radiative modes in a black hole merger, for example. The consequence of the effect generated is a non-trivial vacuum transition, which is result of the application of a supertranslation depicted above. For completeness, is important to stress that the overall global translation [19],

$$f_{global} = c_0 + \frac{c_1(1 - z\bar{z}) + c_2 z + c_3\bar{z} + \bar{c}_3 z}{1 + z\bar{z}}, \tag{4.20}$$

does not alter the radiative effect just discussed. A gravitational memory effect can be determined as the difference between gravitational wave detector positions before and after the interaction of gravitational radiation positions determined through their approximation as inertial observers, subject to motion along geodesics described in terms of geodesic equation of motion

$$\partial_s^2 X^\mu_{geo}(s) + \Gamma^\mu_{\nu\lambda}\partial_s X^\nu_{geo}(s) X^\lambda_{geo}(s) = 0, \tag{4.21}$$

where $X^\mu_{geo}(s)$ is the four vector describing the worldline of the inertial detector evolving along a geodesic. The distance change between two detectors is determined as we locate them at initial positions $(r_0, z_0, \bar{z}_0)$,

$$X^\mu_{BMS}(s) = (r_0, z_0, \bar{z}_0). \tag{4.22}$$

The latter can be approximated to be an inertial observer as far as $u < r_0$, then at leading order

$$X^{u,r}_{BMS}(s) = X^{u,r}_{geo} + \mathcal{O}(r_0^{-1}), \tag{4.23}$$

$$X^{z}_{BMS}(s) = X^{z}_{geo}(s) + \mathcal{O}(1/r_0^2). \tag{4.24}$$

Our original distance is given by

$$L = \frac{2r_0|\delta z|}{1 + z_1\bar{z}_1}. \tag{4.25}$$

After the radiation pulse

$$\Delta L = \frac{r_0}{2L}\Delta C_{zz}(z_1, \bar{z}_1)\delta z^2 + c.c., \tag{4.26}$$

where $\delta z \equiv z_1 - z_2$. We interpret then $\delta z$ as the distance between detectors, whose variation is considered up to $\frac{1}{r_0}$ [19]. Higher order contributions would be relevant for the consideration of other effects beyond the scope of our current study. Having given an overview of the effect, is necessary to determine how to measure this experimentally. In a experimental setup, like the one offered by LIGO observatory, this can be measured as relative distance change between detectors as well as relative change between clock time measurements attached to them due to supertranslation transformations.

So far we have connected the *BMS symmetry* for vacuum with the *gravitational memory effect*. However this is not a clear signal for identification with the *soft graviton* *theorem*. The proper way of developing this map is through the study of "bursts with memory" (BWM), a particular type of gravitational wave worked out by Braginsky and Thorne [36].

They define BWM's as gravitational perturbations characterized by the tranverse-traceless linearized metric $h_{ij}^{TT}$, such that they raise up from zero perturbation, then have a finite time-lapse of oscillation $\Delta t$, and finally settling down to a non-zero perturbation value for the metric. This non-zero left over corresponds to the memory effect generated by the radiation propagation. On the other hand, from an astrophysical point of view, the emission of these signals shall correspond to a system composed by two or more objects such as star binary systems, black hole mergers, etc.

To perform the derivation of the connection between *gravitational memory* and *soft graviton theorem*, first we define $h_{ij}^{TT}$, the transverse-trace-less linear gravitational field perturbation. From [37] we know that transverse-trace-less ($TT$) gauge is mathematically defined as the following set of constraints

$$h_{\mu 0} = 0, \tag{4.27}$$

$$\partial_j h_{kj} = 0, \tag{4.28}$$

$$h_{kk} = 0. \tag{4.29}$$

As consequence, it can be proven that only pure waves can be susceptible to the application of $TT$ *gauge* and satisfy Einstein's equations

$$\Box h_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{4.30}$$

Under these constraints, the geometrical imprint of burst gravitational waves can be expressed in momentum space as

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{r_0}\sqrt{\frac{G}{2\pi}}\Big[\sum_{j=1}^{n}\frac{p'_{j,\mu}p'_{j,\nu}}{\omega k\cdot p'_j} - \sum_{j=1}^{m}\frac{p_{j,\mu}p_{j,\nu}}{\omega k\cdot p_j}\Big]^{TT}. \tag{4.31}$$

On the other hand, the soft limit for $\Delta h_{\mu\nu}^{TT}(\omega,\vec{k})$ reached when $\omega \to 0$, so that

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{4\pi i r_0}\lim_{\omega\to 0}\Big[-i\omega h_{\mu\nu}^{TT}(\omega,\vec{k})\Big]. \tag{4.32}$$

From (4.31) we identify the presence of the soft graviton factor, originally introduced from the scattering process

$$\lim_{\omega\to 0}\mathcal{A}_{m+n+1}(\omega k,\epsilon_{\mu\nu}) = \sqrt{8\pi G}S_{\mu\nu}\epsilon^{\mu\nu}\mathcal{A}_{m+n}, \tag{4.33}$$

where

$$S_{\mu\nu} = \sum_{j=1}^{m}\frac{p_{j,\mu}p_{j,\nu}}{\omega k\cdot p_j} - \sum_{j=1}^{n}\frac{p'_{j,\mu}p'_{j,\nu}}{\omega k\cdot p'_j}, \tag{4.34}$$

finally, we get the proper mapping between the two infrared phenomena [19]

$$\lim_{\omega\to 0} \omega h^{TT}_{\mu\nu}(\omega, k)\epsilon_{\mu\nu} = \sqrt{8\pi G}\epsilon^{\mu\nu} \lim_{\omega\to 0} \omega S_{\mu\nu}(\omega k). \tag{4.35}$$

In analogy to the electromagnetic case, we obtain a correspondence between the gravitational memory effect and soft graviton theorem on momentum space.

The possible measurement of memory effect contribution for gravitational waves is a possibility contemplated by different studies. Two possible candidates for the generation of such effect are the supermassive black hole mergers and pulsar time arrays if considered the use of orbital detectors such as LISA. A numerical estimate on its contribution towards the wave profile is hard to determine, due to the fact that is amplitude is not weaker in comparison to other contributions, see for example Favata [38] or van Haasteren [39].

## 4.3 The infrared triangle for gauge fields and gravity

As expressed by Pasterski, Strominger, Mitra [23][33][40] a correspondence exists between asymptotic symmeries, soft theorems and memory effects, all three phenomena corresponding to infrared behavior of gauge and gravity theories as presented on this dissertation. It can be better illustrated in the Figure 2.

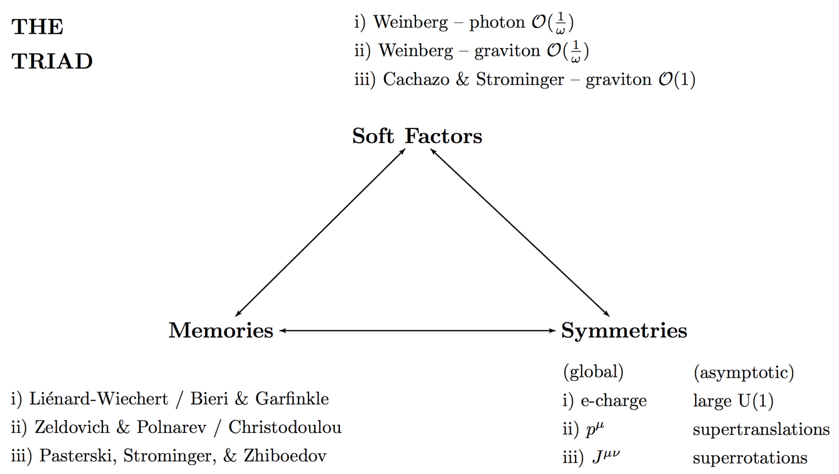
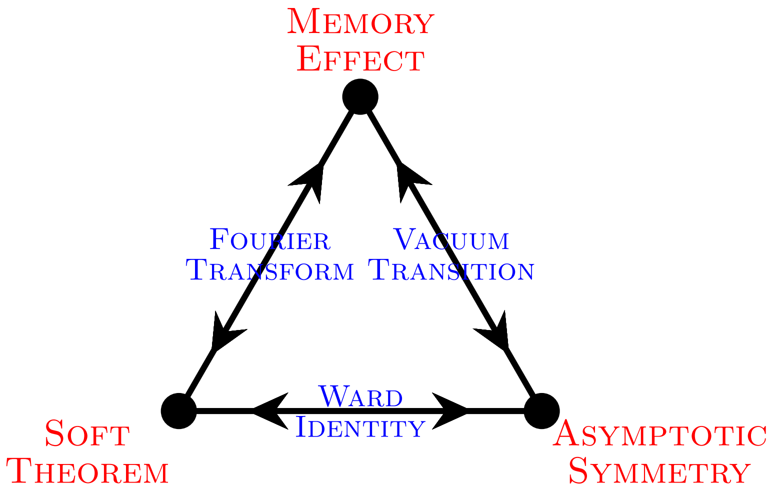



Figure 2 – Infrared triangle big picture. On these graph are explicitly shown the pionering developments on the subject. Extracted from Pasterski's [41]

Prior to the developments explored and cited along our work, there was no clear evidence of the existence of identification between infrared phenomena on gauge and gravity theories. Causal behavior of physical systems in asymptotically flat spacetimes leads to the requirement of continuity conditions around spacelike infinity, leading to the identification of gauge symmetries of the system. The latter guarantee the existence of some sort of spacetime symmetries, the well-known BMS symmetries. In the end, these conditions are responsible for constraining the production of zero-energy radiative modes of either gauge or gravity fields, the soft theorems governing scattering processes. Meanwhile, the application of symmetry transformations leads to the transition between vacuums in the correspondent theory, which due to degeneracy, gives rise to the memory effect. This can be summarized as presented in the following diagram:




Finally we want to make some comments about further developments of the infrared triangle. The development of this infrared correspondence is grounded on the Ashtekar & Hansen [29] protocol for asymptotically flat spacetime, which establishes the boundary conditions to be obeyed by the fields on the null and spacelike infinities. However there has been the inconvenience of needing to define every relationship for past and future nulls separate. In order to improve this and lead to a unified picture of asymptotically flat spacetimes, Krishnan & Pereira [42] redefine this protocol. In addition, all the identifications presented on this dissertation are defined at leading order. To improve this, and determine its universality at subleading terms, for both gauge fields and gravity, we can consider the seminal works of Strominger [19], Pasterski [43], Cachazo [44].

# 5 Conclusions

The main objective of the present work is to make a pedagogical presentation of the concept of Infrared Divergences for Gauge and Gravity theories. Being a frontier research topic under constant development, our presentation expects to serve as a bridge between the basic concepts behind the phenomena under study and the abundant literature published lately.

Soft theorems [1][2] are built as constraints on scattering amplitudes for the creation of soft particles either on the $|\text{in}\rangle$ or $|\text{out}\rangle$ states. Their pole structure is modified because of the extra interaction terms on the asymptotic states due to these new soft particles. Then this leads to the appearance of divergences, which end up being cancelled out due to the contribution of virtual soft modes as well. On the other hand, asymptotic symmetries set the arena for causally consistent physical processes on asymptotically flat spacetimes. In particular, it is the continuity as well the analytical identifications between past and future null infinities that imposes the existence of large gauge symmetries for field theories, and a set of spacetime isometries, the BMS symmetries, "dressing up" the Hilbert space states. On third place, the memory effect produces a change on the field configuration due to the interaction with infrared modes of radiation, generating a non-trivial vacuum transition which is detected as impulses, relative distance changes, etc. Miraculously they are all connected by employing different kinds of transformations.

These transformations are responsible for linking the mathematical expressions describing these phenomena on both the coordinate and momentum spaces. Indeed what we can claim is that the infrared triangle provide us with a new example of how symmetries of gauge and gravitational systems explain around the common framework of asymptotic symmetries. This ultimately explains the existence of conservation laws and vacuum characterization on the same energy scale, the infrared scale.

Initial development of the infrared triangle has led to appearance of different research lines. Of particular interest to us are the advent of *Celestial Holography* program and the study of the *Black Hole Information Paradox*. Celestial Holography in particular, is a new realization of the Holographic Principle proposed by Susskind and others [45] (see references therein) for the case of asymptotically flat spacetimes on the bulk theory. Being the Anti de-Sitter/Conformal Field Theory (AdS/CFT) correspondence by Maldacena [46], and then expanded by Witten et. al. [47] [48], this is a new paradigmatic advance on the description of nature in holographic terms. It aims to describe the dynamics of a 2-dimensional CFT by its 4-dimensional gravitational dual at null infinities $\mathcal{I}^{\pm}$. Then the programs objective in simple words would be to link a bulk gravitational theory on 4-dimensional to an 2-dimensional conformal field theory living on the so called *celestial sphere*, which was not the case of Maldacena and Witten's original proposal. For a more detailed study, please refer to [49, 50, 51, 52, 53, 54, 55, 56].

Beyond, the infrared triangle development has found fertile soil on its possible application for solving the Black Hole (BH) information paradox. The problem can be phrased as follows: given a physical system thrown into a black hole, is its information completely lost from the entire universe or could be recovered in some way? Considering both the system falling into the BH, as well as the BH itself our current understanding of this situation tells us that the total amount of information can be quantified in terms of the overall entropy of the full system. Due to our knowledge of BH physics, its behavior is constrained by three conserved quantities known as the "Black hole hair": Mass, Charge and Angular momentum. In an scenario of BH evaporation, the process is analytically determined in terms of those parameters. However, considering the event horizon (EH) an asymptotically flat region, is possible that there could information "trapped" in that region which would correspond to new hair, corresponding to the conserved quantities associated to the asymptotic symmetries of the region. This has been extensively discussed in the literature, as can be seen in [57, 58, 59, 60, 61, 62, 63, 64], but there is no agreement yet about its role on solving or even describing more efficiently the problem.

Any of the previous research lines would be of interest for future projects, being both linked to a better description of how gravitational theories can be understood at small scales.

# A Penrose diagrams

Let us start with the line element We introduce our first change of coordinates, whose objective is to rotate the lightcone

$$p = t - r, \tag{A.1}$$

$$q = t + r, \tag{A.2}$$

Besides $q - p = 2r \rightarrow r = \frac{q-p}{2}$ with the range $q - p \geq 0, r \geq 0$. Second, we introduce the compactification by employing hyperbolic functions

$$u = \tanh q, \tag{A.3}$$

$$v = \tanh p. \tag{A.4}$$

And then, the line element becomes:

$$ds^2 = -\frac{dudv}{(1-u^2)(1-v^2)}. \tag{A.5}$$

A second rotation

$$u = T + X, \tag{A.6}$$

$$v = T - X, \tag{A.7}$$

is applied to finally obtain:

$$ds^2 = \frac{-dT^2 + dX^2}{(1-(T+X)^2)(1-(T-X)^2)}. \tag{A.8}$$

Extracting the conformal factor $\frac{1}{(1-(T+X)^2)(1-(T-X)^2)}$, we recover a non-physical metric given by

$$ds^2 = -dT^2 + dX^2. \tag{A.9}$$

The Penrose diagram, which contains the particle propagating in Minkowski spacetime, is presented in Figure 3. In terms of the new variables, the conformal infinities of Minkowski spacetime are

- $i^- \rightarrow$ past timelike infinity $R = 0, T = -\pi$
- $i^+ \rightarrow$ future timelike infinity $R = 0, T = \pi$
- $\mathcal{I}^- \rightarrow$ past lightlike infinity $R \in (0, \pi), T = -\pi + R$
- $\mathcal{I}^+ \rightarrow$ future lightlike infinity $R \in (0, \pi), T = \pi - R$
- $i^0 \rightarrow$ spatial infinity $R = \pi, T = 0$

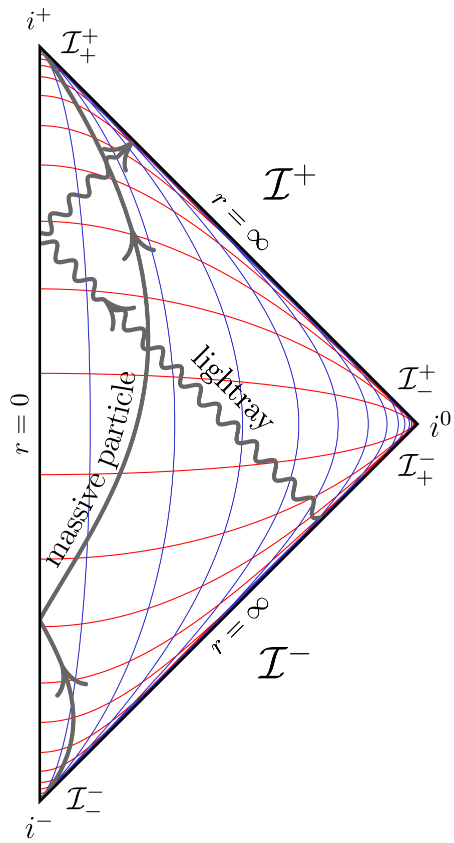



Figure 3 – Penrose diagram of Minkowski spacetime. Extracted from *Strominger, A. (2018). Lectures on the infrared structure of gravity and gauge theory. Princeton University Press.*

# B Noether's theorem and Local Gauge Invariance

The association between symmetries and conservation laws in physics is due to Noether's theorem. For the case of our present study, is of our interest in particular Local Gauge invariance. Our aim then will be to present the derivation of the general expression for Noether's conserved current and charge. Our main reference will be [65]. Given a general lagrangian density $\mathcal{L}(x) = \mathcal{L}\Big(\phi(x), \frac{\partial\phi(x)}{\partial x_\mu}\Big)$, so then the action is defined as

$$S = \int d^4x \mathcal{L}(x). \tag{B.1}$$

Our interest is to introduce the following coordinate and field variations

$$x'_\mu = x_\mu + \delta x_\mu, \tag{B.2}$$

$$\phi'(x') = \phi(x) + \delta\phi(x), \tag{B.3}$$

$$\phi'(x) = \phi(x) + \bar{\delta}\phi(x), \tag{B.4}$$

where the latter is named sometimes as *modified field variation*. Both regular and modified field variations happen to be related as

$$\begin{aligned}
\bar{\delta}\phi(x) &= \phi'(x) - \phi(x) \\
&= \phi'(x) - \phi'(x') + \phi'(x') - \phi(x) \\
&= (\phi'(x') - \phi(x)) - (\phi'(x') - \phi'(x)) \\
&= \delta\phi(x) - (\phi'(x') - \phi'(x)) \\
&= \delta\phi(x) - \Big(\phi'(x) + \frac{\partial\phi'(x)}{\partial x_\mu}\delta x_\mu - \phi'(x)\Big) \\
&= \delta\phi(x) - \frac{\partial\phi'(x)}{\partial x_\mu}\delta x_\mu \\
&= \delta\phi(x) - \frac{\partial\phi(x)}{\partial x_\mu}\delta x_\mu.
\end{aligned} \tag{B.5}$$

Important properties associated to both field variations is their behavior under derivation. First the modified variation

$$\frac{\partial}{\partial x_\mu}\bar{\delta}\phi(x) = \bar{\delta}\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big). \tag{B.6}$$

Second, the regular variation

$$\begin{aligned}
\frac{\partial}{\partial x_\mu}\delta\phi(x) &= \frac{\partial}{\partial x_\mu}\Big(\phi'(x') - \phi(x)\Big)\\
&= \frac{\partial\phi'(x')}{\partial x_\mu} - \frac{\partial\phi(x)}{\partial x_\mu} + \frac{\partial\phi'(x')}{\partial x'_\mu} - \frac{\partial\phi'(x')}{\partial x'_\mu}\\
&= \Big(\frac{\partial\phi'(x')}{\partial x'_\mu} - \frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial\phi'(x')}{\partial x_\mu} - \frac{\partial\phi'(x')}{\partial x'_\mu}\\
&= \delta\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial\phi'(x')}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x_\mu} - \frac{\partial\phi'(x')}{\partial x'_\mu}\\
&= \delta\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial\phi'(x')}{\partial x'^\nu}\Big(g^{\nu\mu} + \frac{\partial\delta x'^\nu}{\partial x_\mu}\Big) - \frac{\partial\phi'(x')}{\partial x'_\mu}\\
&= \delta\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial\phi'(x')}{\partial x'_\mu} + \frac{\partial\phi'(x')}{\partial x'^\nu}\frac{\partial\delta x'^\nu}{\partial x_\mu} - \frac{\partial\phi'(x')}{\partial x'_\mu}\\
&= \delta\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial\phi'(x')}{\partial x'^\nu}\frac{\partial\delta x'^\nu}{\partial x_\mu}\\
&= \delta\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial\phi(x)}{\partial x^\nu}\frac{\partial\delta x^\nu}{\partial x_\mu}.
\end{aligned} \tag{B.7}$$

With all these ingredientes, finally we can address the variation of the action under all the set of transformations (B.2). The action variation is given by

$$\begin{aligned}
\delta S &= \int d^4x'\mathcal{L}'(x') - \int d^4x\mathcal{L}(x)\\
&= \int\Big(1 + \frac{\partial\delta x^\mu}{\partial x^\mu}\Big)d^4x\Big(\delta\mathcal{L}(x) + \mathcal{L}(x)\Big) - \int d^4x\mathcal{L}(x)\\
&= \int d^4x\delta\mathcal{L}(x) + \int d^4x\frac{\partial\delta x^\mu}{\partial x^\mu}\mathcal{L}(x) + \mathcal{O}(\delta^2)\\
&= \int d^4x\Big(\bar\delta\mathcal{L}(x) + \frac{\partial\mathcal{L}(x)}{\partial x^\mu}\delta x^\mu\Big) + \int d^4x\mathcal{L}(x)\frac{\partial\delta x^\mu}{\partial x^\mu}\\
&= \int d^4x\Big[\bar\delta\mathcal{L}(x) + \frac{\partial}{\partial x^\mu}\Big(\mathcal{L}(x)\delta x^\mu\Big)\Big]\\
&= \int d^4x\Big[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)}\bar\delta\phi(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\bar\delta\Big(\frac{\partial\phi(x)}{\partial x_\mu}\Big) + \frac{\partial}{\partial x^\mu}\Big(\mathcal{L}(x)\delta x^\mu\Big)\Big]\\
&= \int d^4x\Big[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)}\bar\delta\phi(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\frac{\partial}{\partial x_\mu}\Big(\bar\delta\phi(x)\Big) + \frac{\partial}{\partial x^\mu}\Big(\mathcal{L}(x)\delta x^\mu\Big)\Big]\\
&= \int d^4x\Big[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)}\bar\delta\phi(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\frac{\partial}{\partial x_\mu}\Big(\bar\delta\phi(x)\Big) + \frac{\partial}{\partial x^\mu}\Big(\mathcal{L}(x)\delta x^\mu\Big)\\
&+\frac{\partial}{\partial x_\mu}\Big(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\Big)\bar\delta\phi(x) - \frac{\partial}{\partial x_\mu}\Big(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\Big)\bar\delta\phi(x)\Big]\\
&= \int d^4x\Big[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)} - \frac{\partial}{\partial x_\mu}\Big(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\Big)\Big]\bar\delta\phi(x)\\
&+ \int d^4x\Big[\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\frac{\partial}{\partial x_\mu}\Big(\bar\delta\phi(x)\Big) + \frac{\partial}{\partial x_\mu}\Big(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\Big)\bar\delta\phi(x) + \frac{\partial}{\partial x^\mu}\Big(\mathcal{L}(x)\delta x^\mu\Big)\Big]\\
&= \int d^4x\Big[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)} - \frac{\partial}{\partial x_\mu}\Big(\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\Big)\Big]\bar\delta\phi(x) + \int d^4x\frac{\partial}{\partial x_\mu}\Big[\frac{\partial\mathcal{L}(x)}{\partial(\partial^\mu\phi)}\bar\delta\phi(x) + \mathcal{L}(x)\delta x^\mu\Big]
\end{aligned} \tag{B.8}$$

Due to *Euler-Lagrange eqs.*, in order to make vanish the variation of the action requires the second integral to be zero. In consequence, this condition can be addressed as

$$\partial^\mu f_\mu = 0 \tag{B.9}$$

where we identify as the *conserved current* $f_\mu$, defined as

$$\begin{aligned} f_\mu &= \frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \phi)} \bar{\delta}\phi(x) + \mathcal{L}(x)\delta x^\mu \\ &= \frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \phi)} \delta\phi(x) - \left[\frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \phi)} \frac{\partial \phi}{\partial x^\nu} - g_{\mu\nu}\mathcal{L}(x)\right]\delta x^\nu. \end{aligned} \tag{B.10}$$

Complementary, the *conserved charge* associated to it is defined as

$$Q = \int d^3x f_0(x) \tag{B.11}$$

In particular, we consider the case of local gauge transformations. Those are defined as

$$A'_\mu = A_\mu + \partial_\mu \Lambda(x), \tag{B.12}$$

that applied to *electromagnetic field lagrangian density* $\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu$, leads to

$$\begin{aligned} \mathcal{L}'(x) &= -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + j_\mu A'^\mu \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu(A^\mu + \partial^\mu\Lambda) \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu + j_\mu\partial^\mu\Lambda \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu + \left[\partial^\mu(j_\mu\Lambda) - \Lambda\partial^\mu j_\mu\right] \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu + \partial^\mu(j_\mu\Lambda) \\ &= \mathcal{L}(x) + \partial^\mu(j_\mu\Lambda), \end{aligned} \tag{B.13}$$

being invariant up to a divergence term, and considering the condition $\partial_\mu j^\mu = 0$.

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