Universidade Federal de Itajubá
Programa de Pós–Graduação em Matemática

DINÂMICA NÃO LINEAR E APLICAÇÕES EM MODELOS ECONÔMICOS

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Orientador: Prof. Dr. Luis Fernando de Osório Mello

Itajubá, fevereiro de 2016

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Dedico este trabalho aos meus amados pais

Amaury e Rosy pois, sem vocês eu não estaria aqui.

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"Vamos viver nossos sonhos,
temos tão pouco tempo."

Chorão

Resumo

Estudos de sistemas dinâmicos não lineares têm sido amplamente aplicados para análise econômica atualmente. O objetivo é compreender fenômenos econômicos através de ferramentas da dinâmica não linear, tais como estabilidade, ciclos limites e bifurcações de Hopf procurando situações que não podem ser facilmente modeladas por outras ferramentas.

Palavras-chave: Estabilidade, Ciclo limite, Bifurcação de Hopf, Modelo Econômico.

Abstract

Studies of nonlinear dynamical systems have been widely applied to current economic analysis. The aim is to comprehend economic phenomena through the nonlinear dynamics tools, such as stability, limit cycles and Hopf bifurcations, looking for situations that can not easily be modeled by other tools.

Keywords: Stability, Limit Cycle, Hopf Bifurcation, Economic Model.

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Introdução

Estudos de sistemas dinâmicos não lineares têm sido amplamente aplicados para análise econômica atualmente. A modelagem não linear pode ser útil para investigar o comportamento e encontrar soluções para os mais sofisticados problemas econômicos que será o foco deste trabalho.

Mostraremos como a teoria da bifurcação de Hopf pode ser aplicada para entender esses modelos, e utilizaremos um método para verificar as condições de Hopf de não degenerescência e transversalidade em sistemas n–dimensionais, garantindo a existência de ciclos limites.

O nosso objetivo é ter uma melhor compreensão de fenômenos econômicos tais como o crescimento econômico, ciclos, análise de mercado, entre outros, a fim de prevenir e controlar seu comportamento.

Em uma família de equações diferenciais, a bifurcação de Hopf normalmente ocorre quando um par de autovalores complexos conjugados do campo linearizado em um ponto de equilíbrio se torna puramente imaginário. Isto implica que uma bifurcação de Hopf só pode ocorrer em sistemas de duas dimensões ou mais.

O primeiro capítulo mostrará alguns dos principais teoremas e definições que serão usados nos próximos capítulos.

O Capítulo 2 consiste no estudo de um sistema tridimensional, dado por

$$\begin{cases} \dot{x} = \nu x - y^2, \\ \dot{y} = \mu(z - y), \\ \dot{z} = ay - bz + xy, \end{cases}$$

no qual analisaremos os valores de parâmetros necessários para a existência de uma bifurcação de Hopf, fazendo uma simulação numérica com tais parâmetros que pode ser encontrado em [1]. Os principais teoremas deste capítulo são o teorema da Bifurcação de Hopf genérica, o Teorema da Bifurcação de Hopf degenerada e a Condição de Transversalidade.

O tema principal deste trabalho encontra-se no Capítulo 3, no qual estudaremos modelos acoplados que podem ser encontrados em [10] e [11]. Consideraremos o sistema com duas equações diferenciais ordinárias

$$\begin{cases} \dot{x} = f_1(x, y, \beta), \\ \dot{y} = f_2(x, y, \beta) \end{cases} \quad \beta \in \mathbb{R}, \;\; x, y \in \mathbb{R}. \tag{1}$$

Com o intuito de estudar a intersecção entre os processos de evolução são utilizados os sistemas acoplados como em [5] e [6]. Considere dois sistemas idênticos ao sistema (1), não simétricos e linearmente acoplados

$$\begin{cases} \dot{x} = f_1(x, y, \beta) + c_1(x - z), \\ \dot{y} = f_2(x, y, \beta), \\ \dot{z} = f_1(z, w, \beta) + c_2(z - x), \\ \dot{w} = f_2(z, w, \beta). \end{cases} \quad c_1, c_2 \in \mathbb{R} \tag{2}$$

O sistema de acoplamento acima é útil para modelar a evolução de dois fenômenos semelhantes que interagem de forma linear através de uma de suas variáveis de estado. Tal acoplamento pode ser utilizado em uma vasta variedade de sistemas que são de interesses químicos, físicos, biológicos ou econômicos, podendo ser encontrados em [2] e [18].

O modelo econômico estudado será um sistema bem conhecido da economia que pode ser encontrado também em [3], [4] e [16],

$$\begin{cases} \dfrac{dx}{d\tau} = k - \alpha x y^2 + \beta y, \\ \\ \dfrac{dy}{d\tau} = \alpha x y^2 - \delta y, \end{cases} \tag{3}$$

o qual modela a dinâmica do número de usuários de uma marca de acordo com a publicidade.

A partir deste sistema, analisaremos as condições necessárias para ocorrer uma bifurcação de Hopf e ver como esta afeta economicamente o modelo.

Os principais teoremas e resultados deste capítulo são os seguintes: Teorema da Bifurcação de Hopf genérica, Teorema da Bifurcação de Hopf degenerada, a Condição de Transversalidade e o Método da projeção.

Por fim, temos uma breve conclusão do que tiramos de todo o trabalho.

Capítulo 1

A bifurcação de Hopf

Neste capítulo apresentaremos algumas definições e resultados básicos de Equações Diferenciais Ordinárias (EDO's), com o foco na bifurcação de Hopf, que serão necessários para os próximos capítulos. As definições e o método da projeção que serão apresentados foram baseados, e podem ser encontrados em [7] e [13]. Estudaremos fenômenos que podem ser modelados por EDO's da forma

$$\dot{x} = \frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n \tag{1.1}$$

sendo que f é uma função suave, ou seja, a classe de diferenciabilidade é suficientemente grande (f é suave $\iff$ $f \in \mathcal{C}^n$) e com o tempo variando continuamente em $I \subset \mathbb{R}$ não vazio e não degenerado a um ponto.

Definição 1.0.1. *Uma função diferenciável $\varphi : I \subset \mathbb{R} \to \mathbb{R}^n$ tal que $\dot{\varphi}(t) = f(\varphi(t))$, para todo $t \in I$ é dita uma **solução** do sistema (1.1).*

Quando f é uma função linear de x, a solução geral do sistema (1.1) é facilmente encontrada em qualquer instante do tempo t, podendo esta ser vista melhor em [8]. No entanto, na maioria das vezes em que modelamos um fenômeno, f é não linear. Linearizando o sistema (1.1) determinamos a matriz Jacobiana $J(x)$, $n \times n$ dada por

$$J(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}.$$

Definição 1.0.2. *Dizemos que $x_0 \in \mathbb{R}^n$ é um* ***ponto de equilíbrio*** *do sistema (1.1) se $f(x_0) = 0$.*

Definição 1.0.3. *Um ponto de equilíbrio x_0 do sistema (1.1) é* ***estável*** *quando para toda vizinhança U_1 de x_0, existir uma vizinhança U_2 de x_0 tal que toda solução $\varphi(t)$ de (1.1), com $\varphi(0) \in U_2$, está definida em U_1, para todo $t \geq 0$. Caso contrário dizemos que x_0 é* ***instável****. Se um equilíbrio for estável e, além disso, $\lim\limits_{t \to +\infty} \varphi(t) = x_0$, então x_0 é* ***assintoticamente estável****.*

Definição 1.0.4. *Um ponto de equilíbrio x_0 do sistema (1.1) é chamado* ***hiperbólico*** *se todos os autovalores de $J(x_0)$ têm partes reais diferentes de zero, onde $J(x_0)$ representa a matriz Jacobiana de (1.1) no ponto x_0. Se a parte real de algum autovalor for nula, o equilíbrio será dito* ***não hiperbólico*** *ou* ***degenerado****.*

Definição 1.0.5. *Um ponto de equilíbrio hiperbólico x_0 do sistema (1.1) é chamado de* ***atrator*** *se todos os autovalores de $J(x_0)$ tiverem partes reais negativas, e* ***repulsor*** *se todos os autovalores de $J(x_0)$ tiverem partes reais positivas.*

Definição 1.0.6. *Um ponto de equilíbrio hiperbólico x_0 do sistema (1.1) é chamado de* ***sela-hiperbólica*** *se todos os autovalores de $J(x_0)$ tiverem partes reais diferentes de zero e pelo menos dois deles possuírem partes reais com sinais opostos.*

Considere os seguintes sistemas de EDO's dependendo do parâmetro $\xi \in \mathbb{R}$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \pm (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.2}$$

O ponto $(x_1, x_2) = (0, 0)$, para qualquer $\xi \in \mathbb{R}$, é um equilíbrio desse sistema sendo

$$A(0,0) = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix}$$

a matriz Jacobiana deste sistema, que possui autovalores $\lambda_1 = \xi + i$ e $\lambda_2 = \xi - i$. Introduzindo a variável complexa $z = x_1 + ix_2$ e usando

$$\dot{x_1} = \xi x_1 - x_2 \pm x_1(x_1^2 + x_2^2)$$

e

$$\dot{x_2} = x_1 + \xi x_2 \pm x_2(x_1^2 + x_2^2),$$

temos

$$\begin{aligned} \dot{z} &= \dot{x_1} + i\dot{x_2} \\ &= \xi(x_1 + ix_2) + i(x_1 + ix_2) \pm (x_1 + ix_2)(x_1^2 + x_2^2). \end{aligned}$$

Assim, obtemos o sistema (1.2) na sua forma complexa

$$\dot{z} = (\xi + i)z \pm z\,|z|^2\,. \tag{1.3}$$

Para a representação $z = \rho e^{i\theta}$, obtemos

$$\dot{z} = \dot{\rho} e^{i\theta} + \rho i \dot{\theta} e^{i\theta}$$

e, portanto,

$$\dot{\rho} e^{i\theta} + \rho i \dot{\theta} e^{i\theta} = \rho e^{i\theta}(\xi + i \pm \rho^2).$$

Assim podemos escrever a equação (1.3) na sua forma polar

$$\begin{cases} \dot{\rho} = \rho(\xi \pm \rho^2), \\ \dot{\theta} = 1. \end{cases} \tag{1.4}$$

Da primeira equação de (1.4), podemos perceber que $\rho = 0$ é um ponto de equilíbrio para qualquer valor de ξ (só faz sentido para $\rho \geq 0$). Dependendo do sinal do termo cúbico em (1.4), para determinados valores de ξ, outro ponto de equilíbrio surgirá. Trabalharemos, por exemplo, com o sinal negativo do termo cúbico de (1.4), ou seja, o sistema

$$\begin{cases} \dot{\rho} = \rho(\xi - \rho^2), \\ \dot{\theta} = 1. \end{cases} \tag{1.5}$$

Sendo assim, para $\xi > 0$, $\rho(\xi) = \sqrt{\xi}$ é um ponto de equilíbrio que descreve uma órbita periódica circular com velocidade constante. Como foi observado acima, $\rho = 0$ é sempre um equilíbrio, que neste caso, é um foco atrator se $\xi < 0$, um foco repulsor se $\xi > 0$ e um foco **atrator fraco** (um equilíbrio não linear e topologicamente equivalente ao foco atrator. Veja Definição 1.0.8) se $\xi = 0$. Para $\xi > 0$, a origem está no interior de uma região limitada por uma órbita isolada fechada (ciclo limite), que é única e atratora. Neste caso, o ciclo limite é uma circunferência de raio $\rho(\xi) = \sqrt{\xi}$. Qualquer órbita externa ou interna a este ciclo, exceto a origem, tende para ele quando $t \to \infty$, vide Figura 1.1.

A este fenômeno de geração de uma órbita periódica e a mudança de estabilidade do foco a partir de uma pertubação no parâmetro ξ chamamos de **bifurcação de Hopf** ou **bifurcação de Andronov-Hopf**.

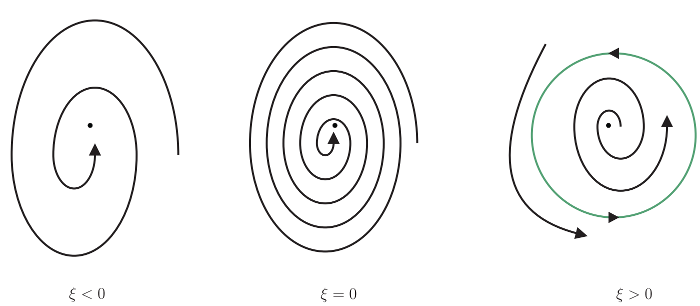

Figura 1.1: Retratos de fase do modelo (1.5) com uma bifurcação de Hopf.

Para o sinal positivo do termo cúbico em (1.4), obtemos o sistema

$$\begin{cases} \dot{\rho} = \rho(\xi + \rho^2), \\ \dot{\theta} = 1, \end{cases} \tag{1.6}$$

podendo ser analisado da mesma maneira. Teremos uma bifurcação de Hopf para $\xi = 0$, porém, ao contrário do sistema (1.5), o ciclo limite surgirá para $\xi < 0$ e é repulsor. Para valores de $\xi > 0$ a origem é um foco repulsor e não possui ciclo limite, quando $\xi = 0$ será um foco **repulsor fraco** (não linear) e para $\xi < 0$ um foco atrator. Neste caso, o ciclo limite é uma circunferência de raio $\rho(\xi) = \sqrt{-\xi}$. Qualquer órbita externa ou interna a este ciclo, exceto a origem, tende para ele quando $t \to -\infty$.

Definição 1.0.7. *Denominamos de **forma normal da bifurcação de Hopf** o sistema (1.2), ou equivalentemente, (1.3) e (1.4).*

Uma vez definida a forma normal da bifurcação de Hopf, precisaremos da seguinte definição, que será usada na próxima seção, na qual estudaremos as condições que um sistema bidimensional deve cumprir para ser topologicamente equivalente a esta forma normal.

Definição 1.0.8. *Os sistemas*

$$\dot{\mathbf{x}} = f(\mathbf{x}, \xi), \quad \mathbf{x} \in \mathbb{R}^n, \xi \in \mathbb{R}^m, \tag{1.7}$$

$$\dot{\boldsymbol{y}} = f(\boldsymbol{y}, \zeta), \quad \boldsymbol{y} \in \mathbb{R}^n, \zeta \in \mathbb{R}^m, \tag{1.8}$$

*são ditos **localmente topologicamente equivalentes** em torno da origem se existir uma aplicação $(\mathbf{x}, \xi) \longmapsto (h_\xi(\mathbf{x}), k(\xi))$, definida na vizinhança $V = U_0 \times V_0$ de $(\mathbf{x}, \xi) = (0, 0)$, contida em $\mathbb{R}^n \times \mathbb{R}^m$, satisfazendo:*

(i) *$k : \mathbb{R}^m \to \mathbb{R}^m$ é um homeomorfismo definido em V_0;*

(ii) *$h_\xi : \mathbb{R}^n \to \mathbb{R}^n$ é um homeomorfismo para cada ξ, definido na vizinhança U_0 de $\mathbf{x} = 0$, $h_0(0) = 0$, levando órbitas de (1.7) contidas em U_0 em órbitas de (1.8) em $h_\xi(U_0)$, preservando a direção do tempo.*

1.1 O caso genérico da Bifurcação de Hopf em um sistema bidimensional

Nesta seção veremos um sistema de EDO's bidimensional e estudaremos as condições que devem ser impostas sobre ele, para que o mesmo seja topologicamente equivalente à forma normal da bifurcação de Hopf. Estas condições serão dadas no Teorema 1.1.1.

Observação 1.1.1. *Para os próximos lemas consideraremos o sistema*

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1.9}$$

que, como vimos na seção anterior, representa a forma normal da bifurcação de Hopf cujo sinal dos termos cúbicos é negativo e, consequentemente, apresenta uma órbita periódica atratora. Para o outro sistema

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

os reultados são análogos.

Lema 1.1.1. *O sistema*

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \xi & -1 \\ 1 & \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mathcal{O}(|\mathrm{x}|^4), \tag{1.10}$$

onde $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, $\xi \in \mathbb{R}$ *e* $\mathcal{O}(|\mathrm{x}|^4)$ *representa os termos de ordem 4 e superiores, e dependem suavemente de* ξ*, é localmente topologicamente equivalente em torno da origem ao sistema (1.9).*

Demonstração. A demonstração deste lema será feita em duas partes: A existência e unicidade do ciclo e depois a construção do homeomorfismo.

Parte I. (Existência e unicidade do ciclo).

Fazendo a mudança polar $x_1 = \rho\cos\theta$ e $x_2 = \rho\mathrm{sen}\theta$, no sistema (1.10) obtemos

$$\begin{cases} \dot{\rho}\cos\theta - \rho\dot{\theta}\mathrm{sen}\theta = \rho\xi\cos\theta - \rho\mathrm{sen}\theta - \rho^3\cos\theta + f(\rho,\theta) \\ \dot{\rho}\mathrm{sen}\theta + \rho\dot{\theta}\cos\theta = \rho\cos\theta + \rho\xi\mathrm{sen}\theta - \rho^3\mathrm{sen}\theta + g(\rho,\theta). \end{cases}$$

Multiplicando a primeira e segunda linhas do sistema acima por $\cos\theta$ e $\mathrm{sen}\theta$ respectivamente e, somando os resultados, temos

$$\dot{\rho} = \rho(\xi - \rho^2) + \mathcal{O}(|\rho|^4).$$

Agora, se multiplicarmos a primeira e a segunda linhas do mesmo sistema por $-\mathrm{sen}\theta$ e $\cos\theta$, respectivamente e somar os resultados, temos

$$\dot{\theta} = 1 + \mathcal{O}(|\rho|^3).$$

Logo, o sistema (1.10) nas coordenadas polares (ρ, θ), é dado por

$$\begin{cases} \dot{\rho} = \rho(\xi - \rho^2) + \Phi(\rho,\theta), \\ \dot{\theta} = 1 + \Psi(\rho,\theta), \end{cases} \tag{1.11}$$

onde $\Phi = \mathcal{O}(|\rho|^4)$ e $\Psi = \mathcal{O}(|\rho|^3)$ e dependem de ξ, porém, não indicaremos a dependência a fim de facilitar a notação.

A existência de um ciclo limite em (1.11), por [12], é equivalente a existência de um ponto fixo na **transformação de Poincaré** (ou transformação de primeiro retorno) do mesmo sistema. Sendo assim passaremos a analisar a transformação de Poincaré de tal sistema.

Uma órbita de (1.11) partindo de $(\rho, \theta) = (\rho_0, 0)$ tem a representação dada pela Figura 1.2.

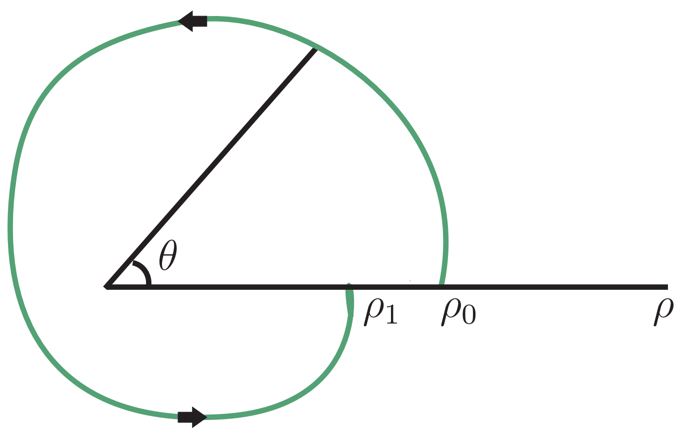

Figura 1.2: Transformação de Poincaré para o estudo da bifurcação de Hopf.

Escrevendo $\rho = \rho(\theta, \rho_0)$ e $\rho_0 = \rho(0, \rho_0)$, pela regra da cadeia, obtemos,

$$\dot{\rho} = \frac{d\rho}{d\theta}\dot{\theta}.$$

Daí, substituindo nesta última equação $\dot{\rho}$ e $\dot{\theta}$ pelas suas expressões dadas por (1.11), temos

$$\frac{d\rho}{d\theta} = \frac{\rho(\xi - \rho^2) + \Phi(\rho, \theta)}{1 + \Psi(\rho, \theta)} = \rho(\xi - \rho^2) + R(\rho, \theta), \tag{1.12}$$

onde $R = \mathcal{O}(|\rho|^4)$. Note que a transformação de (1.11) para (1.12) é equivalente a uma reparametrização do tempo com $\dot{\theta} = 1$, implicando que o tempo de retorno para o semieixo $\theta = 0$ é o mesmo para todas as órbitas que partem desse eixo com $\rho_0 > 0$. Como $\rho(\theta, 0) \equiv 0$, a expansão de Taylor para $\rho(\theta, \rho_0)$, é

$$\rho = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + u_3(\theta)\rho_0^3 + \mathcal{O}(|\rho_0|^4). \tag{1.13}$$

Substituindo (1.13) em (1.12), obtemos

$$\begin{aligned} \frac{d\rho}{d\theta} &= \frac{d}{d\theta}(u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + u_3(\theta)\rho_0^3 + ...) \\ &= (u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + u_3(\theta)\rho_0^3 + ...)[\xi - (u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + u_3(\theta)\rho_0^3 + ...)^2] + R(\rho, \theta) \\ &= u_1(\theta)\rho_0\xi + u_2(\theta)\rho_0^2\xi + u_3(\theta)\rho_0^3\xi - u_1^3(\theta)\rho_0^3 + ... + R(\rho, \theta), \end{aligned}$$

e das correspondentes potências de ρ_0 vêm as seguintes EDO's

$$\frac{du_1}{d\theta} = u_1\xi, \qquad \frac{du_2}{d\theta} = u_2\xi, \qquad \frac{du_3}{d\theta} = u_3\xi - u_1^3.$$

Para obtermos $\rho = \rho_0$ para $\theta = 0$, devemos estabelecer as condições inicias $u_1(0) = 1$, $u_2(0) = u_3(0) = 0$. Assim resolvendo os problemas de valores iniciais resultantes (PVI's), obtemos

$$u_1(\theta) = e^{\xi\theta}, \qquad u_2(\theta) \equiv 0, \qquad u_3(\theta) = e^{\xi\theta}\frac{1 - e^{2\xi\theta}}{2\xi}.$$

Observe que essas expressões são independentes de $R(\rho, \theta)$. Como na expressão de $u_3(2\pi)$ vale a igualdade

$$e^{2\pi\xi}\frac{1 - e^{2(2\pi)\xi}}{2\xi} = \frac{e^{2\pi\xi}}{2\xi}\left[1 - \left(1 + 2\,(2\pi)\,\xi + \frac{(2\,(2\pi))^2\,\xi^2}{2!} + ...\right)\right] = -e^{2\pi\xi}\left[2\pi + \mathcal{O}(\xi)\right],$$

podendo assim concluir que a transformação de retorno $\rho_0 \longmapsto \rho_1 = \rho(2\pi, \rho_0)$ tem a forma

$$\rho_1 = e^{2\pi\xi}\rho_0 - e^{2\pi\xi}[2\pi + \mathcal{O}(\xi)]\rho_0^3 + \mathcal{O}(|\rho_0|^4), \tag{1.14}$$

para todo $R = \mathcal{O}(|\rho_0|^4)$. A transformação (1.14) pode ser facilmente analisada para ρ_0 e $|\xi|$ suficientemente pequenos. Existe uma vizinhança da origem onde essa transformação tem somente o ponto fixo trivial para pequenos valores de $\xi < 0$ e um ponto fixo extra, $\rho_0^* = \sqrt{\xi} + ...$, para pequenos valores de $\xi > 0$, veja Figura 1.3. De fato, sendo $\xi > 0$, escrevendo a transformação (1.14) na forma

$$\rho_1 = \rho_0 \widetilde{S}(\xi, \rho_0), \tag{1.15}$$

onde

$$\widetilde{S}(\xi, \rho_0) = e^{2\pi\xi}(1 - [2\pi + \mathcal{O}(\xi)]\rho_0^2) + \mathcal{O}(|\rho_0|^3),$$

percebemos que a transformação terá um ponto fixo $\rho_0 > 0$ se, e somente se, $\widetilde{S}(\xi, \rho_0) = 1$ tiver solução. Isto é,

$$\begin{array}{lrcl} & \widetilde{S}(\xi, \rho_0) & = & 1 \\ \Leftrightarrow & e^{2\pi\xi}(1 - [2\pi + \mathcal{O}(\xi)]\rho_0^2) + \mathcal{O}(|\rho_0^3|) & = & 1 \\ \Leftrightarrow & 1 - [2\pi + \mathcal{O}(\xi)]\rho_0^2 + e^{-2\pi\xi}\mathcal{O}(|\rho_0^3|) & = & e^{-2\pi\xi} \\ \Leftrightarrow & 1 - [2\pi + \mathcal{O}(\xi)]\rho_0^2 + e^{-2\pi\xi}\mathcal{O}(|\rho_0^3|) - e^{-2\pi\xi} & = & 0. \end{array}$$

Tome

$$S(\xi, \rho_0) = 1 - [2\pi + \mathcal{O}(\xi)]\rho_0^2 + e^{-2\pi\xi}\mathcal{O}(|\rho_0|^3) - e^{-2\pi\xi}.$$

Aplicando o Teorema da Função Implícita na função $S(\xi, \rho_0)$, para $(\xi, \rho_0) = (0, 0)$ comprovamos a afirmação. De fato,

$$S(0,0) = 0 \quad e \quad S_\xi(0,0) = \frac{\partial S}{\partial \xi}(0,0) = 2\pi \neq 0,$$

assim podemos escrever ξ como função de ρ_0 numa vizinhança de $\rho_0 = 0$ e calcular

$$\xi'(\rho_0) = -\frac{S_{\rho_0}(\rho_0, \xi(\rho_0))}{S_\xi(\rho_0, \xi(\rho_0))} = \frac{2(2\pi + \mathcal{O}(\xi))\rho_0 + e^{-2\pi\xi}\mathcal{O}(|\rho_0|^2)}{\rho_0^2 - 2\pi e^{-2\pi\xi}\mathcal{O}(|\rho_0|^3) + 2\pi e^{-2\pi\xi}}.$$

Portanto, temos que

$$\xi'(0) = 0, \qquad \xi''(0) = 2,$$

implicando, pela expansão de Taylor em torno de $\rho_0 = 0$, $\xi(0) = 0$, que

$$\xi(\rho_0) = \rho_0^2 + \dots ,$$

que é uma função injetora no domínio $\rho_0 \geq 0$.

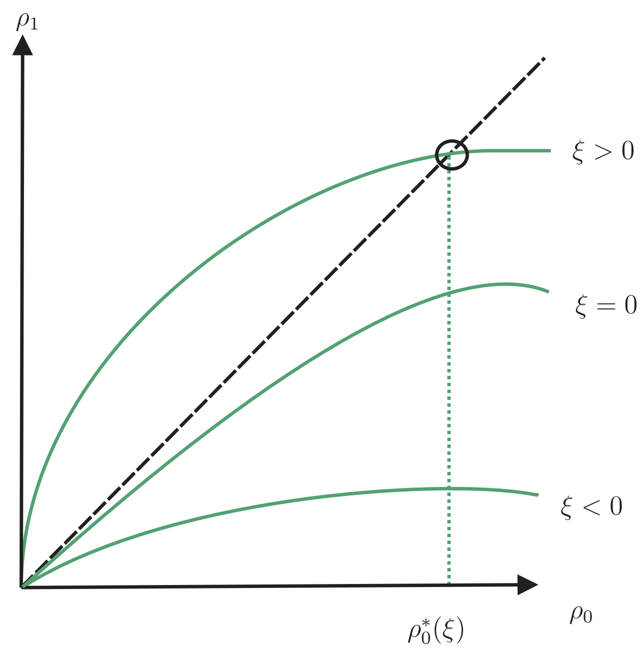

Figura 1.3: Ponto fixo da transformação de retorno.

A estabilidade dos pontos fixos também é obtida de (1.14). Derivando (1.15) com relação a ρ_0, obtemos

$$\frac{d\rho_1}{d\rho_0} = \widetilde{S}(\xi, \rho_0) + \rho_0 \widetilde{S}_{\rho_0}(\xi, \rho_0).$$

Para provarmos a estabilidade de ρ_0^* basta mostrarmos que

$$\frac{d\rho_1}{d\rho_0}(\rho_0^*) < 1.$$

De fato, como $\widetilde{S}(\xi, \rho_0) = 1$ para $\rho_0 = \rho_0^*$; $\xi = \xi(\rho_0^*)$, só falta observar que $\rho_0 \widetilde{S}_{\rho_0}(\xi(\rho_0^*), \rho_0^*)$ é negativo. Fazendo os cálculos

$$\rho_0 \widetilde{S}_{\rho_0}(\xi, \rho_0) = \rho_0 \frac{\partial \widetilde{S}}{\partial \rho_0}(\xi, \rho_0),$$

obtemos

$$\rho_0 \widetilde{S}_{\rho_0}(\xi, \rho_0) = \rho_0^2[-2e^{2\pi\xi}[2\pi + \mathcal{O}(\xi)] + \mathcal{O}(|\rho_0|)],$$

que, para pequenos valores de $\rho_0^* > 0$; $\xi(\rho_0^*) > 0$, satisfaz o esperado.

Levando em conta que o ponto fixo positivo da função corresponde a um ciclo limite do sistema, podemos concluir que (1.11), ou (1.10), com quaisquer termos $\mathcal{O}(|\rho|^4)$, tem um único (e estável) ciclo limite bifurcando na origem quando $\xi > 0$ do mesmo modo que (1.9). Portanto, em outras palavras, os termos de ordem superior não afetam o surgimento do ciclo limite numa vizinhança de $(x_1, x_2) = (0, 0)$ com $|\xi|$ suficientemente pequeno.

Parte II. (Construção do homeomorfismo)

Estabelecida a existência e unicidade do ciclo limite, mostraremos agora como proceder para se obter os homeomorfismos necessários e concluir a equivalência topológica dos retratos de fase.

Fixemos ξ pequeno, porém positivo. Ambos os sistemas (1.9) e (1.10) possuem um ciclo limite em alguma vizinhança da origem. Podemos assumir que a reparametrização do tempo já tenha sido realizado no sistema (1.10), resultando num tempo de retorno constante 2π, (vide Parte I). E ainda, que fizemos um escalonamento linear nas coordenadas do sistema (1.10) de modo que o ponto de intersecção do ciclo e o semieixo horizontal seja $x_1 = \sqrt{\xi}$.

Defina uma função $\mathbf{x} \longmapsto \mathbf{x}^*$ do seguinte modo: Tome $\mathbf{x} = (x_1, x_2)$ e encontre os valores (ρ_0, τ_0), onde τ_0 é o tempo mínimo que uma órbita do sistema (1.9) leva para alcançar

$\mathbf{x}$ partindo do semieixo horizontal com $\rho = \rho_0$. Para o ponto deste eixo com $\rho = \rho_0$, construa uma órbita do sistema (1.10) no intervalo $[0, \tau_0]$ partindo desse ponto. Denote o ponto resultante por $\mathbf{x}^* = (x_1^*, x_2^*)$, veja Figura 1.4.

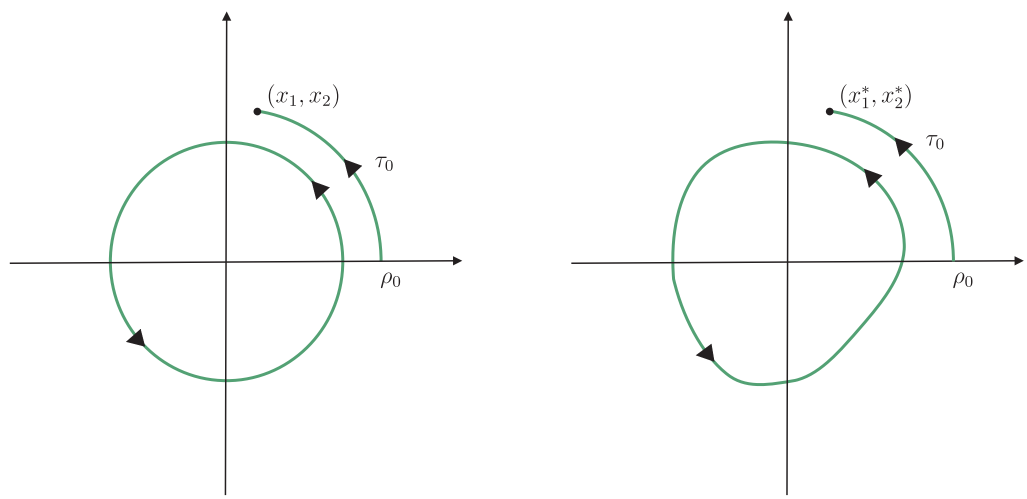

Figura 1.4: Construção do homeomorfismo.

Assuma $\mathbf{x}^* = 0$ para $\mathbf{x} = 0$. A função assim construída é um homeomorfismo que, para $\xi > 0$, leva órbitas do sistema (1.9), em alguma vizinhança da origem, em órbitas de (1.10), preservando a direção do tempo. O caso para $\xi < 0$ pode ser considerado da mesma maneira com uma nova mudança de coordenadas.

□

Ficou provado com o Lema 1.1.1 que os termos de ordem superior a três não afetam o comportamento da bifurcação. Nosso passo seguinte, é impor condições sobre um sistema de EDO's bidimensional de modo a transformá-lo no sistema (1.10) e, podendo assim aplicar o Lema 1.1.1 concluindo a prova do Teorema da Bifurcação de Hopf genérica (Teorema 1.1.1).

Considere o sistema

$$\dot{\mathbf{x}} = f(\mathbf{x}, \xi), \qquad \mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2,\ \xi \in \mathbb{R},$$

com f suave, tendo para $\xi = 0$ o equilíbrio $e = 0$ com autovalores $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$. Pelo Teorema da Função Implícita, como $\lambda = 0$ não é um autovalor da matriz Jacobiana, o sistema tem um único equilíbrio $e_0(\xi)$ em alguma vizinhança da origem para todo $|\xi|$ suficientemente pequeno. Podemos, então, através de uma mudança de coordenadas, levar este equilíbrio para a origem. Portanto, devemos assumir sem perda de generalidade que $e = 0$ é um ponto de equilíbrio do sistema para $|\xi|$ suficientemente pequeno.

Então o sistema pode ser escrito como

$$\dot{\mathbf{x}} = F(\mathbf{x}, \xi), \tag{1.16}$$

onde F é uma função suave com componentes $F_{1,2}$ tendo a expansão de Taylor em $\mathbf{x}$ iniciando com os termos de primeira ordem, $F = \mathcal{O}(|\mathbf{x}|)$. A matriz Jacobiana $A(\xi) = f_{\mathbf{x}}(0, \xi_0)$ possui dois autovalores

$$\lambda_1(\xi) = \lambda(\xi), \qquad \lambda_2(\xi) = \bar{\lambda}(\xi),$$

onde

$$\lambda(\xi) = \gamma(\xi) + i\omega(\xi),$$

e a condição para a bifurcação de Hopf é

$$\gamma(0) = 0, \omega(0) = \omega_0 > 0.$$

Seja $q(\xi) \in \mathbb{C}^2$ o autovetor correspondente ao autovalor $\lambda(\xi)$ e dado por

$$A(\xi)q(\xi) = \lambda(\xi)q(\xi),$$

e seja $p(\xi) \in \mathbb{C}^2$ o autovetor da matriz $A^\top(\xi)$ correspondente ao autovalor $\bar{\lambda}(\xi)$,

$$A^\top(\xi)p(\xi) = \bar{\lambda}(\xi)p(\xi).$$

É sempre possível normalizar p com respeito a q, tal que

$$\langle p(\xi), q(\xi) \rangle = 1,$$

onde $\langle p, q \rangle = \bar{p_1}q_1 + \bar{p_2}q_2$ é o produto escalar em $\mathbb{C}^2$. Qualquer vetor $\mathbf{x} \in \mathbb{R}^2$ pode ser representado unicamente para todo ξ pequeno como

$$\mathbf{x} = zq(\xi) + \bar{z}\,\bar{q}(\xi),$$

para algum complexo z. Assim temos a seguinte fórmula explícita para se determinar z

$$z = \langle p(\xi), \mathbf{x} \rangle . \tag{1.17}$$

Para verificar esta fórmula observemos que

$$\langle p, \mathbf{x} \rangle = \langle p, zq + \bar{z}\,\bar{q} \rangle = \langle p, zq \rangle + \langle p, \bar{z}\,\bar{q} \rangle$$
$$\Leftrightarrow \langle p, \mathbf{x} \rangle = z \langle p, q \rangle + \bar{z} \langle p, \bar{q} \rangle .$$

Como $\langle p, q \rangle = 1$, basta vermos que $\langle p, \bar{q} \rangle = 0$. De fato,

$$\langle p, \bar{q} \rangle = \left\langle p, \frac{1}{\bar{\lambda}} A\bar{q} \right\rangle = \frac{1}{\bar{\lambda}} \left\langle A^\top p, \bar{q} \right\rangle = \frac{\lambda}{\bar{\lambda}} \langle p, \bar{q} \rangle$$

$$\Leftrightarrow \left(1 - \frac{\lambda}{\bar{\lambda}}\right) \langle p, \bar{q} \rangle = 0.$$

Como $\lambda \neq \bar{\lambda}$, pois para $|\xi|$ suficientemente pequeno temos $\omega(\xi) > 0$, concluímos que

$$\langle p, \bar{q} \rangle = 0.$$

Lema 1.1.2. *O sistema (1.16) pode ser escrito, para $|\xi|$ suficientemente pequeno, na forma*

$$\dot{z} = \lambda(\xi) z + g(z, \bar{z}, \xi), \tag{1.18}$$

onde $g = \mathcal{O}(|z|^2)$ é uma função suave de $(z, \bar{z}, \xi)$, dada por

$$g(z, \bar{z}, \xi) = \langle p(\xi), F^*(zq(\xi) + \bar{z}\, \bar{q}(\xi), \xi) \rangle ,$$

com $F^*(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^2)$.

Demonstração. No sistema (1.16) temos $\dot{\mathbf{x}} = F(\mathbf{x}, \xi)$, podendo este ser reescrito como,

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathcal{O}(|\mathbf{x}|^2).$$

sendo $A = f_{\mathbf{x}}(0, \xi_0)$ e $\mathcal{O}(|\mathbf{x}|^2)$ representando a expansão de Taylor em $\mathbf{x}$ iniciando com os termos quadráticos (no mínimo). Temos assim que $F(\mathbf{x}) - A\mathbf{x} = \mathcal{O}(|\mathbf{x}|^2)$, porém, para simplificar a notação tome $F^*(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^2)$. Assim de (1.17) temos que a variável complexa z satisfaz a equação

$$\begin{aligned}
\dot{z} &= \langle p(\xi), \dot{\mathbf{x}} \rangle \\
&= \langle p, A\mathbf{x} + F^*(\mathbf{x}) \rangle \\
&= \langle p, A\mathbf{x} \rangle + \langle p, F^*(\mathbf{x}) \rangle \\
&= \langle p, A(zq + \bar{z}\, \bar{q}) \rangle + \langle p, F^*(zq + \bar{z}\, \bar{q}) \rangle \\
&= \langle p, A(zq) \rangle + \langle p, A(\bar{z}\, \bar{q}) \rangle + \langle p, F^*(zq + \bar{z}\, \bar{q}) \rangle \\
&= \lambda z \langle p, q \rangle + \bar{\lambda}\, \bar{z} \langle p, \bar{q} \rangle + \langle p, F^*(zq + \bar{z}\, \bar{q}) \rangle \\
&= \lambda(\xi) z + \langle p(\xi), F^*(zq(\xi) + \bar{z}\, \bar{q}(\xi), \xi) \rangle ,
\end{aligned}$$

obtendo assim (1.18). □

Escrevendo o desenvolvimento de Taylor para g nas variáveis complexas z e $\bar{z}$ temos

$$g(z, \bar{z}, \xi) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(\xi) z^k \bar{z}^l,$$

onde

$$g_{kl}(\xi) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \left\langle p(\xi), F^*(zq(\xi) + \bar{z}\,\bar{q}(\xi), \xi)\right\rangle \bigg|_{z=0},$$

para $k + l \geq 2$, $k, l = 0, 1, 2, ...$.

Suponha que, para $\xi = 0$, a função $F(\mathbf{x}, \xi)$ de (1.16) seja representada na forma

$$F(\mathbf{x}, 0) = A\mathbf{x} + \frac{1}{2}B(\mathbf{x}, \mathbf{x}) + \frac{1}{6}C(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{24}D(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{120}E(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \mathcal{O}(|\mathbf{x}|^6), \qquad (1.19)$$

onde $A = f_{\mathbf{x}}(0, \xi_0)$ e B, C, D e E são funções multilineares simétricas de $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Em coordenadas, temos

$$B_i(\mathbf{x},\ \mathbf{y}) = \sum_{j,k=1}^{2} \frac{\partial^2 F_i(\eta, 0)}{\partial \eta_j \partial \eta_k}\bigg|_{\eta=0} \mathbf{x}_j \mathbf{y}_k,$$

$$C_i(\mathbf{x},\ \mathbf{y},\ \mathbf{z}) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F_i(\eta, 0)}{\partial \eta_j \partial \eta_k \partial \eta_l}\bigg|_{\eta=0} \mathbf{x}_j \mathbf{y}_k \mathbf{z}_l,$$

$$D_i(\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{u}) = \sum_{j,k,l,r=1}^{2} \frac{\partial^4 F_i(\eta, 0)}{\partial \eta_j \partial \eta_k \partial \eta_l \partial \eta_r}\bigg|_{\eta=0} \mathbf{x}_j \mathbf{y}_k \mathbf{z}_l \mathbf{u}_r,$$

$$E_i(\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{u},\ \mathbf{v}) = \sum_{j,k,l,r,s=1}^{2} \frac{\partial^5 F_i(\eta, 0)}{\partial \eta_j \partial \eta_k \partial \eta_l \partial \eta_r \partial \eta_s}\bigg|_{\eta=0} \mathbf{x}_j \mathbf{y}_k \mathbf{z}_l \mathbf{u}_r \mathbf{v}_s,$$

para $i = 1, 2$.

Então,

$$B(zq+\bar{z}\,\bar{q}, zq+\bar{z}\,\bar{q}) = z^2B(q,q) + 2z\bar{z}B(q,\bar{q}) + \bar{z}^2B(\bar{q},\bar{q}),$$

onde $q = q(0)$, $p = p(0)$, e os coeficientes de Taylor g_{kl}, $k+l=2$, dos termos quadráticos em $g(z,\bar{z},0)$ podem ser escritos, agora, pelas fórmulas

$$\begin{aligned} g_{20} &= \langle p, B(q,q)\rangle\,, \\ g_{11} &= \langle p, B(q,\bar{q})\rangle\,, \\ g_{02} &= \langle p, B(\bar{q},\bar{q})\rangle\,. \end{aligned}$$

Com C, D e E, analogamente calculamos

$$\begin{aligned} g_{30} &= \langle p, C(q,q,q)\rangle\,, & g_{21} &= \langle p, C(q,q,\bar{q})\rangle\,, \\ g_{12} &= \langle p, C(q,\bar{q},\bar{q})\rangle\,, & g_{03} &= \langle p, C(\bar{q},\bar{q},\bar{q})\rangle\,, \end{aligned}$$

$$\begin{aligned} g_{40} &= \langle p, D(q,q,q,q)\rangle\,, & g_{31} &= \langle p, D(q,q,q,\bar{q})\rangle\,, \\ g_{22} &= \langle p, D(q,q,\bar{q},\bar{q})\rangle\,, & g_{13} &= \langle p, D(q,\bar{q},\bar{q},\bar{q})\rangle\,, \\ g_{04} &= \langle p, D(\bar{q},\bar{q},\bar{q},\bar{q})\rangle\,, \end{aligned}$$

$$\begin{aligned} g_{50} &= \langle p, E(q,q,q,q,q)\rangle\,, & g_{41} &= \langle p, E(q,q,q,q,\bar{q})\rangle\,, \\ g_{32} &= \langle p, E(q,q,q,\bar{q},\bar{q})\rangle\,, & g_{23} &= \langle p, E(q,q,\bar{q},\bar{q},\bar{q})\rangle\,, \\ g_{14} &= \langle p, E(q,\bar{q},\bar{q},\bar{q},\bar{q})\rangle\,, & g_{05} &= \langle p, E(\bar{q},\bar{q},\bar{q},\bar{q},\bar{q})\rangle\,. \end{aligned}$$

Lema 1.1.3. *A equação*

$$\dot{z} = \lambda z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + \mathcal{O}(|z|^3), \tag{1.20}$$

onde $\lambda = \lambda(\xi) = \gamma(\xi) + i\omega(\xi)$, $\gamma(0) = 0$, $\omega(0) = \omega_0 > 0$, *e* $g_{ij} = g_{ij}(\xi)$, *pode ser transformada, pela mudança de coordenada complexa*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2,$$

para $|\xi|$ suficientemente pequeno, na equação sem termos quadráticos

$$\dot{w} = \lambda w + \mathcal{O}(|w|^3).$$

Demonstração. A mudança de variável inversa é dada pela expressão

$$w = z - \frac{h_{20}}{2}z^2 - h_{11}z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + \mathcal{O}(|z|^3),$$

a qual derivando temos

$$\dot{w} = \dot{z} - h_{20}z\dot{z} - h_{11}(\dot{z}\bar{z} + z\dot{\bar{z}}) - h_{02}\bar{z}\dot{\bar{z}} + \ldots .$$

Substituindo $\dot{z}$ pela equação (1.20) e organizando os coeficientes, temos

$$\dot{w} = \lambda z + \left(\frac{g_{20}}{2} - \lambda h_{20}\right)z^2 + \left(g_{11} - \lambda h_{11} - \bar{\lambda}h_{11}\right)z\bar{z} + \left(\frac{g_{02}}{2} - \bar{\lambda}h_{02}\right)\bar{z}^2 + \ldots .$$

Assim,

$$\dot{w} = \lambda w + \left(\frac{g_{20}}{2} - \lambda h_{20} + \frac{1}{2}\lambda h_{20}\right)w^2 + \left(g_{11} - \lambda h_{11} - \bar{\lambda}h_{11} + \lambda h_{11}\right)w\bar{w}$$

$$+ \left(\frac{g_{02}}{2} - \bar{\lambda}h_{02} + \frac{1}{2}\lambda h_{02}\right)\bar{w}^2 + \mathcal{O}\left(|w|^3\right)$$

$$= \lambda w + \frac{1}{2}\left(g_{20} - \lambda h_{20}\right)w^2 + \left(g_{11} - \bar{\lambda}h_{11}\right)w\bar{w} + \frac{1}{2}\left(g_{02} - \left(2\bar{\lambda} - \lambda\right)h_{02}\right)\bar{w}^2 + \mathcal{O}\left(|w|^3\right).$$

Como $\lambda(0) = i\omega_0$, $\omega_0 > 0$, podemos escolher

$$h_{20} = \frac{g_{20}}{\lambda},\ h_{11} = \frac{g_{11}}{\bar{\lambda}},\ h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda}$$

e obter assim

$$\dot{w} = \lambda w + \mathcal{O}\left(|w|^3\right)$$

como queríamos. □

Lema 1.1.4. *A equação*

$$\dot{z} = \lambda z + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} + \frac{g_{12}}{2} z \bar{z}^2 + \frac{g_{03}}{6} \bar{z}^3 + \mathcal{O}\left(|z|^4\right), \tag{1.21}$$

onde $\lambda = \lambda(\xi) = \gamma(\xi) + i\omega(\xi)$, $\gamma(0) = 0$, $\omega(0) = \omega_0 > 0$, *e* $g_{ij} = g_{ij}(\xi)$, *pode ser transformada, pela mudança de coordenada complexa*

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} w \bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3,$$

para $|\xi|$ *suficientemente pequeno, na equação com apenas um termo cúbico*

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + \mathcal{O}\left(|w|^4\right),$$

onde $c_1 = c_1(\xi)$.

Demonstração. A transformação inversa é

$$w = z - \frac{h_{30}}{6} z^3 - \frac{h_{21}}{2} z^2 \bar{z} - \frac{h_{12}}{2} z \bar{z}^2 - \frac{h_{03}}{6} \bar{z}^3 + \mathcal{O}\left(|z|^4\right),$$

derivando temos

$$\dot{w} = \dot{z} - \frac{h_{30}}{2} z^2 \dot{z} - \frac{h_{21}}{2}\left(2z\bar{z}\dot{z} + z^2 \dot{\bar{z}}\right) - \frac{h_{12}}{2}\left(\dot{z}\bar{z}^2 + 2z\bar{z}\dot{\bar{z}}\right) - \frac{h_{03}}{2} \bar{z}^2 \dot{\bar{z}} + \ldots,$$

substituindo $\dot{z}$ pela equação (1.21) e reorganizando seus termos, temos

$$\dot{w} = \lambda z + \left(\frac{g_{30}}{6} - \frac{\lambda h_{30}}{2}\right) z^3 + \left(\frac{g_{21}}{2} - \lambda h_{21} - \frac{\bar{\lambda} h_{21}}{2}\right) z^2 \bar{z} + \left(\frac{g_{12}}{2} - \frac{\lambda h_{12}}{2} - \bar{\lambda} h_{12}\right) z \bar{z}^2$$

$$+ \left(\frac{g_{03}}{6} - \frac{\bar{\lambda} h_{03}}{2}\right) \bar{z}^3 + \mathcal{O}\left(|z|^4\right),$$

assim

$$\begin{aligned} \dot{w} = \lambda w &+ \frac{1}{6}\left(g_{30} - 2\lambda h_{30}\right) w^3 + \frac{1}{2}\left(g_{21} - \left(\lambda + \bar{\lambda}\right) h_{21}\right) w^2 \bar{w} + \frac{1}{2}\left(g_{12} - 2\bar{\lambda} h_{12}\right) w \bar{w}^2 \\ &+ \frac{1}{6}\left(g_{03} + \left(\lambda - 3\bar{\lambda}\right) h_{03}\right) \bar{w}^3 + \mathcal{O}\left(|w|^4\right). \end{aligned}$$

Fazendo

$$h_{30} = \frac{g_{30}}{2\lambda},\ h_{12} = \frac{g_{12}}{2\bar{\lambda}},\ h_{03} = \frac{g_{03}}{3\bar{\lambda} - \lambda},$$

eliminando todos os termos cúbicos com exceção do termo $w^2\bar{w}$, que será tratado separadamente. As substituições são válidas, pois, os denominadores envolvidos são diferentes de zero para todo $|\xi|$ suficientemente pequeno.

Uma tentativa de eliminar o termo $w^2\bar{w}$ seria escolher

$$h_{21} = \frac{g_{21}}{\lambda + \bar{\lambda}}.$$

Isso é possível para $\xi \neq 0$ pequeno, mas quando $\xi = 0$ o denominador se anula, pois $\lambda(0) + \bar{\lambda}(0) = i\omega_0 - i\omega_0 = 0$. Para obtermos então uma transformação que dependa suavemente de ξ, escolhemos $h_{21} = 0$, no que resulta

$$c_1 = \frac{g_{21}}{2}.$$

□

O termo $w^2\bar{w}$ é chamado de *termo ressonante*. Note que o seu coeficiente é o mesmo coeficiente do termo cúbico de $z^2\bar{z}$ na equação (1.21).

Lema 1.1.5. *A equação*

$$\dot{z} = \lambda z + \frac{g_{40}}{24}z^4 + \frac{g_{31}}{6}z^3\bar{z} + \frac{g_{22}}{4}z^2\bar{z}^2 + \frac{g_{13}}{6}z\bar{z}^3 + \frac{g_{04}}{24}\bar{z}^4 + \mathcal{O}(|z|^5), \tag{1.22}$$

onde $\lambda = \lambda(\xi) = \gamma(\xi) + i\omega(\xi)$, $\gamma(0) = 0$, $\omega(0) = \omega_0 > 0$, *e* $g_{ij} = g_{ij}(\xi)$, *pode ser transformada, pela mudança de coordenada complexa*

$$z = w + \frac{h_{40}}{24}w^4 + \frac{h_{31}}{6}w^3\bar{w} + \frac{h_{22}}{4}w^2\bar{w}^2 + \frac{h_{13}}{6}w\bar{w}^3 + \frac{h_{04}}{24}\bar{w}^4,$$

para $|\xi|$ *suficientemente pequeno, na equação sem termos de quarta ordem*

$$\dot{w} = \lambda w + \mathcal{O}(|w|^5).$$

Demonstração. A transformação inversa é

$$w = z - \frac{h_{40}}{24}z^4 - \frac{h_{31}}{6}z^3\bar{z} - \frac{h_{22}}{4}z^2\bar{z}^2 - \frac{h_{13}}{6}z\bar{z}^3 - \frac{h_{04}}{24}\bar{z}^4 + \mathcal{O}(|z|^5).$$

Sendo assim

$$\dot{w} = \dot{z} - \frac{h_{40}}{6} z^3 \dot{z} - \frac{h_{31}}{6}\left(3z^2\bar{z}\dot{z} + z^3\dot{\bar{z}}\right) - \frac{h_{22}}{4}\left(2z\bar{z}^2\dot{z} + 2z^2\bar{z}\dot{\bar{z}}\right) - \frac{h_{13}}{6}\left(\dot{z}\bar{z}^3 + 3z\bar{z}^2\dot{\bar{z}}\right) - \frac{h_{04}}{6}\bar{z}^3\dot{\bar{z}} + ...$$

$$= \lambda z + \left(\frac{g_{40}}{24} - \frac{h_{40}}{6}\lambda\right) z^4 + \left(\frac{g_{31}}{6} - \frac{h_{31}}{2}\lambda - \frac{h_{31}}{6}\bar{\lambda}\right) z^3\bar{z} + \left(\frac{g_{22}}{4} - \frac{h_{22}}{2}\lambda - \frac{h_{22}}{2}\bar{\lambda}\right) z^2\bar{z}^2$$

$$+ \left(\frac{g_{13}}{6} - \frac{h_{13}}{6}\lambda - \frac{h_{13}}{6}\bar{\lambda}\right) z\bar{z}^3 + \left(\frac{g_{04}}{24} - \frac{h_{04}}{6}\bar{\lambda}\right)\bar{z}^4 + ...$$

$$= \lambda w + \frac{1}{24}\left(g_{40} - 3\lambda h_{40}\right) w^4 + \frac{1}{6}\left(g_{31} - \left(2\lambda + \bar{\lambda}\right) h_{31}\right) w^3\bar{w} + \frac{1}{4}\left(g_{22} - \left(\lambda + 2\bar{\lambda}\right) h_{22}\right) w^2\bar{w}^2$$
$$+ \frac{1}{6}\left(g_{13} - 3\bar{\lambda} h_{13}\right) w\bar{w}^3 + \frac{1}{24}\left(g_{04} - \left(4\bar{\lambda} - \lambda\right) h_{04}\right)\bar{w}^4 + \mathcal{O}\left(|w|^5\right).$$

Fazendo

$$h_{40} = \frac{g_{40}}{3\lambda}, \quad h_{31} = \frac{g_{31}}{2\lambda + \bar{\lambda}}, \quad h_{22} = \frac{g_{22}}{\lambda + 2\bar{\lambda}},$$

$$h_{13} = \frac{g_{13}}{3\bar{\lambda}}, \quad h_{04} = \frac{g_{04}}{4\bar{\lambda} - \lambda},$$

eliminamos assim, todos os termos de ordem quatro. Temos que essas substituições são válidas, uma vez que, para $|\xi|$ suficientemente pequeno, os denominadores envolvidos são diferentes de zero, afinal $\lambda(0) = i\omega_0$, com $\omega_0 > 0$. □

Lema 1.1.6. *A equação*

$$\dot{z} = \lambda z + \frac{g_{50}}{120} z^5 + \frac{g_{41}}{24} z^4\bar{z} + \frac{g_{32}}{12} z^3\bar{z}^2 + \frac{g_{23}}{12} z^2\bar{z}^3 + \frac{g_{14}}{24} z\bar{z}^4 + \frac{g_{05}}{120}\bar{z}^5 + \mathcal{O}(|z|^6), \qquad (1.23)$$

onde $\lambda = \lambda(\xi) = \gamma(\xi) + i\omega(\xi)$, $\gamma(0) = 0$, $\omega(0) = \omega_0 > 0$, e $g_{ij} = g_{ij}(\xi)$, pode ser transformada, pela mudança de coordenada complexa

$$z = w + \frac{h_{50}}{120} w^5 + \frac{h_{41}}{24} w^4\bar{w} + \frac{h_{32}}{12} w^3\bar{w}^2 + \frac{h_{23}}{12} w^2\bar{w}^3 + \frac{h_{14}}{25} w\bar{w}^4 + \frac{h_{05}}{120}\bar{w}^5,$$

para $|\xi|$ suficientemente pequeno, na equação com apenas um termo de quinta ordem

$$\dot{w} = \lambda w + c_2 w^3 \bar{w}^2 + \mathcal{O}(|w|^6),$$

onde $c_2 = c_2(\xi)$.

Demonstração. A transformação inversa é dada por

$$w = z - \frac{h_{50}}{120} z^5 - \frac{h_{41}}{24} z^4 \bar{z} - \frac{h_{32}}{12} z^3 \bar{z}^2 - \frac{h_{23}}{12} z^2 \bar{z}^3 - \frac{h_{14}}{24} z \bar{z}^4 - \frac{h_{05}}{120} \bar{z}^5 + \mathcal{O}(|z|^6).$$

De onde temos que

$$\begin{aligned}
\dot{w} &= \dot{z} - \frac{h_{50}}{24} z^4 \dot{z} - \frac{h_{41}}{24}\left(4z^3 \bar{z} \dot{z} + z^4 \dot{\bar{z}}\right) - \frac{h_{32}}{12}\left(3z^2 \bar{z}^2 \dot{z} + 2z^3 \bar{z} \dot{\bar{z}}\right) - \frac{h_{23}}{12}\left(2z\bar{z}^3 \dot{z} + 3z^2 \bar{z}^2 \dot{\bar{z}}\right) \\
&\quad - \frac{h_{14}}{24}\left(\dot{z}\bar{z}^4 + 4z\bar{z}^3 \dot{\bar{z}}\right) - \frac{h_{05}}{24} \bar{z}^4 \dot{\bar{z}} + \mathcal{O}\left(|z|^6\right) \\
&= \lambda z + \left(\frac{g_{50}}{120} - \frac{h_{50}}{24}\lambda\right) z^5 + \left(\frac{g_{41}}{24} - \frac{h_{41}}{6} - \frac{h_{41}}{24}\bar{\lambda}\right) z^4 \bar{z} + \left(\frac{g_{32}}{12} - \frac{h_{32}}{4}\lambda - \frac{h_{32}}{6}\bar{\lambda}\right) z^3 \bar{z}^2 \\
&\quad + \left(\frac{g_{23}}{12} - \frac{h_{23}}{6}\lambda - \frac{h_{23}}{4}\bar{\lambda}\right) z^2 \bar{z}^3 + \left(\frac{g_{14}}{24} - \frac{h_{14}}{24}\lambda - \frac{h_{14}}{6}\bar{\lambda}\right) z\bar{z}^4 + \left(\frac{g_{05}}{120} - \frac{h_{05}}{24}\bar{\lambda}\right) \bar{z}^5 + \mathcal{O}\left(|z|^6\right) \\
&= \lambda w + \frac{1}{120}\left(g_{50} - 4\lambda h_{50}\right) w^5 + \frac{1}{24}\left(g_{41} - \left(3\lambda + \bar{\lambda}\right) h_{41}\right) w^4 \bar{w} + \frac{1}{12}\left(g_{32} - 2\left(\lambda + \bar{\lambda}\right) h_{32}\right) w^3 \bar{w}^2 \\
&\quad + \frac{1}{12}\left(g_{23} - \left(\lambda + 3\bar{\lambda}\right) h_{23}\right) w^2 \bar{w}^3 + \frac{1}{24}\left(g_{14} - 4\bar{\lambda} h_{14}\right) w\bar{w}^4 + \frac{1}{120}\left(g_{05} - \left(5\bar{\lambda} - \lambda\right) h_{05}\right) \bar{w}^5 \\
&\quad + \mathcal{O}\left(|w|^6\right).
\end{aligned}$$

Fazendo

$$h_{50} = \frac{g_{50}}{4\lambda}, \quad h_{41} = \frac{g_{41}}{3\lambda + \bar{\lambda}}, \quad h_{23} = \frac{g_{23}}{\lambda + 3\bar{\lambda}},$$

$$h_{14} = \frac{g_{14}}{4\bar{\lambda}}, \quad h_{05} = \frac{g_{05}}{5\bar{\lambda} - \lambda},$$

eliminamos assim, todos os termos de ordem cinco, com exceção do termo $w^3\bar{w}^2$, que trataremos separadamente. As substituições são válidas, pois, os denominadores envolvidos são diferentes de zero para todo $|\xi|$ suficientemente pequeno.

Uma tentativa de eliminar o termo $w^3\bar{w}^2$ seria escolher

$$h_{32} = \frac{g_{32}}{2\left(\lambda + \bar{\lambda}\right)}.$$

Isso é possível para $\xi \neq 0$ pequeno, mas quando $\xi = 0$ o denominador se anula, pois $\lambda(0) + \bar{\lambda}(0) = i\omega_0 - i\omega_0 = 0$. Para obtermos então uma transformação que dependa suavemente de ξ, escolhemos $h_{32} = 0$, no que resulta

$$c_2 = \frac{g_{32}}{12}.$$

□

O termo $w^3\bar{w}^2$ é chamado de *termo ressonante*. Note que o seu coeficiente é o mesmo coeficiente do termo de quinta ordem $z^3\bar{z}^2$ na equação (1.23).

Lema 1.1.7. *A equação*

$$\dot{z} = \lambda z + \sum_{2\leq k+l\leq 5} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l + \mathcal{O}(|z|^6), \tag{1.24}$$

onde $\lambda = \lambda(\xi) = \gamma(\xi) + i\omega(\xi)$, $\gamma(0) = 0$, $\omega(0) = \omega_0 > 0$, *e* $g_{ij} = g_{ij}(\xi)$, *pode ser transformada, pela mudança de coordenada complexa*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2 + \frac{h_{30}}{6}w^3 + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3 + \frac{h_{40}}{24}w^4 + \frac{h_{31}}{6}w^3\bar{w}$$

$$+\frac{h_{22}}{4}w^2\bar{w}^2 + \frac{h_{13}}{6}w\bar{w}^3 + \frac{h_{04}}{24}\bar{w}^4 + \frac{h_{50}}{120}w^5 + \frac{h_{41}}{24}w^4\bar{w} + \frac{h_{23}}{12}w^2\bar{w}^3 + \frac{h_{14}}{24}w\bar{w}^4 + \frac{h_{05}}{120}\bar{w}^5,$$

para $|\xi|$ *suficientemente pequeno, na equação com apenas um termo cúbico e um termo de quinta ordem*

$$\dot{w} = \lambda w + c_1 w^2\bar{w} + c_2 w^3\bar{w}^2 + \mathcal{O}(|w|^6), \tag{1.25}$$

onde $c_1 = c_1(\xi)$ *e* $c_2 = c_2(\xi)$.

Demonstração. Obviamente a suposição das transformações definidas nos lemas anteriores, nos levam a este resultado. As transformações

$$\begin{aligned} z &= w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2, \\ z &= w + \frac{h_{40}}{24}w^4 + \frac{h_{31}}{6}w^3\bar{w} + \frac{h_{22}}{4}w^2\bar{w}^2 + \frac{h_{13}}{6}w\bar{w}^3 + \frac{h_{04}}{24}\bar{w}^4, \end{aligned} \tag{1.26}$$

com

$$h_{20} = \frac{g_{20}}{\lambda}, \qquad h_{11} = \frac{g_{11}}{\bar{\lambda}}, \qquad h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda}, \quad h_{40} = \frac{g_{40}}{3\lambda}$$

$$h_{31} = \frac{g_{31}}{2\lambda + \bar{\lambda}}, \quad h_{22} = \frac{g_{22}}{\lambda + 2\bar{\lambda}}, \quad h_{13} = \frac{g_{13}}{3\bar{\lambda}}, \qquad h_{04} = \frac{g_{04}}{4\bar{\lambda} - \lambda},$$

definidas nos Lemas 1.1.3 e 1.1.5, anulam os respectivos termos, mas também alteram os outros termos. Os coeficientes $g_{21}/2$ e $g_{32}/12$ dos termos $z^2\bar{z}$ e $z^3\bar{z}^2$ respectivamente na equação (1.24) foram modificados pelas transformações de (1.26). Os termos de ordem 6 ou maiores, afetam somente $\mathcal{O}(|w|^6)$ e podem ser truncados. □

É necessário agora calcular os coeficientes c_1 e c_2 em termos da equação (1.24). O valor de c_1 e c_2 serão dados pelos novo coeficientes $g_{21}^*/2$ e $g_{32}^*/12$ dos termos $w^2\bar{w}$ e $w^3\bar{w}^2$ após as transformações de (1.26). Sendo assim seguem os lemas.

Lema 1.1.8. *O coeficiente $c_1(\xi)$ da equação (1.25), para $\xi = 0$, é dado por*

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}. \tag{1.27}$$

Demonstração. Derivando a primeira expressão de (1.26), obtemos

$$\dot{z} = \dot{w} + h_{20}w\dot{w} + h_{11}(w\dot{\bar{w}} + \bar{w}\dot{w}) + h_{02}\bar{w}\dot{\bar{w}}.$$

Substituindo $\dot{w}$ e seu complexo conjugado $\dot{\bar{w}}$, usando (1.25), obtemos

$$\dot{z} = \lambda w + \lambda h_{20} w^2 + (\lambda + \bar{\lambda}) h_{11} w \bar{w} + \bar{\lambda} h_{02} \bar{w}^2 + c_1 \ + \ldots .$$

Por outro lado, na equação (1.24),

$$\dot{z} = \lambda z + \frac{1}{2} g_{20} z^2 + g_{11} z \bar{z} + \frac{1}{2} g_{02} \bar{z}^2 + \frac{1}{6} g_{30} z^3 + \frac{1}{2} g_{21} z^2 \bar{z} + \frac{1}{2} g_{12} z \bar{z}^2 + \frac{1}{6} g_{03} \bar{z}^3 + \ldots ,$$

se substituirmos z e $\bar{z}$, dados pela primeira expressão de (1.26), escrevemos apenas os termos que nos interessam, tendo assim

$$\dot{z} = \lambda w + \frac{1}{2}(\lambda h_{20} + g_{20}) w^2 + (\lambda h_{11} + g_{11}) w \bar{w} + \frac{1}{2}(\lambda h_{02} + g_{02}) \bar{w}^2$$

$$+ \left(g_{20} h_{11} + g_{11} \left(\frac{h_{20}}{2} + \bar{h}_{11} \right) + \frac{g_{02} \bar{h}_{02}}{2} + \frac{g_{21}}{2} \right) w^2 \bar{w} + \ldots .$$

Comparando os coeficientes $w^2 \bar{w}$ nas duas equações obtidas, e usando

$$h_{20} = \frac{g_{20}}{\lambda}, \qquad h_{11} = \frac{g_{11}}{\bar{\lambda}}, \qquad h_{02} = \frac{g_{02}}{2\bar{\lambda} - \lambda},$$

temos

$$c_1 = g_{20} \frac{g_{11}}{\bar{\lambda}} + g_{11} \left(\frac{g_{20}}{2\lambda} + \frac{\bar{g}_{11}}{\lambda} \right) + \frac{g_{02} \bar{g}_{02}}{2(2\lambda - \bar{\lambda})} + \frac{g_{21}}{2}$$

$$\Rightarrow c_1 = \frac{g_{20} g_{11} (2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{g_{21}}{2}.$$

Essa fórmula nos dá a dependência de c_1 em relação a ξ, lembrando que λ e g_{ij} são funções suaves do parâmetro. No valor da bifurcação $\xi = 0$, a última equação se reduz a

$$c_1(0) = \frac{g_{20} g_{11} (2i\omega_0 - i\omega_0)}{2\omega_0^2} + \frac{|g_{11}|^2}{i\omega_0} + \frac{|g_{02}|^2}{2(2i\omega_0 - i\omega_0)} + \frac{g_{21}}{2}.$$

Finalmente temos

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},$$

como queríamos.

□

Lema 1.1.9. *A parte real do coeficiente $c_2(\xi)$ da equação (1.25), para $\xi = 0$ é dada por*

$$\begin{aligned}\mathrm{Re}\, c_2(0) = \frac{1}{12}\Bigg\{&\mathrm{Re}\, g_{32} + \frac{1}{\omega_0}\mathrm{Im}\left[g_{20}\bar{g}_{31} - g_{11}\left(4g_{31} + 3\bar{g}_{22}\right) - \frac{1}{3}g_{02}\left(g_{40} + \bar{g}_{13}\right) - g_{30}g_{12}\right] \\ &+\frac{1}{\omega_0^2}\left[\mathrm{Re}\,\left(g_{20}\left(\bar{g}_{11}\left(3g_{12} - \bar{g}_{30}\right) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30}\right) + \frac{1}{3}\bar{g}_{02}g_{03}\right)\right.\right. \\ &\left.\left.+g_{11}\left(\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12}\right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30}\right)\right) + 3\mathrm{Im}\left(g_{20}g_{11}\right)\mathrm{Im} g_{21}\right] \\ &+\frac{1}{\omega_0^3}\left[\mathrm{Im}\left(g_{11}\bar{g}_{02}\left(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\right)\right) + \mathrm{Im}\left(g_{20}g_{11}\right)\left(3\mathrm{Re}\,\left(g_{20}g_{11}\right) - 2|g_{02}|^2\right)\right]\Bigg\}.\end{aligned}$$

Demonstração. A demonstração deste lema é análoga a do Lema 1.1.8, no entanto, usaremos os termos de até ordem 5 e depois tomaremos a sua parte real.

□

Lema 1.1.10. *Considere a equação*

$$\frac{dw}{dt} = (\gamma(\xi) + i\omega(\xi))w + c_1(\xi)w|w|^2 + \mathcal{O}(|w|^4),$$

onde $\gamma(0) = 0$ e $\omega(0) = \omega_0 > 0$. Suponha $\gamma'(0) \neq 0$ e $\mathrm{Re}\, c_1(0) \neq 0$. Então a equação acima pode ser transformada, por uma mudança de coordenadas, na equação

$$\frac{du}{d\theta} = (\chi + i)u + su|u|^2 + \mathcal{O}(|u|^4), \tag{1.28}$$

onde u é a nova coordenada complexa, θ e χ são, respectivamente, os novos tempo e parâmetro e $s =$ sinal $\mathrm{Re}\, c_1(0) = \pm 1$.

Demonstração. Como $\omega(\xi) > 0$ para todo $|\xi|$ suficientemente pequeno, introduzindo o novo tempo $\tau = \omega(\xi)t$, a direção do tempo será preservada. Daí

$$\frac{dw}{d\tau} = \frac{\gamma(\xi) + i\omega(\xi)}{\omega(\xi)} w + \frac{c_1(\xi)}{\omega(\xi)} w|w|^2 + \mathcal{O}(|w|^4)$$

$$\Leftrightarrow \frac{dw}{d\tau} = (\chi + i)w + d_1(\chi)w|w|^2 + \mathcal{O}(|w|^4),$$

onde

$$\chi = \chi(\xi) = \frac{\gamma(\xi)}{\omega(\xi)}, \;\; d_1 = \frac{c_1(\xi(\chi))}{\omega(\xi(\chi))}.$$

Como

$$\chi(0) = 0, \;\; \chi'(0) = \frac{\gamma'(0)}{\omega(0)} \neq 0,$$

podemos considerar χ como o novo parâmetro, mais ainda, pelo Teorema da Função Inversa, ξ pode ser escrito como função suave de χ para $|\chi|$ suficientemente pequeno.

Faremos agora uma reparametrização do tempo ao longo das órbitas com a nova mudança de tempo $\theta = \theta(\tau, \chi)$, tal que

$$d\theta = (1 + e_1(\chi)|w|^2)d\tau,$$

sendo $e_1(\chi) = \mathrm{Im}\, d_1(\chi)$. Numa pequena vizinhança da origem, essa mudança é próxima da identidade. Para esse novo valor de tempo, temos

$$\frac{dw}{d\theta} = (\chi + i)w + l_1(\chi)w|w|^2 + \mathcal{O}(|w|^4),$$

onde $l_1(\chi) = \text{Re}\, d_1(\chi) - \chi e_1(\chi)$ é um número real para cada valor de χ suficientemente pequeno, e

$$l_1(0) = \frac{\text{Re}\, c_1(0)}{\omega(0)}. \tag{1.29}$$

Com efeito,

$$\frac{dw}{d\theta} = \frac{dw}{(1 + e_1(\chi)|w|^2)d\tau} = (\chi + i)w + l_1(\chi)w|w|^2 + \dots\,.$$

Mas, isso é equivalente a escrevermos

$$\begin{aligned}
\frac{dw}{d\tau} &= (1 + e_1(\chi)|w|^2)[(\chi + i)w + l_1(\chi)w|w|^2 + \dots] \\
&= (\chi + i)w + [l_1(\chi) + e_1(\chi)(\chi + i)w|w|^2 + \dots] \\
&= (\chi + i)w + [\text{Re}\, d_1 - \chi e_1 + \chi e_1 + ie_1]w|w|^2 + \dots] \\
&= (\chi + i)w + [\text{Re}\, d_1 + i\text{Im} d_1]w|w|^2 + \dots] \\
&= (\chi + i)w + d_1(\chi)w|w|^2 + \dots\,.
\end{aligned}$$

Como $\text{Re}\, c_1(0) \neq 0$, segue que $l_1(0) \neq 0$, e assim podemos introduzir a nova variável complexa u de modo que

$$w = \frac{u}{\sqrt{|l_1(\chi)|}}.$$

Substituindo na equação de $dw/d\theta$, temos

$$\frac{1}{\sqrt{|l_1(\chi)|}}\frac{du}{d\theta} = (\chi + i)\frac{u}{\sqrt{|l_1(\chi)|}} + l_1(\chi)\frac{u}{\sqrt{|l_1(\chi)|}}\left|\frac{u}{\sqrt{|l_1(\chi)|}}\right|^2 + \dots\,.$$

E assim,

$$\frac{du}{d\theta} = (\chi + i)u + \frac{l_1(\chi)}{|l_1(\chi)|}u|u|^2 + \mathcal{O}(|u|^4) = (\chi + i)u + su|u|^2 + \mathcal{O}(|u|^4),$$

onde $s =$ sinal $l_1(0) =$ sinal $\operatorname{Re} c_1(0)$.

□

Lema 1.1.11. *Considere a equação*

$$\frac{dw}{dt} = (\gamma(\xi) + i\omega(\xi))w + c_1(\xi)w|w|^2 + c_2(\xi)w|w|^4 + \mathcal{O}(|w|^6),$$

onde $\gamma(0) = 0$ *e* $\omega(0) = \omega_0 > 0$. *Suponha* $\gamma'(0) \neq 0$, $\operatorname{Re} c_1(0) = 0$ *e* $\operatorname{Re} c_2(0) \neq 0$. *Então a equação acima pode ser transformada, por uma mudança de coordenadas, na equação*

$$\frac{du}{d\theta} = (\chi + i)u + \zeta u|u|^2 + su|u|^4 + \mathcal{O}(|u|^6), \tag{1.30}$$

onde u *é a nova coordenada complexa,* θ *e* χ *são, respectivamente, os novos tempo e parâmetro,*

$$\zeta = \frac{d_1(0)}{\sqrt{|\operatorname{Re} c_2(0)|}}$$

e $s = \text{sinal} \operatorname{Re} c_2(0) = \pm 1$.

Demonstração. Como $\omega(\xi) > 0$ para todo $|\xi|$ suficientemente pequeno, introduzindo o novo tempo $\tau = \omega(\xi)t$, a direção do tempo será preservada. Sendo assim,

$$\frac{dw}{d\tau} = \frac{\gamma(\xi) + i\omega(\xi)}{\omega(\xi)}w + \frac{c_1(\xi)}{\omega(\xi)}w|w|^2 + \frac{c_2(\xi)}{\omega(\xi)}w|w|^4 + \mathcal{O}(|w|^6)$$

$$\Leftrightarrow \frac{dw}{d\tau} = (\chi + i)w + d_1(\chi)w|w|^2 + d_2(\chi)w|w|^4 + \mathcal{O}(|w|^6),$$

onde

$$\chi = \chi(\xi) = \frac{\gamma(\xi)}{\omega(\xi)}, \;\; d_1 = \frac{c_1(\xi(\chi))}{\omega(\xi(\chi))}, \;\; d_2 = \frac{c_2(\xi(\chi))}{\omega(\xi(\chi))}.$$

Podemos considerar χ como um novo parâmetro, pois

$$\chi(0) = 0, \;\; \chi'(0) = \frac{\gamma'(0)}{\omega(0)} \neq 0,$$

e, portanto, o Teorema da Função Inversa nos garante a existência local e suave de ξ como função de χ.

Faremos agora uma reparametrização do tempo ao longo das órbitas com a nova mudança de tempo $\theta = \theta(\tau, \chi)$, tal que

$$d\theta = (1 + e_1(\chi)|w|^2 + e_2(\chi)|w|^4)d\tau,$$

sendo $e_1(\chi) = \text{Im}\, d_1(\chi)$ e $e_2(\chi) = \text{Im}\, d_2(\chi)$. Numa pequena vizinhança da origem, essa mudança é próxima da identidade. Para esse novo valor de tempo, temos

$$\frac{dw}{d\theta} = (\chi + i)w + \eta(\chi)w|w|^2 + l_2(\chi)w|w|^4 + \mathcal{O}(|w|^6),$$

onde $\eta(\chi) = -\chi e_1(\chi)$, $l_2(\chi) = \text{Re}\, d_2(\chi) + \chi(e_1(\chi)^2 - e_2(\chi))$, é real e

$$l_1(0) = \frac{\text{Re}\, c_1(0)}{\omega(0)} = 0, \;\; l_2(0) = \frac{\text{Re}\, c_2(0)}{\omega(0)}. \tag{1.31}$$

De fato,

$$\frac{dw}{d\theta} = \frac{dw}{(1 + e_1(\chi)|w|^2 + e_2(\chi)|w|^4)d\tau} = (\chi + i)w + \eta(\chi)w|w|^2 + l_2(\chi)w|w|^4 + \ldots,$$

o que é equivalente a escrevermos

$$\begin{aligned}\frac{dw}{d\tau} &= (1 + e_1(\chi)|w|^2 + e_2(\chi)w|w|^4)[(\chi + i)w + \eta(\chi)w|w|^2 + l_2(\chi)w|w|^4 + \cdots] \\ &= (\chi + i)w + [\eta(\chi) + e_1(\chi)(\chi + i)]w|w|^2 + [l_2(\chi) + e_1(\chi)\eta + e_2(\chi)(\chi + i)]w|w|^4 + \cdots \\ &= (\chi + i)w + [-\chi e_1 + \chi e_1 + ie_1]w|w|^2 + [\text{Re}\, d_2 + \chi e_1^2 - \chi e_2 - \chi e_1^2 + \chi e_2 + ie_2]w|w|^4 + \cdots \\ &= (\chi + i)w + i\text{Im}\, d_1 w|w|^2 + [\text{Re}\, d_2 + i\text{Im}\, d_2]w|w|^4 + \cdots \\ &= (\chi + i)w + d_1(\chi)w|w|^2 + d_2(\chi)w|w|^4 + \cdots\end{aligned}$$

já que neste caso $\text{Re}\, d_1 = 0$ e sendo assim $i\,\text{Im} d_1$ é o próprio d_1. Assim, podemos introduzir a nova variável complexa u de modo que

$$w = \frac{u}{\sqrt[4]{|l_2(\chi)|}},$$

que é possível, pois $\text{Re}\, c_2(0) \neq 0$ e, portanto, $l_2(0) \neq 0$. A equação toma então a forma

$$\frac{1}{\sqrt[4]{|l_2(\chi)|}}\frac{du}{d\theta} = (\chi+i)\frac{u}{\sqrt[4]{|l_2(\chi)|}} + d_1\frac{u}{\sqrt[4]{|l_2(\chi)|}}\left|\frac{u}{\sqrt[4]{|l_2(\chi)|}}\right|^2 + l_2(\chi)\frac{u}{\sqrt[4]{|l_2(\chi)|}}\left|\frac{u}{\sqrt[4]{|l_2(\chi)|}}\right|^4 + \ldots.$$

E assim,

$$\frac{du}{d\theta} = (\chi+i)u + \frac{d_1(\chi)}{\sqrt{|l_2(\chi)|}}u|u|^2 + \frac{l_2(\chi)}{|l_2(\chi)|}u|u|^4 + \mathcal{O}(|u|^6) = (\chi+i)u + \zeta u|u|^2 + su|u|^4 + \mathcal{O}(|u|^6),$$

onde $s = \text{sinal}\, l_2(0) = \text{sinal Re}\, c_2(0)$.

□

Definição 1.1.1. *As funções $l_1(\chi)$ e $l_2(\chi)$ são chamadas, respectivamente, de* ***primeiro e segundo coeficientes de Lyapunov****.*

As equações de (1.29) e (1.31) nos dizem que o primeiro e o segundo coeficiente de Lyapunov, para $\chi = 0$, podem ser calculados pelas fórmulas

$$l_1(0) = \frac{1}{2\omega_0^2}\text{Re}\,(ig_{20}g_{11} + \omega_0 g_{21}) \qquad (1.32)$$

e

$$
\begin{aligned}
l_2(0) = \frac{1}{12}\Bigg\{ & \frac{1}{\omega_0}\text{Re}\, g_{32} \\
& + \frac{1}{\omega_0^2}\text{Im}\, \left[g_{20}\bar{g}_{31} - g_{11}\left(4g_{31} + 3\bar{g}_{22}\right) - \frac{1}{3}g_{02}\left(g_{40} + \bar{g}_{13}\right) - g_{30}g_{12} \right] \\
& + \frac{1}{\omega_0^3}\left[\text{Re}\, \left(g_{20}\left(\bar{g}_{11}\left(3g_{12} - \bar{g}_{30}\right) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30} \right) + \frac{1}{3}\bar{g}_{02}g_{03} \right) \right.\right. \\
& \left. + g_{11}\left(\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12} \right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30} \right) \right) \\
& \left. + 3\text{Im}\, \left(g_{20}g_{11}\right)\text{Im}\, g_{21} \right] \\
& + \frac{1}{\omega_0^4}\left[\text{Im}\, \left(g_{11}\bar{g}_{02}\left(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\right)\right) \right. \\
& \left. + \text{Im}\, \left(g_{20}g_{11}\right)\left(3\text{Re}\, \left(g_{20}g_{11}\right) - 2|g_{02}|^2\right) \right] \Bigg\},
\end{aligned} \tag{1.33}
$$

respectivamente. Isto significa que é necessário somente as derivadas parciais de segunda, terceira, quarta e quinta ordens no ponto de bifurcação para calcularmos $l_1(0)$ e $l_2(0)$.

Observação 1.1.2. *Os valores de $l_1(0)$ e $l_2(0)$ dependerão da normalização dos autovetores q e p, enquanto que seu sinal é invariante pela escolha de q e p, obviamente considerando a normalização $\langle p, q\rangle = 1$.*

Observação 1.1.3. *O sinal dos coeficientes de Lyapunov nos mostra que, sendo*

$$
\lambda_{1,2}(\xi) = \gamma(\xi) \pm i\omega(\xi),
$$

onde $\gamma(0) = 0$ e $\omega(0) = \omega_0 > 0$, e sendo $l_1 > 0$ ($l_1 < 0$), então temos um foco repulsor (atrator) fraco.

Note que se a equação (1.28) com sinal $s = -1$ for escrita na sua forma real, ela coincidirá com o sistema (1.10). Assim podemos agora resumir os resultados obtidos nos seguintes teoremas.

Teorema 1.1.1. (**Teorema da bifurcação de Hopf genérica**) *Qualquer sistema dinâmico da forma*

$$\dot{\mathbf{x}} = f(\mathbf{x}, \xi), \tag{1.34}$$

onde f é uma função suave, $\mathbf{x} \in \mathbb{R}^2$ e $\xi \in \mathbb{R}$, tendo para todo $|\xi|$ suficientemente pequeno, o equilíbrio $e = 0$ com os autovalores

$$\lambda_{1,2}(\xi) = \gamma(\xi) \pm i\omega(\xi),$$

onde $\gamma(0) = 0$ e $\omega(0) = \omega_0 > 0$, satisfazendo:

1. *$l_1(0) \neq 0$ (condição de não degenerescência);*
2. *$\gamma'(0) \neq 0$ (condição de transversalidade),*

é localmente topologicamente equivalente, em torno da origem, a uma das seguintes formas normais

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} \zeta & -1 \\ 1 & \zeta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm (y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Demonstração. Utilizando os Lemas 1.1.3, 1.1.4, 1.1.7, 1.1.8 e 1.1.10, trasformamos o sistema (1.34) na equação (1.28), então pelo Lema 1.1.1, concluímos o resultado.

□

Portanto, o Teorema 1.1.1 nos garante que um sistema em duas dimensões que possui autovalores imaginários puros e satisfaz as condições 1 e 2 desse mesmo teorema, possui uma bifurcação de Hopf.

Teorema 1.1.2. (Teorema da bifurcação de Hopf degenerada) *Considere o sistema planar*

$$\dot{\mathbf{x}} = f(\mathbf{x}, \xi),$$

onde f é uma função suave, $\mathbf{x} \in \mathbb{R}^2$ e $\xi \in \mathbb{R}^2$, tendo o equilíbrio $e_0 = 0$ com os autovalores

$$\lambda_{1,2}(\xi) = \gamma(\xi) \pm i\omega(\xi),$$

para todo $|\xi|$ suficientemente pequeno, onde $\omega(0) = \omega_0 > 0$. Para $\xi = 0$, sejam as condições para a bifurcação de Hopf degenerada

$$\gamma(0) = 0, \;\; l_1(0) = 0,$$

onde $l_1(\xi)$ é o primeiro coeficiente de Lyapunov. Assuma que as seguintes condições genéricas sejam satisfeitas:

1. *$l_2(0) \neq 0$, onde $l_2(0)$ é o segundo coeficiente de Lyapunov dado por (1.33);*
2. *a função $\xi \longmapsto (\gamma(\xi), l_1(\xi))^\top$ é regular em $\xi = 0$.*

Então, pela introdução de uma variável complexa e aplicando uma transformação de coordenadas que dependa suavemente da escolha do parâmetro e do tempo, o sistema pode ser reduzido à seguinte forma complexa

$$\dot{z} = (\chi + i)z + \zeta z|z|^2 + sz|z|^4 + \mathcal{O}(|z|^6), \tag{1.35}$$

onde $s = \text{sinal}\, l_2(0) = \pm 1$.

Demonstração. A demonstração do teorema pode ser encontrada em [7].

□

O Teorema 1.1.2 nos garante que um sistema em duas dimensões que possui autovalores imaginários puros e satisfaz as condições 1 e 2 desse mesmo teorema, possui uma bifurcação de Hopf degenerada. Veja Figura 1.5.

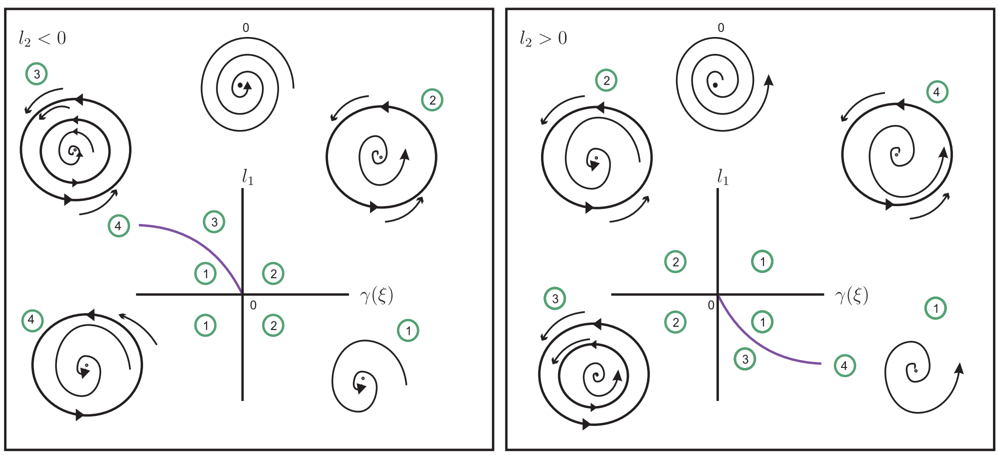

Figura 1.5: Diagramas de bifurcações da Bifurcação de Hopf degenerada. Na figura da esquerda $l_2 < 0$ e na figura da direita $l_2 > 0$.

1.2 O Método da Projeção

O método em questão tem como base a transformação do sistema,

$$\dot{\mathbf{x}} = f(\mathbf{x}, \xi), \;\; \mathbf{x} \in \mathbb{R}^n \;\; e \;\; \xi \in \mathbb{R}^m,$$

escrevendo-o numa base formada por seus autovetores generalizados e, posteriormente, na projeção deste sistema usando apenas os autovetores correspondentes aos autovalores críticos (único par de autovalores com partes reais nulas) para restringí-lo ao caso bidimensional já estudado.

Antes de escrevermos o Método da Projeção, faremos um breve resumo de alguns resultados de Álgebra Linear que serão necessários nesta seção.

Sejam, A uma matriz quadrada e λ um autovalor de A com multiplicidade algébrica m, com v_1, v_2, ...,v_l, $1 \leq l \leq m$, autovetores linearmente independentes correspondentes a λ. Para cada autovetor v_j, existe uma escolha maximal de vetores $w_1^{(j)}$, $w_2^{(j)}$, ..., $w_k^{(j)}$, onde $k = k(j) \in \mathbb{N}$, tal que

$$\begin{aligned} Aw_1 &= \lambda w_1, \\ Aw_2 &= \lambda w_2 + w_1, \\ &\vdots \\ Aw_k &= \lambda w_k + w_{k-1}. \end{aligned}$$

Não há nenhum problema em escolhermos $w_1 = w_1^{(j)}$ como o próprio v_j.

Definição 1.2.1. *Os vetores $w_i^{(j)}$, com $i \geq 2$, são chamados* ***autovetores generalizados*** *de A correspondentes ao autovalor λ.*

Os autovetores generalizados $w_1^{(j)}$, $w_2^{(j)}$, ..., $w_k^{(j)}$, relativos a um autovalor λ são sempre linearmente independentes e o subespaço

$$X = \{\mathbf{x} \in \mathbb{C}^n : \mathbf{x} = \alpha_1 w_1^{(j)} + \alpha_2 w_2^{(j)} + ... + \alpha_k w_k^{(j)}, \alpha_i \in \mathbb{C}\}$$

é A-invariante.

Pelas formas canônicas de Jordan, podemos decompor o espaço vetorial $\mathbb{C}^n$ em subespaços A-invariantes correspondentes aos autovalores de A e gerados pelos respectivos autovetores e autovetores generalizados. A esses subespaços denominamos ***autoespaços generalizados*** de A. Quando a matriz A é real, esses subespaços A-invariantes do $\mathbb{R}^n$ são gerados pelos autovetores e autovetores generalizados de A, correspondente aos autovalores reais e às partes real e imaginária dos autovalores complexos. Para uma demonstração dessa última afirmação veja Pontryagin [9].

Seja e_0 um ponto de equilíbrio não-hiperbólico de

$$\dot{\mathbf{x}} = F(\mathbf{x}, 0), \;\; \mathbf{x} \in \mathbb{R}^n, \tag{1.36}$$

onde $F(\mathbf{x}, 0)$, dada por (1.19) é uma função suave, $A = f_{\mathbf{x}}(0, \xi_0)$ correspondente à parte linear do sistema e possui um par de autovalores imaginários puros $\lambda = i\omega_0$ e $\bar{\lambda} = -i\omega_0$, $\omega_0 > 0$ e não admite outro autovalor com parte real nula.

Seja $q \in \mathbb{C}^n$ correspondente à λ. Então

$$A(\xi_0)q(\xi_0) = i\omega_0 q(\xi_0), \;\; A(\xi_0)\bar{q}(\xi_0) = -i\omega_0 \bar{q}(\xi_0).$$

Introduzindo agora o autovetor adjunto $p \in \mathbb{C}^n$ com a propriedade

$$A^\top(\xi_0)p(\xi_0) = -i\omega_0 p(\xi_0), \;\; A^\top(\xi_0)\bar{p}(\xi_0) = i\omega_0 \bar{p}(\xi_0)$$

satisfazendo a normalização

$$\langle p(\xi_0), q(\xi_0)\rangle = \sum_{i=1}^{n} \bar{p}_i(\xi_0) q_i(\xi_0) = 1,$$

onde $A^\top(\xi_0)$ é a matriz transposta de $A(\xi_0)$ e $\langle p(\xi_0), q(\xi_0)\rangle$ é o produto escalar usual em $\mathbb{C}^n$. Considere o autoespaço real T^c, correspondente a λ e $\bar{\lambda}$. T^c tem dimensão dois e é gerado por $\{\mathrm{Re}\, q, \; \mathrm{Im}\, q\}$. O autoespaço real generalizado T^{su}, correspondente a todos os outros autovalores de A, tem dimensão $n-2$.

Sempre podemos decompor $\mathbf{x} \in \mathbb{R}^n$ em

$$\mathbf{x} = zq + \bar{z}\bar{q} + \mathbf{y}_{su},$$

onde $z \in \mathbb{C}$, $zq + \bar{z}\bar{q} \in T^c$ e $\mathbf{y}_{su} \in T^{su}$, uma vez que $T^{su} \oplus T^c = \mathbb{R}^n$.

Lema 1.2.1. *Seja $\boldsymbol{y} \in \mathbb{R}^n$. $\boldsymbol{y} \in T^{su}$ se, e somente se, $\langle p, \boldsymbol{y} \rangle = 0$.*

Demonstração. A demostração será feita em duas partes:

Parte I ($\boldsymbol{y} \in T^{su} \Rightarrow \langle p, \boldsymbol{y} \rangle = 0$).

Sejam μ_1, μ_2, ..., μ_l os autovalores reais de A e η_1, $\bar{\eta}_1$; η_2, $\bar{\eta}_2$; ...;η_k, $\bar{\eta}_k$, os autovalores complexos de A, diferentes de λ e $\bar{\lambda}$.

Sejam T_{μ_i} o autoespaço generalizado correspondente ao autovalor μ_i e $T_{\eta_j,\bar{\eta}_j}$ o autoespaço real generalizado correspondente aos autovalores η_j, $\bar{\eta}_j$.

Temos,então, que

$$T^{su} = T_{\mu_1} \oplus T_{\mu_2} \oplus ... \oplus T_{\mu_l} \oplus T_{\eta_1,\bar{\eta}_1} \oplus T_{\eta_2,\bar{\eta}_2} \oplus ... \oplus T_{\eta_k,\bar{\eta}_k}.$$

Como T_{μ_i} são espaços generalizados, é verdade que para cada i existe um $N_{\mu_i} \in \mathbb{N}$, tal que, se $\boldsymbol{y} \in T_{\mu_i}$, então $(A - \mu_i I_n)^{N_{\mu_i}} \boldsymbol{y} = 0$. Portanto,

$$\begin{aligned} 0 &= \left\langle p, (A - \mu_i I_n)^{N_{\mu_i}} \boldsymbol{y} \right\rangle = \left\langle (A^\top - \bar{\mu}_i I_n)^{N_{\mu_i}} p, \boldsymbol{y} \right\rangle \\ &= \left\langle (\bar{\lambda} - \bar{\mu}_i)^{N_{\mu_i}} p, \boldsymbol{y} \right\rangle \quad = (\lambda - \mu_i)^{N_{\mu_i}} \langle p, \boldsymbol{y} \rangle \end{aligned}$$

e, como $\lambda \neq \mu_i$, temos que

$$\langle p, \boldsymbol{y} \rangle = 0.$$

Do mesmo modo, como $T_{\eta_j,\bar{\eta}_j}$, são espaços generalizados, para cada j existe um $N_{\eta_j} \in \mathbb{N}$, tal que, se $\boldsymbol{y} \in T_{\eta_j,\bar{\eta}_j}$, então $(A - \eta_j I_n)^{N_{\eta_j}} (A - \bar{\eta}_j I_n)^{N_{\eta_j}} \boldsymbol{y} = 0$. Portanto

$$\begin{aligned}
0 &= \left\langle p, (A-\eta_j I_n)^{N_{\eta_j}}(A-\bar{\eta}_j I_n)^{N_{\eta_j}}\boldsymbol{y}\right\rangle \\
&= \left\langle (A^\top-\bar{\eta}_j I_n)^{N_{\eta_j}}p, (A-\bar{\eta}_j I_n)^{N_{\eta_j}}\boldsymbol{y}\right\rangle \\
&= \left\langle (A^\top-\eta_j I_n)^{N_{\eta_j}}(A^\top-\bar{\eta}_j I_n)^{N_{\eta_j}}p, \boldsymbol{y}\right\rangle \\
&= \left\langle (\bar{\lambda}-\eta_j)^{N_{\eta_j}}(\bar{\lambda}-\eta_j)^{N_{\eta_j}}p, \boldsymbol{y}\right\rangle \\
&= (\lambda-\bar{\eta}_j)^{N_{\eta_j}}(\lambda-\eta_j)^{N_{\eta_j}}\left\langle p, \boldsymbol{y}\right\rangle .
\end{aligned}$$

e como $\lambda \neq \eta_j$ e $\lambda \neq \bar{\eta}_j$ temos que

$$\langle p, \boldsymbol{y}\rangle = 0.$$

Portanto, para qualquer $\boldsymbol{y} \in T^{su}$, como podemos escrever

$$\boldsymbol{y} = \sum_{i=1}^{l} \boldsymbol{y}_{\mu_i} + \sum_{j=1}^{k} \boldsymbol{y}_{\eta_j},$$

com $\boldsymbol{y}_{\mu_i} \in T_{\mu_i}$, para $i = 1, 2, ..., l$ e $\boldsymbol{y}_{\eta_j} \in T_{\eta_j,\bar{\eta}_j}$, para $j = 1, 2, ..., k$, podemos concluir então que

$$\begin{aligned}
\langle p, \boldsymbol{y}\rangle &= \left\langle p, \boldsymbol{y}_{\mu_1} + ... + \boldsymbol{y}_{\mu_l} + \boldsymbol{y}_{\eta_1} + ... + \boldsymbol{y}_{\eta_k}\right\rangle \\
&= \left\langle p, \boldsymbol{y}_{\mu_1}\right\rangle + ... + \left\langle p, \boldsymbol{y}_{\mu_l}\right\rangle + \left\langle p, \boldsymbol{y}_{\eta_1}\right\rangle + ... + \left\langle p, \boldsymbol{y}_{\eta_k}\right\rangle \\
&= 0.
\end{aligned}$$

Parte II ($\langle p, \mathbf{y}\rangle = 0,\ \mathbf{y} \in \mathbb{R}^n \Rightarrow \mathbf{y} \in T^{su}$).

Seja $\mathbf{y}$ qualquer, tal que $\mathbf{y} \in T^{su} \oplus T^c \subset \mathbb{R}^n$. Portanto podemos escrever

$$\mathbf{y} = \mathbf{y}_{su} + \mathbf{y}_c,$$

com $\mathbf{y}_{su} \in T^{su}$ e $\mathbf{y}_c \in T^c$. Como T^c é gerado por q, $\bar{q}$, mas $\mathbf{y}_c \in \mathbb{R}^n$,

$$\mathbf{y}_c = \alpha q + \bar{\alpha}\bar{q},$$

com $\alpha \in \mathbb{C}$, concluímos que

$$\mathbf{y} = \mathbf{y}_{su} + \alpha q + \bar{\alpha}\bar{q}. \tag{1.37}$$

Queremos mostrar aqui que $\mathbf{y}_c = 0$, o que será feito mostrando que $\alpha = 0$.

Da hipótese, temos que

$$0 = \langle p, \mathbf{y} \rangle = \langle p, \mathbf{y}_{su} + \mathbf{y}_c \rangle = \langle p, \mathbf{y}_{su} \rangle + \langle p, \mathbf{y}_c \rangle .$$

Da Parte I, temos que $\langle p, \mathbf{y}_{su} \rangle = 0$. Portanto,

$$\begin{aligned} & \langle p, \mathbf{y}_c \rangle = 0 \\ \Rightarrow \quad & \langle p, \alpha q + \bar{\alpha}\bar{q} \rangle = 0 \\ \Rightarrow \quad & \alpha \langle p, q \rangle + \bar{\alpha} \langle p, \bar{q} \rangle = 0 \\ \Rightarrow \quad & \alpha = 0, \end{aligned}$$

pois $\langle p, q \rangle = 1$ e $\langle p, \bar{q} \rangle = 0$. De fato,

$$\langle p, \bar{q} \rangle = \left\langle p, \frac{1}{\bar{\lambda}} A\bar{q} \right\rangle = \frac{1}{\bar{\lambda}} \left\langle A^{\top} p, \bar{q} \right\rangle = \frac{\lambda}{\bar{\lambda}} \langle p, \bar{q} \rangle ,$$

$$\left(1 - \frac{\lambda}{\bar{\lambda}}\right) \langle p, \bar{q} \rangle = 0.$$

Como λ não é real, temos $\lambda \neq \bar{\lambda}$ e, portanto $\langle p, \bar{q} \rangle = 0$. □

Utilizando o lema anterior, podemos agora explicitar z e $\mathbf{y}$ com relação a $\mathbf{x}$. Sendo $\mathbf{x} = zq + \bar{z}\bar{q} + \mathbf{y} \in \mathbb{R}^n$, com $zq + \bar{z}\bar{q} \in T^c$ e $\mathbf{y} \in T^{su}$, vale que

$$\langle p, \mathbf{x} \rangle = \langle p, zq + \bar{z}\bar{q} + \mathbf{y} \rangle = \langle p, zq \rangle + \langle p, \bar{z}\bar{q} \rangle + \langle p, \mathbf{y} \rangle .$$

Como $\langle p, \mathbf{y} \rangle = 0$, pois $\mathbf{y} \in T^{su}$ (Lema 1.2.1),

$$\langle p, \mathbf{x} \rangle = \langle p, zq \rangle + \langle p, \bar{z}\bar{q} \rangle = z \langle p, q \rangle + \bar{z} \langle p, \bar{q} \rangle ,$$

e lembrando que $\langle p, q \rangle = 1$ e $\langle p, \bar{q} \rangle = 0$, como visto na demostração do lema anterior

(Parte II), concluímos que

$$\begin{cases} z = \langle p, \mathbf{x} \rangle \,, \\ \mathbf{y} = \mathbf{x} - \langle p, \mathbf{x} \rangle \, q - \langle \bar{p}, \mathbf{x} \rangle \, \bar{q}. \end{cases} \tag{1.38}$$

Teorema 1.2.1. (Teorema da Variedade central) *Localmente, existe um conjunto invariante $W^c(0)$ de (1.36) que é tangente a T^c em $e_0 = 0$. Tal conjunto é o gráfico de uma aplicação suave, cujas derivadas parciais de todas as ordens são unicamente determinadas. Se $\psi^\top$ denota um fluxo associado a (1.36), então existe uma vizinhança U de $e_0 = 0$, tal que se $\psi^\top \mathbf{x} \in U$ para todo $t \geq 0$ ($t \leq 0$), então $\psi^\top \mathbf{x} \to W^c(0)$ para todo $t \to +\infty$ ($t \to -\infty$).*

Para ter uma melhor compreensão olhar Kuznetsov [7].

Definição 1.2.2. *O conjunto W^c é denominado **variedade central**.*

Considere uma variedade central W^c que tenha a mesma classe de diferenciabilidade (finita) que f (se $f \in C^k$ para algum k finito, W^c também é de classe C^k) em alguma vizinhança U de e_0. Contudo, quando $k \to \infty$, a vizinhança U poderá diminuir e, para alguns casos, resultar na não existência de uma variedade W^c de classe C^∞ para algum sistema C^∞.

Assim, o sistema

$$\dot{\mathbf{x}} = f(\mathbf{x}), \;\; \mathbf{x} \in \mathbb{R}^n,$$

pode ser escrito como

$$\begin{cases} \dot{z} = Bz + g(z, \mathbf{y}), \\ \dot{\mathbf{y}} = C\mathbf{y} + h(z, \mathbf{y}), \end{cases} \tag{1.39}$$

onde $z \in T^c$, $\mathbf{y} \in T^{su}$, B é uma matriz 2×2 formada pelos autovalores com partes reais nulas, e C é uma matriz $(n-2) \times (n-2)$ formada pelos autovalores com partes reais não nulas. As funções g e h têm a expansões de Taylor começando com os termos quadráticos. A variedade central W^c do sistema (1.39) pode ser localmente representada como um gráfico de uma função suave

$$W^c = \{(z, \bar{z}, \mathbf{y}) : \mathbf{y} = V(z, \bar{z})\}.$$

Aqui, $V : T^c \to T^{su}$, e devido à propriedade de tangência de W^c, $V(z, \bar{z}) = \mathcal{O}(|z|^2)$.

Qualquer vetor $\mathbf{z} \in T^c$ pode ser representado como $\mathbf{z} = wq + \bar{w}\bar{q}$, onde $w = \langle p, \mathbf{z} \rangle \in \mathbb{C}$. A variedade central bidimensional pode ser parametrizada por w, $\bar{w}$ por meio de uma imersão da forma $\mathbf{x} = H(w, \bar{w})$, onde $H : \mathbb{C}^2 \to \mathbb{R}^n$ tem sua expansão de Taylor da forma

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \le j+k \le 5} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + \mathcal{O}(|w|^6), \tag{1.40}$$

com $h_{jk} \in \mathbb{C}^n$ e $h_{jk} = \bar{h}_{kj}$. Substituindo (1.40) em (1.36), obtemos a seguinte equação diferencial

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})), \tag{1.41}$$

onde F é dada pela expansão (1.19). Assim, temos que o campo restrito a variedade central, de acordo com (1.25), pode ser escrito como

$$w' = i\omega_0 w + \frac{1}{2} g_{21} w|w|^2 + \frac{1}{12} g_{32} w|w|^4 + \mathcal{O}(|w|^6), \tag{1.42}$$

com $g_{jk} \in \mathbb{C}$. Ou seja, estamos projetando o campo de vetores sobre a variedade central. Assim, sobre a variedade central, a equação diferencial se comporta como no plano.

Observação 1.2.1. *Note que a equação (1.42) é exatamente igual à equação (1.25). Vejamos,* $w^2\bar{w} = w|w|^2$, $w^3\bar{w}^2 = w|w|^4$ *e tomando*

$$c_1 = \frac{1}{2} g_{21} \ \ e \ \ c_2 = \frac{1}{12} g_{32}$$

chegamos à equação (1.42).

Temos que

$$H_w = q + h_{20}w + h_{11}\bar{w} + \frac{1}{2}h_{30}w^2 + h_{21}w\bar{w} + \frac{1}{2}h_{12}\bar{w}^2 + \frac{1}{6}h_{40}w^3 + \frac{1}{2}h_{31}w^2\bar{w} + \frac{1}{2}h_{22}w\bar{w}^2$$

$$+\frac{1}{6}h_{13}\bar{w}^3 + \frac{1}{4}h_{32}w^2\bar{w}^2 + \ldots ,$$

$$H_{\bar{w}} = \bar{q} + h_{11}w + h_{02}\bar{w} + h_{12}w\bar{w} + \frac{1}{2}h_{21}w^2 + \frac{1}{2}h_{03}\bar{w}^2 + \frac{1}{2}h_{13}w\bar{w}^2 + \frac{1}{2}h_{22}w^2\bar{w} + \frac{1}{6}h_{31}w^3$$

$$+\frac{1}{6}h_{04}\bar{w}^3 + \frac{1}{6}h_{32}w^3\bar{w} + \ldots .$$

Aplicando H_w, $H_{\bar{w}}$, w' e $\bar{w}'$ em (1.41), obtemos

$$H_w w' + H_{\bar{w}}\bar{w}' = qi\omega_0 w - \bar{q}i\omega_0\bar{w} + h_{20}i\omega_0 w^2 - h_{02}i\omega_0\bar{w}^2 + \frac{1}{2}h_{30}i\omega_0 w^3$$

$$+\left(\frac{1}{2}qg_{21} + \frac{1}{2}h_{21}i\omega_0\right)w^2\bar{w} + \left(\frac{1}{2}\bar{q}\bar{g}_{21} - \frac{1}{2}h_{12}i\omega_0\right)w\bar{w}^2 - \frac{1}{2}h_{03}i\omega_0\bar{w}^3$$

$$+\frac{1}{6}h_{40}i\omega_0 w^4 + \left(\frac{1}{2}g_{21}h_{20} + \frac{1}{3}h_{31}i\omega_0\right)w^3\bar{w} + \left(\frac{1}{2}g_{21}h_{11} + \frac{1}{2}\bar{g}_{21}h_{11}\right)w^2\bar{w}^2$$

$$+\left(\frac{1}{2}h_{02}\bar{g}_{21} - \frac{1}{3}h_{13}i\omega_0\right)w\bar{w}^3 - \frac{1}{6}h_{04}i\omega_0\bar{w}^4$$

$$+\left(\frac{1}{12}qg_{32} + \frac{1}{2}g_{21}h_{21} + \frac{1}{12}h_{32}i\omega_0 + \frac{1}{4}h_{21}\bar{g}_{21}\right)w^3\bar{w}^2 + \ldots .$$

Por outro lado,

$$F(H(w,\bar{w})) = A(q)w + A(\bar{q})\bar{w} + w^2(\frac{1}{2}B(q,q) + \frac{1}{2}A(h_{20})) + \bar{w}^2(\frac{1}{2}B(\bar{q},\bar{q}) + \frac{1}{2}A(h_{02}))+$$

$$w\bar{w}(B(q,\bar{q}) + A(h_{11})) + w^3(\frac{1}{6}C(q,q,q) + \frac{1}{2}B(h_{20},q) + \frac{1}{6}A(h_{30})) + w^2\bar{w}(\frac{1}{2}C(\bar{q},q,q)+$$

$$B(h_{11},q) + \frac{1}{2}B(\bar{q},h_{20}) + \frac{1}{2}A(h_{21})) + w\bar{w}^2(\frac{1}{2}C(q,\bar{q},\bar{q}) + B(h_{11},\bar{q}) + \frac{1}{2}B(q,h_{02}) + \frac{1}{2}A(h_{12}))+$$

$$\bar{w}^3(\frac{1}{6}C(\bar{q},\bar{q},\bar{q}) + \frac{1}{2}B(h_{02},\bar{q}) + \frac{1}{6}A(h_{03})) + w^4(\frac{1}{24}D(q,q,q,q) + \frac{1}{4}C(h_{20},q,q) + \frac{1}{6}B(h_{30},q)+$$

$$\frac{1}{8}B(h_{20},q,q) + \frac{1}{24}A(h_{40})) + w^3\bar{w}(\frac{1}{6}D(\bar{q},q,q,q) + \frac{1}{2}C(h_{11},q,q) + \frac{1}{2}C(\bar{q},h_{20},q) + \frac{1}{2}B(h_{21},q)+$$

$$\frac{1}{2}B(h_{11},h_{20}) + \frac{1}{6}B(\bar{q},h_{30}) + \frac{1}{6}A(h_{31})) + w^2\bar{w}^2(\frac{1}{4}D(\bar{q},\bar{q},q,q) + \frac{1}{4}C(h_{02},q,q) + C(\bar{q},h_{11},q)+$$

$$\frac{1}{2}B(h_{12},q) + \frac{1}{2}B(h_{11},h_{11}) + \frac{1}{4}C(\bar{q},\bar{q},h_{20}) + \frac{1}{4}B(h_{02},h_{20}) + \frac{1}{2}B(\bar{q},h_{21}) + \frac{1}{4}A(h_{22}))+$$

$$\bar{w}^4(\frac{1}{24}D(\bar{q},\bar{q},\bar{q},\bar{q}) + \frac{1}{4}C(h_{02},\bar{q},\bar{q}) + \frac{1}{6}B(h_{03},\bar{q}) + \frac{1}{8}B(h_{02},h_{02}) + \frac{1}{24}A(h_{04})) + w\bar{w}^3(\frac{1}{6}D(q,\bar{q},\bar{q},\bar{q})+$$

$$\frac{1}{2}C(h_{11},\bar{q},\bar{q}) + \frac{1}{2}C(q,h_{02},\bar{q}) + \frac{1}{2}B(h_{12},\bar{q}) + \frac{1}{6}B(q,h_{03}) + \frac{1}{2}B(h_{02},h_{11}) + \frac{1}{6}A(h_{13}))+$$

$$w^3\bar{w}^2(\frac{1}{12}E(\bar{q},\bar{q},q,q,q) + \frac{1}{12}D(h_{02},q,q,q) + \frac{1}{2}D(\bar{q},h_{11},q,q) + \frac{1}{4}(D(\bar{q},\bar{q},h_{20},q) + \frac{1}{4}C(h_{12},q,q)+$$

$$\frac{1}{2}C(h_{11},h_{11},q) + \frac{1}{4}C(h_{02},h_{20},q) + \frac{1}{2}C(\bar{q},h_{21},q) + \frac{1}{2}C(\bar{q},h_{11},h_{20}) + \frac{1}{12}C(\bar{q},\bar{q},h_{30}) + \frac{1}{4}B(h_{22},q)+$$

$$\frac{1}{4}B(h_{12},h_{20}) + \frac{1}{2}B(h_{11},h_{21}) + \frac{1}{12}B(h_{02},h_{30}) + \frac{1}{6}B(\bar{q},h_{31}) + \frac{1}{12}A(h_{32})).$$

Aplicando $(H_w w' + H_{\bar{w}}\bar{w}')$ e $F(H(w,\bar{w}))$ em (1.41), obtemos

$$\begin{cases} q i\omega_0 &= A(q), \\ \bar{q} i\omega_0 &= -A(\bar{q}), \\ h_{20} &= (2i\omega_0 I_n - A)^{-1} B(q,q), \\ h_{11} &= -A^{-1}(B(q,\bar{q})), \\ h_{02} &= (-2i\omega_0 I_n - A)^{-1} B(\bar{q},\bar{q}), \\ h_{30} &= (3i\omega_0 I_n - A)^{-1}(C(q,q,q) + 3B(h_{20},q)), \\ h_{03} &= (-3i\omega_0 I_n - A)^{-1}(C(\bar{q},\bar{q},\bar{q}) + 3B(h_{02},\bar{q})), \end{cases} \tag{1.43}$$

onde I_n é a matriz identidade $n \times n$.

Para o termo h_{21} obtemos um sistema singular

$$(i\omega_0 I_n - A)h_{21} = C(\bar{q},q,q) - g_{21}q + 2B(h_{11},q) + B(\bar{q},h_{20}), \tag{1.44}$$

que possuirá solução se, e somente se,

$$\langle p, C(\bar{q},q,q) - g_{21}q + 2B(h_{11},q) + B(\bar{q},h_{20}) \rangle = 0.$$

Deste modo,

$$g_{21} = \langle p, C(\bar{q},q,q) + 2B(h_{11},q) + B(\bar{q},h_{20}) \rangle ,$$

onde h_{11} e h_{20} são dados em (1.43).

Sendo assim, de acordo com (1.31), o primeiro coeficiente de Lyapunov é dado por

$$l_1 = \frac{\mathrm{Re}\, c_1(0)}{\omega_0} = \frac{1}{2\omega_0} \mathrm{Re}\, g_{21},$$

ou seja,

$$l_1 = \frac{1}{2\omega_0}\mathrm{Re}\,[\langle p, C(\bar{q}, q, q)\rangle + 2\,\langle p, B(h_{11}, q)\rangle + \langle p, B(\bar{q}, h_{20})\rangle]. \tag{1.45}$$

Para encontrarmos o valor de h_{21} basta resolver o seguinte sistema

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} C(\bar{q}, q, q) - g_{21}q + 2B(h_{11}, q) + B(\bar{q}, h_{20}) \\ 0 \end{pmatrix}, \tag{1.46}$$

tal que $\langle p, h_{21}\rangle = 0$.

Lema 1.2.2. *O sistema (1.46) é não singular, e se (ν, r) é solução, tal que $\langle p, \nu\rangle = 0$, ν é solução de (1.44).*

Demonstração. Sabemos que T^c e T^{su} são, respectivamente, autoespaço generalizado de A correspondente aos autovalores com parte real nula e autovalores com parte real não nula, ambos invariantes por A. Sendo assim, escrevamos $\mathbb{R}^n = T^c \oplus T^{su}$, pelo Lema 1.2.1 temos que $\nu \in T^{su}$ se, e somente se, $\langle p, \nu\rangle = 0$.

Defina

$$v = C(\bar{q}, q, q) - g_{21}q + 2B(h_{11}, q) + B(\bar{q}, h_{20}).$$

Seja (ν, r) a solução da equação obtida a partir de (1.46). Equivalentemente,

$$\begin{aligned} (i\omega_0 I_n - A)\nu + rq &= 0, \\ \langle p, \nu\rangle &= 0. \end{aligned} \tag{1.47}$$

Da segunda equação de (1.47) segue que $\nu \in T^{su}$, e consequentemente, $(i\omega_0 I_n - A)\nu \in T^{su}$. Portanto, $\langle p, (i\omega_0 I_n - A)\nu\rangle = 0$.

Agora, do produto interno de p com o primeiro termo de (1.47), temos

$$\begin{aligned}
&\langle p, (i\omega_0 I_n - A)\nu + rq \rangle = 0 \\
\Rightarrow &\langle p, (i\omega_0 I_n - A)\nu \rangle + r\langle p, q \rangle = 0 \\
\Rightarrow &\, r\langle p, q \rangle = 0 \\
\Rightarrow &\, r = 0
\end{aligned}$$

pois como sabemos $\langle p, q \rangle = 1$ e $\langle p, (i\omega_0 I_n - A)\nu \rangle = 0$.

Substituindo $r = 0$, na primeira equação de (1.47), temos que

$$(i\omega_0 I_n - A)\nu = 0 \quad \Rightarrow \nu = \alpha q, \tag{1.48}$$

onde $\alpha \in \mathbb{C}$. No entanto,

$$0 = \langle p, \nu \rangle = \langle p, \alpha q \rangle = \alpha \langle p, q \rangle = \alpha,$$

o que implica, de (1.48), que $\nu = 0$. Assim, $(\nu, r) = (0, 0)$, mostrando que de fato, o sistema (1.46) é não singular.

Considere agora (ν, r) solução de (1.46). Então temos que

$$\begin{aligned}
&(i\omega_0 I_n - A)\nu + rq = v, \\
&\langle p, \nu \rangle = 0.
\end{aligned} \tag{1.49}$$

Da segunda equação de (1.49), segue que $v \in T^{su}$, e que

$$\begin{aligned}
&(i\omega_0 I_n - A)\nu \in T^{su} \\
\Rightarrow &\langle p, (i\omega_0 I_n - A)\nu \rangle = 0.
\end{aligned}$$

Fazendo o produto interno de p com a primeira equação de (1.49) temos que

$$\begin{aligned}
&\langle p, (i\omega_0 I_n - A)\nu + rq \rangle = \langle p, v \rangle \\
\Rightarrow &\langle p, (i\omega_0 I_n - A)\nu \rangle + r\langle p, q \rangle = \langle p, v \rangle .
\end{aligned}$$

Como $\langle p, v\rangle = 0$, $\langle p, q\rangle = 1$ e $\langle p, (i\omega_0 I_n - A)\nu\rangle = 0$, segue que $r = 0$. Substituindo $r = 0$ na primeira equação de (1.49) obtemos

$$(i\omega_0 I_n - A)\nu = v.$$

Logo, ν é solução de (1.44).

□

Observação 1.2.2. *O termo h_{32} é obtido de forma análoga.*

Os termos seguintes, serão necessários para calcularmos o segundo coeficiente de Lyapunov.

$$\begin{cases} h_{40} = (4i\omega_0 I_n - A)^{-1}(D(q,q,q,q) + 6C(h_{20},q,q) + 4B(h_{30},q) + 3B(h_{20},h_{20}), \\ h_{31} = (2i\omega_0 I_n - A)^{-1}(D(\bar{q},q,q,q) + 3C(h_{11},q,q) + 3C(\bar{q},h_{20},q) + 3B(h_{21},q) \\ \quad -3g_{21}h_{20} + 3B(h_{11},h_{20}) + B(\bar{q},h_{30})), \\ h_{22} = -A^{-1}(D(\bar{q},\bar{q},q,q) + C(h_{02},q,q) + 4C(\bar{q},h_{11},q) + 2B(h_{12},q) + 2B(h_{11},h_{11}) \\ \quad +C(\bar{q},\bar{q},h_{20}) + B(h_{02},h_{20}) + 2B(\bar{q},h_{21}) - 2h_{11}(g_{21}+\bar{g}_{21})), \\ h_{13} = (-2i\omega_0 I_n - A)^{-1}(D(q,\bar{q},\bar{q},\bar{q}) + 3C(h_{11},\bar{q},\bar{q}) + 3C(q,h_{02},\bar{q}) + 3B(h_{12},\bar{q}) \\ \quad +B(q,h_{03}) + 3B(h_{02},h_{11}) - 3h_{02}\bar{g}_{21})), \\ h_{04} = (-4i\omega_0 I_n - A)^{-1}(D(\bar{q},\bar{q},\bar{q},\bar{q}) + 6C(h_{02},\bar{q},\bar{q}) + 4B(h_{03},\bar{q}) + 3B(h_{02},h_{02}). \end{cases} \quad (1.50)$$

Para $l_1 = 0$, devemos ter $g_{21} + \bar{g}_{21} = 0$, donde o último termo de h_{22} se torna nulo.

O termo singular associado a h_{32}, é dado por

$$\begin{aligned}(i\omega_0 I_n - A)h_{32} = {} & E(\bar{q},\bar{q},q,q,q) + D(h_{02},q,q,q) + 6D(\bar{q},h_{11},q,q) + 3C(h_{12},q,q) \\ & +6C(h_{11},h_{11},q) + 3D(\bar{q},\bar{q},h_{20},q) + 3C(h_{02},h_{20},q) + 6C(\bar{q},h_{21},q) \\ & +3B(h_{22},q) + 6C(\bar{q},h_{11},h_{20}) + 3B(h_{12},h_{20}) - 6g_{21}h_{21} + 6B(h_{11},h_{21}) \\ & +6C(\bar{q},\bar{q},h_{30}) + B(h_{02},h_{30}) + 2B(\bar{q},h_{31}) - 3h_{21}\bar{g}_{21} - g_{32}q.\end{aligned}$$

Fazendo

$$\begin{aligned}H_{32} = {} & E(\bar{q},\bar{q},q,q,q) + D(h_{02},q,q,q) + 6D(\bar{q},h_{11},q,q) + 3C(h_{12},q,q) \\ & +6C(h_{11},h_{11},q) + 3D(\bar{q},\bar{q},h_{20},q) + 3C(h_{02},h_{20},q) + 6C(\bar{q},h_{21},q) \\ & +3B(h_{22},q) + 6C(\bar{q},h_{11},h_{20}) + 3B(h_{12},h_{20}) - 6g_{21}h_{21} + 6B(h_{11},h_{21}) \\ & +6C(\bar{q},\bar{q},h_{30}) + B(h_{02},h_{30}) + 2B(\bar{q},h_{31}) - 3h_{21}\bar{g}_{21},\end{aligned}$$

podemos reescrever

$$(i\omega_0 I_n - A)h_{32} = H_{32} - g_{32}q,$$

que possui solução se, e somente se,

$$\begin{aligned}\langle p, H_{32} - g_{32}q\rangle &= 0 \\ g_{32} &= \langle p, H_{32}\rangle\,,\end{aligned}$$

sendo que os termos $-6g_{21}h_{21}$ e $-3h_{21}\bar{g}_{21}$ não entram na última equação pois, $\langle p, h_{21}\rangle = 0$.

Sendo assim, de acordo com a equação (1.31), o segundo coeficiente de Lyapunov é definido por

$$l_2 = \frac{\mathrm{Re}\, c_2(0)}{\omega_0} = \frac{1}{12\omega_0}\mathrm{Re}\, g_{32},$$

ou seja,

$$l_2 = \frac{1}{12\omega_0}\text{Re}\,[\langle p, E(q,q,q,\bar{q},\bar{q}) + D(q,q,q,\bar{h}_{20}) + 3D(q,\bar{q},\bar{q},h_{20}) + 6D(q,q,\bar{q},h_{11})$$
$$+C(\bar{q},\bar{q},h_{30}) + 3C(q,q,\bar{h}_{21}) + 6C(q,\bar{q},h_{21}) + 3C(q,\bar{h}_{20},h_{20})$$
$$+6C(q,h_{11},h_{11}) + 6C(\bar{q},h_{20},h_{11}) + 2B(\bar{q},h_{31}) + 3B(q,h_{22})$$
$$+B(\bar{h}_{20},h_{30}) + 3B(\bar{h}_{21},h_{20}) + 6B(h_{11},h_{21})\rangle]. \tag{1.51}$$

Considere agora a equação diferencial

$$\dot{\mathbf{x}} = f(\mathbf{x},\xi), \tag{1.52}$$

onde $\mathbf{x} \in \mathbb{R}^n$ e $\xi \in \mathbb{R}^m$, são respectivamente vetores representados pelas variáveis e parâmetros. Assuma que f seja de classe $\mathbb{C}^\infty$ em $\mathbb{R}^n \times \mathbb{R}^m$. Suponha que (1.52) tenha um ponto de equilíbrio $\mathbf{x} = e_0$ quando $\xi = \xi_0$ e , denotando a variável $\mathbf{x} - e_0$ também por $\mathbf{x}$, escrevemos

$$F(\mathbf{x}) = f(\mathbf{x},\xi_0).$$

Temos que $F(\mathbf{x})$ é uma função de $\mathbf{x}$ suave, com respeito a ξ com sua expansão de Taylor dada por (1.19) e $A(\xi) = f_{\mathbf{x}}(0,\xi_0)$ corresponde à parte linear do sistema com um par de autovalores complexos

$$\lambda_1(\xi) = \lambda(\xi),\;\; \lambda_2(\xi) = \bar{\lambda}(\xi),$$

onde

$$\lambda(\xi) = \gamma(\xi) + i\omega(\xi),$$

satisfazendo a condição de Hopf para $\xi = 0$

$$\gamma(0) = 0, \;\; \omega(0) = \omega_0 > 0.$$

Sendo assim resumidamente escrevemos:

- Um ponto de Hopf e_0 é um ponto de equilíbrio de (1.52) onde a matriz Jacobiana $A = f_{\mathbf{x}}(e_0, \xi_0)$ tem um par de autovalores puramente imaginários $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, e não tem outros autovalores críticos.

- Num ponto de Hopf, a variedade central bidimensional está bem definida, e é invariante pelo fluxo gerado por (1.52) e pode ser estendida suavemente a valores do parâmetro numa vizinhança deste ponto.

- Um ponto de Hopf é chamado de transversal se as curvas de autovalores complexos intersectarem o eixo imaginário com derivadas não nulas.

- Numa vizinhança de um ponto de Hopf transversal (ponto H1) com $l_1 \neq 0$ o comportamento do sistema dinâmico (1.52), reduzido a família parâmetro-dependente da variedade central, é orbitalmente topologicamente equivalente a forma normal complexa
$$w' = (\gamma + i\omega)w + l_1 w|w|^2,$$
com $w \in \mathbb{C}$.

- Quando $l_1 < 0\,(l_1 > 0)$ uma família de órbitas periódicas estáveis (instáveis) podem ser encontradas nesta família de variedades, reduzindo a um ponto de equilíbrio em H1.

- Um ponto de Hopf de codimensão 2 é um ponto de Hopf onde l_1 se anula. Este é chamado transversal se $\gamma = 0$ e $l_1 = 0$ tem intersecção transversal, onde $\gamma = \gamma(\xi)$ é a parte real do autovalor crítico.

- Em uma vizinhança de um ponto de Hopf transversal de codimensão 2 (ponto H2) com $l_2 \neq 0$ a dinâmica do sistema (1.52), reduz-se a uma família parâmetro-dependente de variedades centrais e é orbitalmente topologicamente equivalente a

  $$w' = (\gamma + i\omega)w + \eta w|w|^2 + l_2 w|w|^4,$$

  onde γ e η podem ser entendidos como parâmetros. Veja [7]. O diagrama de bifurcação para $l_2 \neq 0$ pode ser encontrado em [7], p. 313, e em [15].

Os próximos teoremas nos mostram como verificar a condição de transversalidade para a bifurcação de Hopf genérica e a bifurcação de Hopf degenerada.

Teorema 1.2.2. (**A condição de Transversalidade**) *Considere o sistema (1.52), cuja matriz Jacobiana $A(\xi)$ possui um par de autovalores puramente imaginários para $\xi = 0$, $\lambda_{1,2} = \gamma(\xi) \pm i\omega(\xi)$, $\gamma(0) = 0$, $\omega(0) = \omega_0 > 0$. Então,*

$$\gamma'(0) = \text{Re } \langle p, A'(0)q \rangle ,$$

onde p, $q \in \mathbb{C}^n$ satisfazem

$$\begin{aligned} &A(0)q = i\omega_0 q, \\ &A^\top(0)p = -i\omega_0 p, \\ &\langle p, q \rangle = 1. \end{aligned}$$

Demonstração. Derivando ambos os membros da equação

$$A(\xi)q(\xi) = \lambda(\xi)q(\xi).$$

com relação a ξ, obtemos

$$A'(\xi)q(\xi) + A(\xi)q'(\xi) = \lambda'(\xi)q(\xi) + \lambda(\xi)q'(\xi).$$

Aplicando, agora, o produto escalar por p em ambos os membros, temos

$$\begin{aligned}
&\langle p, A'q + Aq' \rangle = \langle p, \lambda' q + \lambda q' \rangle \\
&\langle p, A'q \rangle + \langle p, Aq' \rangle = \langle p, \lambda' q \rangle + \langle p, \lambda q' \rangle \\
&\langle p, A'q \rangle + \left\langle A^{\top} p, q' \right\rangle = \lambda' \langle p, q \rangle + \lambda \langle p, q' \rangle .
\end{aligned}$$

Para $\xi = 0$, $A^{\top} p = -i\omega_0 p$, portanto

$$\begin{aligned}
&\langle p, A'(0)q \rangle + i\omega_0 \langle p, q' \rangle = (\gamma'(0) + i\omega'(0)) \langle p, q \rangle + i\omega_0 \langle p, q' \rangle \\
\Rightarrow &\langle p, A'(0)q \rangle = (\gamma'(0) + i\omega'(0)) \langle p, q \rangle \\
\Rightarrow &\langle p, A'(0)q \rangle = \gamma'(0) + i\omega'(0)
\end{aligned}$$

pois, $\langle p, q \rangle = 1$. $\square$

Capítulo 2

O estudo da bifurcação de Hopf em um sistema tridimensional

Neste capítulo estudaremos o comportamento de um sistema dinâmico tridimensional onde faremos uma análise das condições que serão necessárias para a existência de uma bifurcação de Hopf e ciclos limites. É importante ressaltar que neste capítulo foi feita uma versão correta de [1]. Na última seção apresentaremos um caso particular do caso geral, onde foi feito uma análise e correção dos resultados de tal artigo.

2.1 Análise do sistema

Considere o sistema dado por

$$\begin{cases} \dot{x} = \nu x - y^2, \\ \dot{y} = \mu(z - y), \\ \dot{z} = ay - bz + xy, \end{cases} \tag{2.1}$$

onde $(x, y, z) \in \mathbb{R}^3$ são as variáveis de estado, e ν, μ, a e b são todos parâmetros reais em $\mathcal{D} = \{(\nu, \mu, a, b), \nu < 0, \mu > 0, a > 0 \text{ e } b > 0\}$.

Note que se $b - a \geq 0$ o sistema (2.1) possuirá um único ponto de equilíbrio em $e_1 = (0, 0, 0)$, de modo que o caso de interesse será para $b - a < 0$. Neste caso o sistema

(2.1) possuirá três equilíbrios, sendo eles

$$e_1 = (0, 0, 0),$$

$$e_2 = (b - a, \sqrt{\nu(b - a)}, \sqrt{\nu(b - a)}),$$

$$e_3 = (b - a, -\sqrt{\nu(b - a)}, -\sqrt{\nu(b - a)}).$$

A matriz Jacobiana do sistema (2.1) é dada por

$$J(x, y, z) = \begin{pmatrix} \nu & -2y & 0 \\ 0 & -\mu & \mu \\ y & a + x & -b \end{pmatrix}.$$

Observação 2.1.1. *Note que:*

(i) Para o ponto $e_1 = (0, 0, 0)$ a matriz jacobiana acima não tem autovalores complexos, de modo que, para este ponto, o sistema (2.1) não possuirá uma bifurcação de Hopf.
(ii) O sistema (2.1) numa vizinhança de e_2 é topologicamente equivalente ao sistema (2.1) numa vizinhança de e_3, sendo assim focaremos os estudos em torno do equilíbrio $e^* = e_2$ e os resultados para $e^* = e_3$ serão análogos.

Calculando a matriz Jacobiana de (2.1) no ponto e^* obtemos

$$J(e^*) = \begin{pmatrix} \nu & -2\sqrt{\nu(b - a)} & 0 \\ 0 & -\mu & \mu \\ \sqrt{\nu(b - a)} & b & -b \end{pmatrix}. \tag{2.2}$$

O polinômio característico de $J(e^*)$ é dado por

$$-P(\lambda) = \lambda^3 + (b + \mu - \nu)\lambda^2 - \nu(b + \mu)\lambda + 2(b - a)\mu\nu.$$

Sejam λ_1, λ_2, λ_3 os autovalores de $J(e^*)$ tais que $\lambda_1 = \bar{\lambda}_2 \in \mathbb{C}$ e $\lambda_3 \in \mathbb{R}$.

Lema 2.1.1. *O ponto e^* é um equilíbrio:*

(i) repulsor, se $a > a_c$,

(ii) atrator, se $a < a_c$,

(iii) não hiperbólico, se $a = a_c$.

onde a_c é dado por

$$a_c = \frac{b^2 + 4b\mu + \mu^2 - \nu(b+\mu)}{2\mu}. \tag{2.3}$$

Demonstração. Visto que $\nu < 0$, $\mu > 0$, $a > 0$ e $b > 0$ e como $\lambda_3 = -\mu - b + \nu < 0$ temos:

(i) se $a > a_c$, então $\operatorname{Re}\lambda_{1,2} > 0$ e teremos um equilíbrio repulsor,

(ii) se $a < a_c$, então $\operatorname{Re}\lambda_{1,2} < 0$ e teremos um equilíbrio atrator,

(iii) se $a = a_c$, então $\operatorname{Re}\lambda_{1,2} = 0$ e teremos um equilíbrio não hiperbólico.

□

Assim para $(\nu,\ \mu,\ a,\ b) \in D_0 = \{(\nu,\ \mu,\ a,\ b),\ a = a_c,\ \nu < 0,\ \mu > 0 \text{ e } b > 0\}$ a matriz $J(e^*)$ possui dois autovalores puramente imaginários. Logo, para os parâmetros em D_0 pode ocorrer uma bifurcação de Hopf.

2.2 O primeiro coeficiente de Lyapunov

Para parâmetros em D_0, temos que a parte linear do sistema (2.1) aplicado no equilíbrio e^* possui os seguintes autovalores

$$\lambda_1 = i\omega_0,\ \ \lambda_2 = -i\omega_0,\ \ \lambda_3 = -b - \mu + \nu,$$

onde $\omega_0 = \sqrt{-\nu(b+\mu)} > 0$.

Denote por q o autovetor de $J(e^*)$ correspondente a $\lambda_1 = i\omega_0$. Assim

$$q = \left(-\frac{\sqrt{2}h\mu}{(\mu + i\omega_0)(-\nu + i\omega_0)},\ \ \frac{\mu}{\mu + i\omega_0},\ \ 1 \right), \tag{2.4}$$

e por p o autovetor de $J^\top(e^*)$ correspondente a $\lambda_2 = -i\omega_0$. Assim, normalizando p para que $\langle p, q\rangle = 1$, obtemos

$$p = \left(\frac{1}{1 + \dfrac{i\omega_0}{\mu + i\omega_0} + \dfrac{b + \dfrac{h^2\mu}{(i\nu + \omega_0)^2}}{\mu + i\omega_0}} \right) \left(\frac{-h}{\sqrt{2}(\nu + i\omega_0)}, \quad \frac{b - i\omega_0}{\mu}, \quad 1 \right), \tag{2.5}$$

onde

$$h = \sqrt{\frac{(b+\mu)(-b-\mu+\nu)\nu}{\mu}}. \tag{2.6}$$

Lema 2.2.1. *Sejam* $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{z} = (z_1, z_2, z_3)$, $\mathbf{u} = (u_1, u_2, u_3)$ *e* $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. *As funções multilineares B, C, D e E para o sistema (2.1), são dadas por*

$$B(\mathbf{x},\mathbf{y}) = \begin{pmatrix} -2x_2y_2 \\ 0 \\ x_1y_2 + x_2y_1 \end{pmatrix},$$

$$C(\mathbf{x},\mathbf{y},\mathbf{z}) = D(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) = E(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Demonstração. Utilizaremos as fórmulas do Capítulo 1 para encontrarmos a funções coordenadas de B. Assim,

$$B_i(\mathbf{x},\ \mathbf{y}) = \sum_{j,k=1}^{3} \left. \frac{\partial^2 F_i(\eta, 0)}{\partial\eta_j \partial\eta_k} \right|_{\eta=0} \mathbf{x}_j \mathbf{y}_k,$$

Para encontrarmos B_1 consideramos $F_1(x,y) = -y^2$ e calculamos

$$\frac{\partial F_1(x,y)}{\partial x} = 0, \qquad \frac{\partial F_1(x,y)}{\partial y} = -2y,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F_1(x,y)}{\partial x}\right) = 0, \qquad \frac{\partial}{\partial x}\left(\frac{\partial F_1(x,y)}{\partial y}\right) = 0,$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F_1(x,y)}{\partial x}\right) = 0, \quad \frac{\partial}{\partial y}\left(\frac{\partial F_1(x,y)}{\partial y}\right) = -2.$$

Assim temos que,

$$B_1(\mathbf{x{,}y}) = 0x_1y_1 + 0x_1y_2 + 0x_2y_1 - 2x_2y_2.$$

Analogamente podemos obter as funções B_2, B_3. As outras funções multilineares C, D e E são nulas, afinal, não há termos de ordem maior que 2, concluindo o resultado. □

A fim de facilitar a leitura, usaremos os cálculos apresentados no Apêndice I para mostrar os resultados relativos ao primeiro e segundo coeficientes de Lyapunov, uma vez que as contas são muito extensas.

Temos de (1.45) que

$$l_1 = \frac{1}{2\omega_0}\text{Re}\left[\langle p, C(\bar{q},q,q)\rangle + 2\langle p, B(h_{11},q)\rangle + \langle p, B(\bar{q},h_{20})\rangle\right].$$

Teorema 2.2.1. *O primeiro coeficiente de Lyapunov do sistema* (2.1) *em* D_0 *e para valores de* q *e* p*, dados em (2.4) e (2.5), é*

$$l_1 = \frac{2\mu^3\nu((b+\mu)^2 + 12(b+\mu)\nu - \nu^2)}{(b+\mu-\nu)(-\mu^2+(b+\mu)\nu)((b+\mu)^2 - 6(b+\mu)\nu + \nu^2)((b+\mu)^2 - 3(b+\mu)\nu + \nu^2)}.$$

Sendo assim, temos que:

(i) se $l_1 > 0$, o equilíbrio e^* será um repulsor fraco. Além disso, para $a < a_c$ surgirá uma órbita periódica repulsora envolvendo o equilíbrio atrator;
(ii) se $l_1 < 0$, o equilíbrio e^* será um atrator fraco. Além disso, para $a > a_c$ surgirá uma órbita periódica atratora envolvendo o equilíbrio repulsor;
(iii) se $l_1 = 0$, nada se pode afirmar.

Vemos através da Figura 2.1, um esboço da região onde l_1 pode se anular, isto é, quando

$$(b+\mu)^2 + 12(b+\mu)\nu - \nu^2 = 0.$$

Na Figura 2.1 a região mostrada tem as seguintes variações dos parâmetros $0 < b \leq 4$, $0 < \mu \leq 4$ e $-1 \leq \nu < 0$. Sobre essa região será necessário calcularmos o segundo coeficiente de Lyapunov.

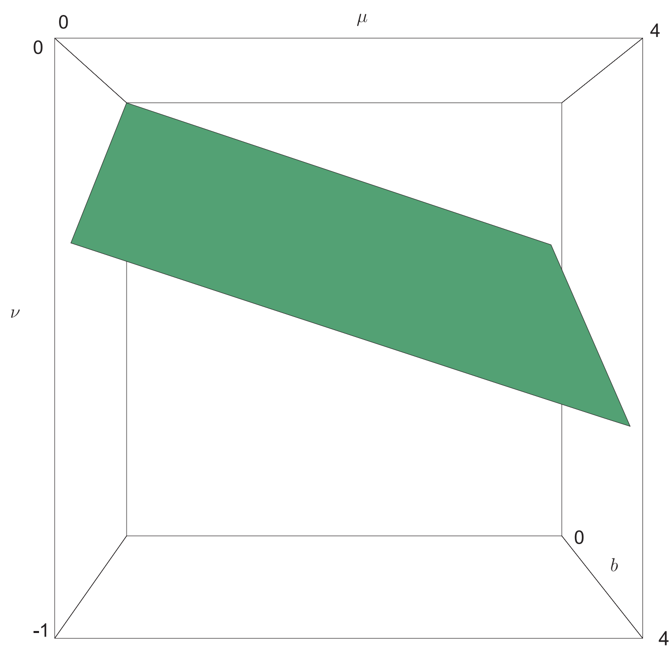

Figura 2.1: Gráfico de $l_1 = 0$ do sistema (2.1).

De acordo com a equação (1.51) temos que

$$l_2 = \frac{1}{12\omega_0}\mathrm{Re}\,[\langle p, E(q,q,q,\bar{q},\bar{q}) + D(q,q,q,\bar{h}_{20}) + 3D(q,\bar{q},\bar{q},h_{20}) + 6D(q,q,\bar{q},h_{11})$$
$$+C(\bar{q},\bar{q},h_{30}) + 3C(q,q,\bar{h}_{21}) + 6C(q,\bar{q},h_{21}) + 3C(q,\bar{h}_{20},h_{20})$$
$$+6C(q,h_{11},h_{11}) + 6C(\bar{q},h_{20},h_{11}) + 2B(\bar{q},h_{31}) + 3B(q,h_{22})$$
$$+B(\bar{h}_{20},h_{30}) + 3B(\bar{h}_{21},h_{20}) + 6B(h_{11},h_{21})\rangle].$$

Teorema 2.2.2. *O segundo coeficiente de Lyapunov do sistema* (2.1) *em* D_0 *e para valores de* q *e* p*, dados em (2.4) e (2.5), é*

$$l_2 = \frac{(\mu^6(9(b+\mu)^{14}-277(b+\mu)^{13}\nu-608(b+\mu)^{12}\nu^2+54271(b+\mu)^{11}\nu^3-522318(b+\mu)^{10}\nu^4}{(9(b+\mu)\nu(\mu^2-(b+\mu)\nu)^2((b+\mu)^2-11(b+\mu)\nu+\nu^2)((b+\mu)^5-10(b+\mu)^4\nu+29(b+\mu)^3\nu^2-29(b+\mu)^2\nu^3+10(b+\mu)\nu^4-\nu^5)^3)}$$

$$\frac{+2396645(b+\mu)^9\nu^5-5068501(b+\mu)^8\nu^6+2025066(b+\mu)^7\nu^7+4809361(b+\mu)^6\nu^8-4193735(b+\mu)^5\nu^9}{(9(b+\mu)\nu(\mu^2-(b+\mu)\nu)^2((b+\mu)^2-11(b+\mu)\nu+\nu^2)((b+\mu)^5-10(b+\mu)^4\nu+29(b+\mu)^3\nu^2-29(b+\mu)^2\nu^3+10(b+\mu)\nu^4-\nu^5)^3)}$$

$$\frac{+601386(b+\mu)^4\nu^{10}+172127(b+\mu)^3\nu^{11}-49228(b+\mu)^2\nu^12+4015(b+\mu)\nu^{13}-117\nu^{14}))}{(9(b+\mu)\nu(\mu^2-(b+\mu)\nu)^2((b+\mu)^2-11(b+\mu)\nu+\nu^2)((b+\mu)^5-10(b+\mu)^4\nu+29(b+\mu)^3\nu^2-29(b+\mu)^2\nu^3+10(b+\mu)\nu^4-\nu^5)^3)}.$$

Sendo assim, temos que para $l_1 = 0$, $l_2 > 0$, o equilíbrio e^* será um repulsor fraco. Veja o diagrama de bifurcação na Figura 1.5.

Observe a Figura 2.2 onde aparece a região de $l_2 = 0$ com as seguintes variações dos parâmetros $0 < b \leq 4$, $0 < \mu \leq 4$ e $-1 \leq \nu < 0$ e note pela Figura 2.3 que as regiões onde l_1 e l_2 se anulam não se interceptam. Concluindo então que a bifurcação de Hopf do sistema (2.1) é não-degenerada.

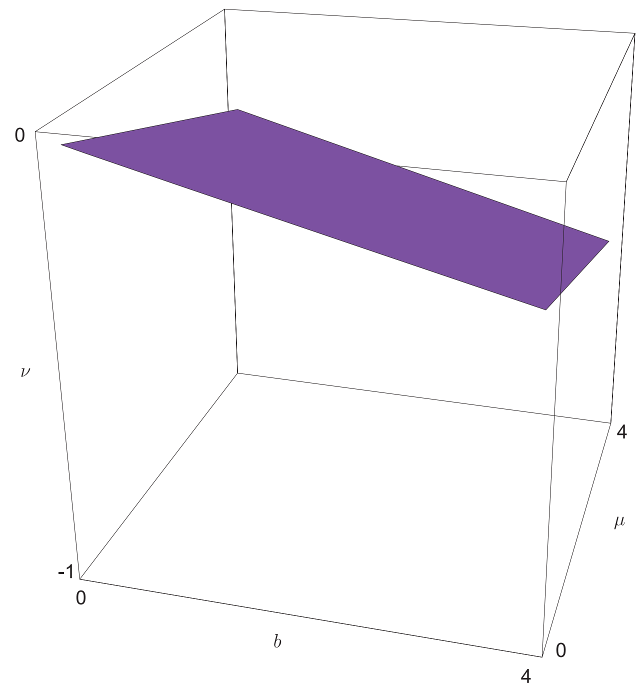

Figura 2.2: Gráfico de $l_2 = 0$ do sistema (2.1).

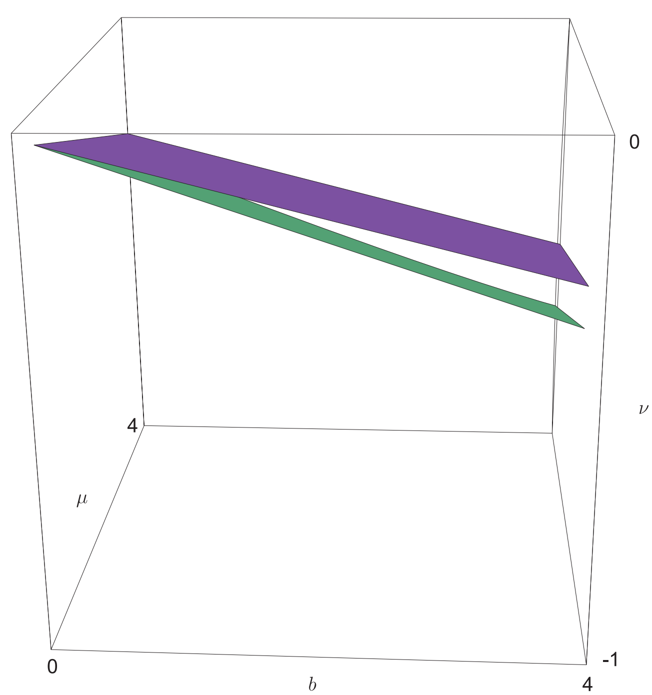

Figura 2.3: Gráfico de $l_1 = 0$ e $l_2 = 0$ do sistema (2.1).

2.3 A condição de Transversalidade

Teorema 2.3.1. *Considere o sistema (2.1) com parâmetros em D, temos então que*

$$\gamma'(a_c) = -\frac{\mu\nu}{(b+\mu)^2 - 3(b+\mu)\nu + \nu^2} \neq 0,$$

satisfazendo a condição de transversalidade.

Demonstração. Temos que

$$\left.\frac{\partial J(e^*)}{\partial a}\right|_{a=a_c} = J'(a_c),$$

é dada por

$$J'(a_c) = \begin{pmatrix} 0 & \dfrac{\sqrt{2}\nu}{\sqrt{-\dfrac{(b+\mu)(b+\mu-\nu)\nu}{\mu}}} & 0 \\ 0 & 0 & 0 \\ \dfrac{\sqrt{-\dfrac{\mu\nu}{(b+\mu)(b+\mu-\nu)}}}{\sqrt{2}} & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Agora fazendo o cálculo para $\gamma'(a_c) = \operatorname{Re}\langle p, J'(a_c)q\rangle$, obtemos

$$\gamma'(a_c) = -\frac{\mu\nu}{(b+\mu)^2 - 3(b+\mu)\nu + \nu^2},$$

como $\mu > 0$, $b > 0$ e $\nu < 0$ temos que $\gamma'(a_c) > 0$, o que implica em $\gamma'(a_c) \neq 0$, satisfazendo assim a condição de transversalidade.

□

Com as condições de Não-Degenerescência e Transversalidade satisfeitas, conclui-se que para o sistema (2.1) existirá uma bifurcação de Hopf para $a = a_c$ com $l_1 \neq 0$.

2.4 Simulação numérica do sistema (2.1)

Considere o sistema (2.1) com os seguintes valores para os parâmetros

$$\nu = -1,$$
$$\mu = b.$$

Substituindo esses novos valores, temos de (2.3) que,

$$a_c = 1 + 3b. \tag{2.8}$$

Temos também que, $\omega_0 = \sqrt{2b}$. Calculando o primeiro coeficiente de Lyapunov obtemos

$$l_1 = \frac{2b^2(-1 + 4(-6 + b)b)}{(2 + b)(1 + 2b)(1 + 6b + 4b^2)(1 + 4b(3 + b))}.$$

Para estes valores temos que l_1 vai se anular somente em $b = 6 + \sqrt{37}/2$. Isso está ilustrado na Figura 2.4.

Suponha então $b = (6 + \sqrt{37})/2$. Assim, para esses valores de parâmetros temos o equilíbrio e^* dado por

$$e^* = \left(\frac{6 + \sqrt{37}}{2} - a, \ \sqrt{\frac{-6 - \sqrt{37}}{2} + a}, \ \sqrt{\frac{-6 - \sqrt{37}}{2} + a} \right)$$

A matriz Jacobiana $J(e^*)$ será

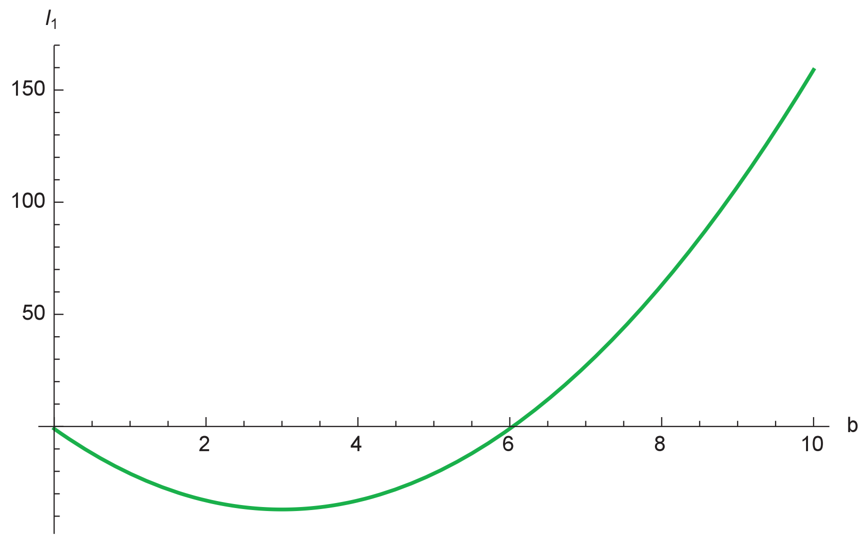

Figura 2.4: Gráfico de l_1 do sistema (2.1) para os valores $\nu = -1$ e $\mu = b$.

$$J(e^*) = \begin{pmatrix} -1 & -2\sqrt{\frac{1}{2}(-6-\sqrt{37})+a} & 0 \\ 0 & \frac{1}{2}(-6-\sqrt{37}) & \frac{1}{2}(6+\sqrt{37}) \\ \sqrt{\frac{1}{2}(-6-\sqrt{37})+a} & \frac{1}{2}(6+\sqrt{37}) & \frac{1}{2}(-6-\sqrt{37}). \end{pmatrix}.$$

Do Lema 2.1.1 temos que para $a = a_c = 10 + ((3\sqrt{37})/2)$ o equilíbrio será não hiperbólico. Assim, os autovalores de $J(e^*)$ são dados por

$$\lambda_1 = i\omega_0, \;\; \lambda_2 = -i\omega_0, \;\; \lambda_3 = -7 - \sqrt{37},$$

onde $\omega_0 = \sqrt{6 + \sqrt{37}} > 0$.

Como vimos na seção anterior, a condição de transversalidade foi satisfeita para qualquer valor dos parâmetros tal que $\nu < 0$, $\mu > 0$, $a > 0$ e $b > 0$, em particular para esses novos valores usados nesta seção. Com os cálculos feitos no Apêndice I, vemos que para tais valores $l_1 = 0$, $l_2 \cong 0.0002527 > 0$. O que implica que teremos uma bifurcação de Hopf.

Teorema 2.4.1. *Considere a família de equações diferenciais ordinárias (2.1). Então para* $\nu = -1$, $\mu = (6+\sqrt{37})/2$, $b = (6+\sqrt{37})/2$ *e* $a = 10 + ((3\sqrt{37})/2)$, *o ponto de Hopf* e^* *é um repulsor fraco. Veja Figura 1.5.*

Capítulo 3

Sistemas Dinâmicos Acoplados

Considere o sistema com duas EDO's

$$\begin{cases} \dot{x} = f_1(x, y, \beta), \\ \dot{y} = f_2(x, y, \beta), \end{cases} \quad \beta \in \mathbb{R}, \;\; x, y \in \mathbb{R} \tag{3.1}$$

em que, para todo β, $f = (f_1, f_2)$ é uma função suave. Assuma que o sistema (3.1) tem um equilíbrio $\mathbf{x} = 0$ para $\beta = 0$. Em torno de 0 o sistema (3.1) pode ser escrito como

$$\dot{\mathbf{x}} = A(\beta) + F(\mathbf{x}, \beta), \tag{3.2}$$

onde $\mathbf{x} = (x, y)^\top$, $A(\beta)$ é a matriz Jacobiana em $\mathbf{x} = 0$ e F é uma função suave cujas componentes $F_{1,2}$ tem expansão de Taylor começando com as condições no mínimo quadráticas.

Agora considere dois sistemas idênticos ao sistema (3.1), não simétricos e linearmente acoplados através de uma das variáveis dado da seguinte forma,

$$\begin{cases} \dot{x} = f_1(x, y, \beta) + c_1(x - z), \\ \dot{y} = f_2(x, y, \beta), \\ \dot{z} = f_1(z, w, \beta) + c_2(z - x), \\ \dot{w} = f_2(z, w, \beta). \end{cases} \quad c_1, c_2 \in \mathbb{R} \tag{3.3}$$

Neste capítulo obteremos novas expressões para os coeficientes de Lyapunov utilizados no estudo da Bifurcação de Hopf em torno de equilíbrios simétricos do sistema (3.3) apenas em termos de vetores 2D relacionados com sistema (3.1) e com os parâmetros de acoplamento c_1 e c_2. Desta forma o estudo da bifurcação de Hopf em (3.3) é simplificado.

3.1 O estudo qualitativo do modelo em $\mathbb{R}^2$

Iniciaremos com um sistema bem conhecido da economia que pode ser encontrado também em [3], [4] e [16],

$$\begin{cases} \dfrac{dx}{d\tau} = k - \alpha x y^2 + \beta y, \\ \\ \dfrac{dy}{d\tau} = \alpha x y^2 - \delta y, \end{cases} \tag{3.4}$$

o qual modela a dinâmica do número de usuários de uma marca de acordo com a publicidade.

Suponha que o número de pessoas no mercado seja dividido em

- $x(\tau)$: O número de potencias clientes de uma marca no tempo τ;
- $y(\tau)$: O número atual de clientes.

Tal modelo assume que a informação se espalha a partir de indivíduos que sabem da existência de uma marca ou produto para as pessoas que não tem conhecimento. Como na teoria de epidemias, a publicidade age de forma semelhante à propagação de germes. Os potencias compradores, $x(\tau)$, "contraem esses germes"por meio de uma propaganda ao entrarem em contato com os usuários da marca, $y(\tau)$.

Assim, o número de potenciais compradores que se tornam usuários passa a ser $a(\tau)x(\tau)y(\tau)$, onde $a(\tau)$ é a taxa de contato com a publicidade no tempo τ.

Seja β uma taxa constante na qual os clientes atuais mudam para uma marca concorrente. Uma vez que o indivíduo pode mudar novamente para a marca original, o mesmo permanece no grupo dos clientes potenciais. Segue que $\beta y(\tau)$ é o número de usuários que param de usar a marca e se tornam esses clientes potenciais.

Denote por k o fluxo de novos potenciais compradores que entram no mercado. Como o número de potenciais compradores cresce com k e com $\beta y(\tau)$, e decresce com $a(\tau)x(\tau)y(\tau)$ obtemos então

$$\frac{dx}{d\tau} = k - axy + \beta y.$$

Assuma que a taxa de compradores atuais que podem deixar o mercado de vez (por exemplo morte, migração) seja ε. Assim, número de tais compradores aumenta com $a(\tau)x(\tau)y(\tau)$ e decresce com $\beta y(\tau)$ e $\varepsilon y(\tau)$. Deste modo,

$$\frac{dy}{d\tau} = axy - \beta y - \varepsilon y.$$

Assuma também que a taxa de contato com a publicidade é proporcional ao número de compradores habituais, isto é, $a(\tau) = \alpha y(\tau)$ e seja $\delta = \beta + \varepsilon$.

Deste modo, explicamos como foi obtido o sistema (3.4).

O diagrama de transição do modelo de publicidade é esboçado na Figura 3.1.

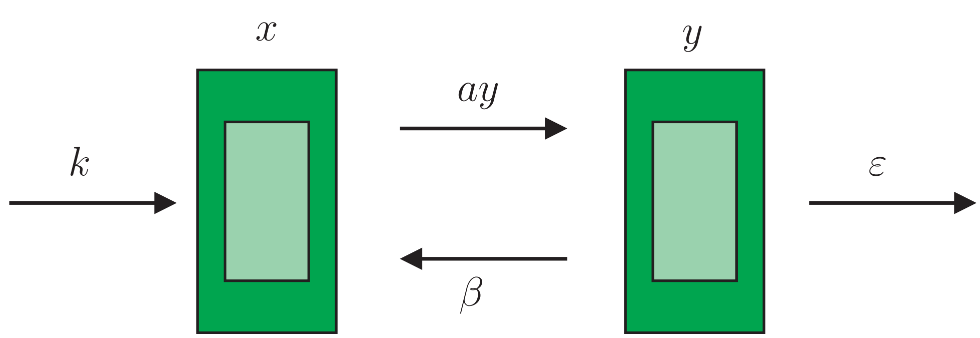

Figura 3.1: Diagrama de transição do modelo de publicidade.

Usando as transformações

$$u = \frac{\alpha k}{\delta \varepsilon} x - 1, \qquad v = \frac{\varepsilon}{k} y - 1, \qquad a = \frac{\alpha k^2}{\delta \varepsilon^2}, \qquad b = 2 - \frac{\beta}{\delta}, \qquad t = \delta \tau,$$

o sistema (3.4) pode ser escrito na forma

$$\begin{cases} \dot{u}(t) = -a(u + bv + 2uv + v^2 + uv^2), \\ \\ \dot{v}(t) = u + v + 2uv + v^2 + uv^2, \end{cases} \tag{3.5}$$

onde o ponto significa a diferenciação de u e v com respeito ao novo tempo t. Quando $\delta > \beta$ temos $b > 1$, de modo que o único caso de interesse é quando $a > 0$, $b > 1$. Neste caso, o único ponto de equilíbrio do sistema (3.5) é $(u, v) = (0, 0)$.

Lema 3.1.1. *A linearização do sistema (3.5) na origem apresenta 2 autovalores dados por*

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(1-a) + \frac{1}{2}\sqrt{(1-a)^2 - 4a(b-1)}, \\ \\ \lambda_2 &= \frac{1}{2}(1-a) - \frac{1}{2}\sqrt{(1-a)^2 - 4a(b-1)}. \end{aligned} \tag{3.6}$$

Demonstração. Seja $J(u, v)$ a matriz Jacobiana do sistema (3.5) calculada em (u, v), assim

$$J(u,v) = \begin{pmatrix} -a - 2av - av^2 & -ab - 2au - 2av - 2auv \\ \\ 1 + 2v + v^2 & 1 + 2u + 2v + 2uv \end{pmatrix}.$$

Aplicando na origem, temos

$$J(0,0) = \begin{pmatrix} -a & -ab \\ 1 & 1 \end{pmatrix}.$$

Deste modo,

$$T = \mathrm{Tr} J(0,0) = 1 - a,$$

e

$$D = \det J(0,0) = a(b-1).$$

Logo, a equação característica fica definida por

$$\lambda^2 - T\lambda + D = 0,$$

de onde os autovalores são dados por

$$\lambda_{1,2} = \frac{T}{2} \pm \frac{\sqrt{T^2 - 4D}}{2},$$

ou seja,

$$\lambda_{1,2} = \frac{1}{2}(1-a) \pm \frac{1}{2}\sqrt{(1-a)^2 - 4a(b-1)}.$$

□

Lema 3.1.2. *A origem* $(0,0)$ *é um equilíbrio:*

(i) repulsor, se $a < 1$,

(ii) atrator, se $a > 1$,

(iii) equilíbrio não hiperbólico, se $a = 1$.

Demonstração. Visto que $a > 0$ e $b > 1$, temos $a(b-1) > 0$. Assim, $-4a(b-1) < 0$, e

(i) se $a < 1$, então $\text{Re}\,\lambda_{1,2} > 0$ e teremos um equilíbrio repulsor,

(ii) se $a > 1$, então $\text{Re}\,\lambda_{1,2} < 0$ e teremos um equilíbrio atrator,

(iii) se $a = 1$, então $\text{Re}\,\lambda_{1,2} = 0$ e teremos um equilíbrio não hiperbólico.

□

Assim, para $(a,b) \in H_0 = \{(a,b), a = 1, b > 1\}$ os autovalores da matriz jacobiana, associados a (3.5), em (0,0) são puramente imaginários. Logo, para parâmetros em H_0 poder ocorrer uma bifurcação de Hopf.

Na Figura (3.2) pode ser observada a reta de Hopf definida para $a = 1$ no plano de parâmetros.

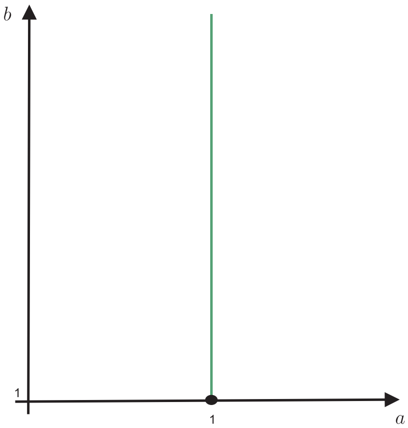

Figura 3.2: Reta de Hopf para $a = 1$.

3.1.1 O Primeiro Coeficiente de Lyapunov e a Condição de Não-Degenerescência

O sistema (3.5) pode ser reescrito como

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} -a & -ab \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -2auv - av^2 - auv^2 \\ 2uv + v^2 + uv^2 \end{pmatrix}, \tag{3.7}$$

na qual a matriz correspondente à parte linear do sistema (3.7) será denotada por

$$A = \begin{pmatrix} -a & -ab \\ 1 & 1 \end{pmatrix},$$

e a função correspondente a parte não linear será

$$F(\mathbf{u}) = \begin{pmatrix} -2auv - av^2 - auv^2 \\ 2uv + v^2 + uv^2 \end{pmatrix}. \tag{3.8}$$

Assim, sendo $\mathbf{u} = (u, v)$, o nosso sistema será dado por

$$\dot{\mathbf{u}} = A\mathbf{u} + F(\mathbf{u}).$$

Nota-se que A é exatamente a matriz Jacobiana do sistema (3.5) aplicada em $(0,0)$, e $F(\mathbf{u})$ os termos de ordem 2 e superiores.

Assim, tomando os parâmetros como em H_0, temos que o sistema (3.5) possui os autovalores dados por

$$\begin{aligned} \lambda_1 &= \quad i\sqrt{b-1} = \quad i\omega_0, \\ \lambda_2 &= -i\sqrt{b-1} = -i\omega_0, \end{aligned} \tag{3.9}$$

onde $\omega_0 = \sqrt{b-1}$.

Denote por q o autovetor de A correspondente a $\lambda_1 = i\omega_0$. Assim,

$$q = (-1 + i\omega_0, 1). \tag{3.10}$$

Na realidade, qualquer múltiplo não nulo de q será autovetor correspondente a $\lambda_1 = i\omega_0$. Assim, o comprimento escolhido pode alterar o valor do coeficiente de Lyapunov, mas não o seu sinal.

Seja agora p o autovetor de $A^\top$ correspondente a $\lambda_2 = -i\omega_0$. Assim, normalizando p para que $< p, q >= 1$, obtemos

$$p = \frac{i}{2\omega_0}(1, 1 - i\omega_0). \tag{3.11}$$

Lema 3.1.3. *Sejam* $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$ *e* $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$. *As funções multilineares* B *e* C*, para o sistema (3.5), são dadas por*

$$B(\mathbf{x},\mathbf{y}) = \begin{pmatrix} -2a(x_1y_2 + x_2y_1 + x_2y_2) \\ 2(x_1y_2 + x_2y_1 + x_2y_2) \end{pmatrix},$$

$$C(\mathbf{x},\mathbf{y},\mathbf{z}) = \begin{pmatrix} -2a(x_2y_1z_2 + x_2y_2z_1 + x_1y_2z_2) \\ 2(x_2y_1z_2 + x_2y_2z_1 + x_1y_2z_2) \end{pmatrix}.$$

Demonstração. Utilizaremos as fórmulas usadas no Capítulo 1 para encontrarmos as funções B e C. Assim,

$$B_i(\mathbf{x},\ \mathbf{y}) = \sum_{j,k=1}^{2} \left.\frac{\partial^2 F_i(\eta, 0)}{\partial\eta_j \partial\eta_k}\right|_{\eta=0} \mathbf{x}_j \mathbf{y}_k,$$

$$C_i(\mathbf{x},\ \mathbf{y},\ \mathbf{z}) = \sum_{j,k,l=1}^{2} \left.\frac{\partial^3 F_i(\eta, 0)}{\partial\eta_j \partial\eta_k \partial\eta_l}\right|_{\eta=0} \mathbf{x}_j \mathbf{y}_k \mathbf{z}_l,$$

para $i = 1, 2$.

Para encontrarmos $B_1(\mathbf{x{,}y})$ consideramos, $F_1(x, y) = -2axy - ay^2 - axy^2$ e calculamos

$$\frac{\partial F_1(x, y)}{\partial x} = -2ay - ay^2, \qquad \frac{\partial F_1(x, y)}{\partial y} = -2ax - 2ay - 2axy,$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F_1(x, y)}{\partial x}\right)\Bigg|_{x,y=0} = 0, \qquad \frac{\partial}{\partial x}\left(\frac{\partial F_1(x, y)}{\partial y}\right)\Bigg|_{x,y=0} = -2a,$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F_1(x, y)}{\partial x}\right)\Bigg|_{x,y=0} = -2a, \quad \frac{\partial}{\partial y}\left(\frac{\partial F_1(x, y)}{\partial y}\right)\Bigg|_{x,y=0} = -2a.$$

Assim temos que,

$$B_1(\mathbf{x{,}y}) = 0x_1y_1 - 2ax_1y_2 - 2ax_2y_1 - 2ax_2y_2.$$

Analogamente podemos obter as funções B_2, C_1, C_2, concluindo o resultado. □

Deste modo, temos que as funções $B(q,q)$, $B(q,\overline{q})$ e $C(q,q,\overline{q})$ para os parâmetros em H_0 ficam sendo

$$B(q,q) = -2(2i\omega_0 - 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B(q,\overline{q}) = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad C(q,q,\overline{q}) = 2(3 - i\omega_0)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Sejam agora

$$h_{11} = -A^{-1}B(q,\overline{q}) \quad e \quad h_{20} = (2i\omega_0 I_2 - A)^{-1}B(q,q).$$

Sendo assim, h_{11} e h_{20} ficam sendo

$$h_{11} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h_{20} = 2\left(\frac{2i\omega_0 - 1}{3\omega_0}\right)\begin{pmatrix} \omega_0 + 2i \\ -2i \end{pmatrix}.$$

Fazendo os cálculos de $B(q,h_{11})$ e $B(\overline{q},h_{20})$, obtemos

$$B(q,h_{11}) = -4\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B(\overline{q},h_{20}) = \frac{4(2i\omega_0 - 1)(\omega_0 - 2i)}{3\omega_0}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Portanto, temos

$$\begin{aligned} \langle p, B(q,h_{11})\rangle &= 2, \\ \langle p, B(\overline{q},h_{20})\rangle &= \frac{-2(3\omega_0 + 2i + 2i\omega_0^2)}{3\omega_0}, \\ \langle p, C(q,q,\overline{q})\rangle &= -3 + i\omega_0. \end{aligned}$$

Como

$$\left\langle p, B(q, -A^{-1}B(q,\overline{q}))\right\rangle = \langle p, B(q,h_{11})\rangle$$

e

$$\left\langle p, B(\overline{q}, (2i\omega_0 I_2 - A)^{-1}B(q,q))\right\rangle = \langle p, B(\overline{q},h_{20})\rangle,$$

a fórmula para o cálculo do coeficiente de Lyapunov, vista em (1.45), é dada por

$$l_1 = \frac{1}{2\omega_0}\text{Re}\left[\langle p, C(q,q,\overline{q})\rangle + 2\langle p, B(q,h_{11})\rangle + \langle p, B(\overline{q},h_{20})\rangle\right].$$

Fazendo essas contas temos que

$$l_1 = -\frac{1}{2\omega_0}.$$

Substituindo $\omega_0 = \sqrt{b-1}$ obtemos então

$$l_1 = -\frac{1}{2\sqrt{b-1}}, \tag{3.12}$$

claramente $l_1 \neq 0$, mais precisamente $l_1 < 0$, concluindo então que a bifurcação de Hopf é não–degenerada.

3.1.2 A condição de Transversalidade

Considere a matriz A do sistema (3.7). Conforme o Teorema 1.4, temos que a derivada parcial

$$\left.\frac{\partial}{\partial a} A\right|_{a=1} = A'(1),$$

é dada por

$$A'(1) = \begin{pmatrix} -1 & -b \\ 0 & 0 \end{pmatrix}.$$

Agora fazendo o cálculo para $\gamma'(1) = \text{Re}\ \langle p, A'(1)q \rangle$, obtemos

$$\gamma'(1) = -\frac{1}{2},$$

o que implica em $\gamma'(1) \neq 0$.

3.1.3 O teorema de Hopf para o sistema (3.5)

Dadas as condições de não degenerescência e transversalidade acima, temos o seguinte teorema:

Teorema 3.1.1. *Considere a família a 2–parâmetros de equações diferenciais ordinárias (3.5). Então para* $a = 1$ *e todo* $b > 1$*, o ponto de Hopf* $(0, 0)$ *é um foco atrator fraco. Além disso, para* $a < 1$ *suficientemente pequeno, existe uma órbita periódica atratora envolvendo o equilíbrio repulsor na origem. Veja Figura 3.3.*

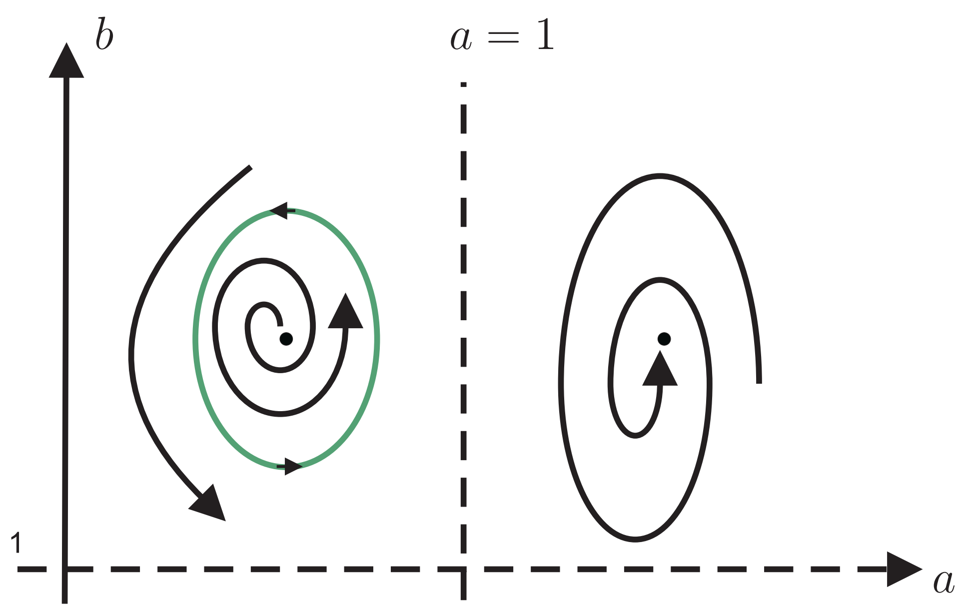

Figura 3.3: Diagrama de bifurcação do sistema (3.5) na origem.

Exemplo 3.1.1. *A Figura 3.4 mostra um ciclo limite estável, corresponde a* $a = 0,9$ *e* $b = 2$*, e suas oscilações. Sendo assim, existem quatro regimes de comportamentos:*

1. *Prosperidade* $(u\uparrow$ *e* $v\uparrow)$;
2. *Saturação* $(u\downarrow$ *e* $v\uparrow)$;
3. *Baixa* $(u\downarrow$ *e* $v\downarrow)$;
4. *Recuperação* $(u\uparrow$ *e* $v\downarrow)$.

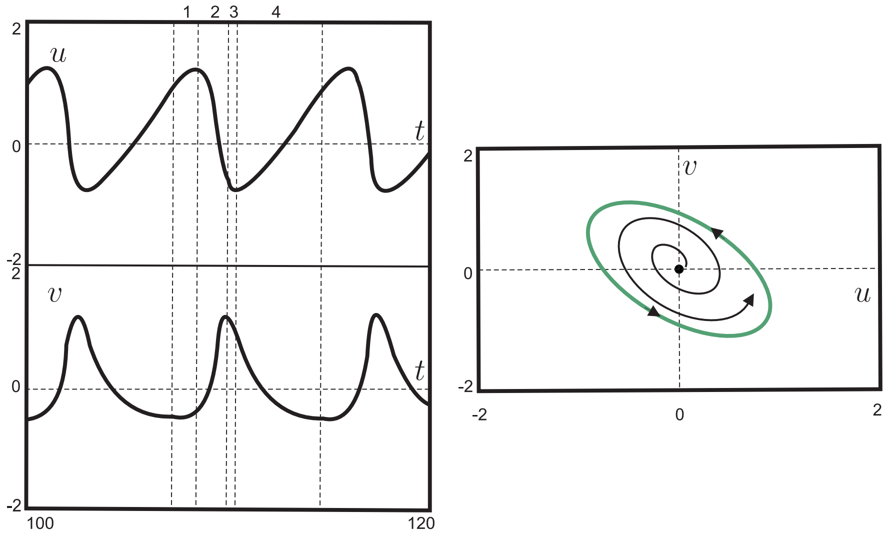

Figura 3.4: Oscilações e ciclo limite para o sistema (3.5).

3.2 O estudo qualitativo do modelo em $\mathbb{R}^4$

A fim de estudar o efeito da publicidade na interação entre o número de clientes potenciais e clientes atuais de dois produtos similares, consideraremos um modelo, como em (3.5), linearmente acoplado através do fluxo de compradores potenciais utilizando os parâmetros c_1 e c_2

$$\begin{cases} \dot{x} = -a(x + by + 2xy + y^2 + xy^2) + c_1(x - z), \\ \dot{y} = x + y + 2xy + y^2 + xy^2, \\ \dot{z} = -a(z + bw + 2zw + w^2 + zw^2) + c_2(z - w), \\ \dot{w} = z + w + 2zw + w^2 + zw^2. \end{cases} \tag{3.13}$$

O sistema (3.13) é da forma (3.3). Estudaremos a bifurcação de Hopf do sistema (3.13) em torno da origem, usando uma nova expressão para os coeficientes de Lyapunov.

O domínio dos parâmetros a serem considerados neste sistema é

$$\mathcal{D} = \{(a, b, c_1, c_2), a > 0, b > 1, c_{1,2} > 0\}.$$

Observe que quando $c_1 = c_2$ o sistema é simetricamente acoplado. Neste caso, o estudo da bifurcação de Hopf em torno da origem foi feito em [14] usando as fórmulas usuais.

3.2.1 O Coeficiente de Lyapunov para sistemas acoplados

Considere o sistema (3.13) e seja $\mathbf{x} = (x, y, z, w)^\top \in \mathbb{R}^4$. Como $\mathbf{x}_0 = \mathbf{0} \in \mathbb{R}^4$ é um ponto de equilíbrio de tal sistema, o mesmo pode ser escrito em torno de $\mathbf{x}_0$ como

$$\dot{\mathbf{x}} = J\mathbf{x} + \widetilde{F}(\mathbf{x}, \beta),$$

onde

$$J = \begin{pmatrix} A + c_1 I_0 & -c_1 I_0 \\ -c_2 I_0 & A + c_2 I_0 \end{pmatrix}, \qquad \text{com} \;\; I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

e A a matriz do sistema (3.7), é a matriz Jacobiana de (3.3) em $\mathbf{x}_0$ e

$$\widetilde{F}(\mathbf{x}, \beta) = (F(x, y, \beta), F(z, w, \beta))^\top.$$

Suponha que F é representado como

$$F(\mathbf{x}, \beta) = \frac{1}{2}B(\mathbf{x}, \mathbf{x}) + \frac{1}{6}C(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{24}D(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{120}E(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \mathcal{O}(|\mathbf{x}|^6), \tag{3.14}$$

onde B, C, D e E são funções multilineares simétricas de $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Note que as funções multilineares B, C, D e E em (3.14), também dependem de β. Omitindo β, segue de (3.14) que $\widetilde{F}$ pode ser escrita como

$$\widetilde{F}(\mathbf{x}) = \frac{1}{2}\widetilde{B}(\mathbf{x}, \mathbf{x}) + \frac{1}{6}\widetilde{C}(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{24}\widetilde{D}(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \frac{1}{120}\widetilde{E}(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}) + \mathcal{O}(|\mathbf{x}|^6),$$

onde

$$\begin{aligned}
&\widetilde{B}(\mathbf{x},\mathbf{y}) = (B(x_1, x_2, y_1, y_2), B(x_3, x_4, y_3, y_4))^\top,\\
&\widetilde{C}(\mathbf{x},\mathbf{y},\mathbf{z}) = (C(x_1, x_2, y_1, y_2, z_1, z_2), C(x_3, x_4, y_3, y_4, z_3, z_4))^\top,\\
&\widetilde{D}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) = (D(x_1, x_2, y_1, y_2, z_1, z_2, u_1, u_2), D(x_3, x_4, y_3, y_4, z_3, z_4, u_3, u_4))^\top,\\
&\widetilde{E}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v}) = (E(x_1, x_2, y_1, y_2, z_1, z_2, u_1, u_2, v_1, v_2), E(x_3, x_4, y_3, y_4, z_3, z_4, u_3, u_4, v_3, v_4))^\top,
\end{aligned}$$

são funções multilineares simétricas com $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^4$.

O polinômio característico de $\mathbf{x}_0 = (0,0,0,0)$ associado à matriz J é dado por

$$\underbrace{[\lambda^2 - \lambda T(A) + \det(A)]}_{(I)}\underbrace{[\lambda^2 - \lambda(T(A) + c_1 + c_2) + \det(A) + c_1 + c_2]}_{(II)} = 0 \qquad (3.15)$$

onde $T(A)$ e $\det(A)$ são, respectivamente, o traço e o determinante da matriz A dada em (3.7). Suponha que os autovalores λ_1 e λ_2 são raízes de (I) e λ_3 e λ_4 são raízes de (II). Uma vez que em $\mathcal{D}$ temos $\det(A) = a(b-1) > 0$ e $\det(A) + c_1 + c_2 > 0$ segue que:

1. Se $1 + c_1 + c_2 < a$, então $\operatorname{Re}\lambda_i < 0$ para $i = 1, ..., 4$ e $\mathbf{x}_0$ é um atrator;
2. Se $1 < a < 1 + c_1 + c_2$, então $\operatorname{Re}\lambda_{1,2} < 0$, $\operatorname{Re}\lambda_{3,4} > 0$ e $\mathbf{x}_0$ é uma sela do tipo $(2,2)$;
3. Se $a < 1$, então $\operatorname{Re}\lambda_i > 0$ para $i = 1, ..., 4$ e $\mathbf{x}_0$ é um repulsor;
4. Se $a = 1$, então $\lambda_{1,2}$ são puramente imaginários e $\operatorname{Re}\lambda_{3,4} > 0$ e, portanto, $\mathbf{x}_0$ é um ponto de Hopf;
5. Se $a = 1 + c_1 + c_2$, então $\lambda_{3,4}$ são puramente imaginários e $\operatorname{Re}\lambda_{1,2} < 0$ e portanto $\mathbf{x}_0$ é um ponto de Hopf.

Note que temos dois casos em que o equilíbrio $\mathbf{x}_0$ é um ponto de Hopf não hiperbólico, são eles:

(i) Para $T(A) = 0$, isto é, $a = 1$ e $\det(A) > 0$;

(ii) Para (β, c_1, c_2) tal que $T(A) + c_1 + c_2 = 0$, ou seja, $a = 1 + c_1 + c_2$, $T(A) \neq 0$, $c_1 + c_2 \neq 0$ e $\det(A) + (c_1 + c_2) > 0$.

Considere (β, c_1, c_2) tal que a matriz J tem um autovalor puramente imaginário $\lambda = i\omega$, $\omega > 0$. Sejam Q o autovetor de J correspondente a λ, e P o autovetor de $J^\top$ correspondente a $\bar{\lambda}$, normalizado com respeito a Q, isto é, $\langle P, Q\rangle = 1$. Aqui o produto interno em $\mathbb{C}^n$ será denotado por $\langle P, Q\rangle = \sum_{j=1}^{n} \bar{P}_j Q_j$.

Caso(i): Essas condições estão satisfeitas em $\beta = 0$ para cada $c_1,\ c_2 \in \mathbb{R}$. Neste caso, $\lambda_1 = -\lambda_2 = i\omega_0$, com $\omega_0 = \sqrt{\det(A)}$, são autovalores para J e A. Assuma $c_1 + c_2 \neq 0$. Temos então os seguintes lemas.

Lema 3.2.1. *Se $q \in \mathbb{C}^2$ é um autovetor de A correspondente a λ_1, e $p \in \mathbb{C}^2$ é o autovetor de $A^\top$ com respeito a λ_2 e normalizado com respeito a q, então $Q = (q, q)^\top \in \mathbb{C}^4$ é um autovetor de J correspondente a λ_1, e $P = \frac{1}{c_1+c_2}(c_2 p, c_1 p)^\top \in \mathbb{C}^4$ é um autovetor de $J^\top$ correspondente a λ_2 normalizado com respeito a Q.*

Demonstração. Se q é um autovetor de A correspondente a λ_1 e p é o autovetor de $A^\top$ com respeito a λ_2 e normalizado com respeito a q então

$$Aq \ = \ \lambda_1 q, \quad A^\top p \ = \ \lambda_2 p, \quad \langle p, q\rangle = \ 1.$$

Assim,

$$\begin{aligned}
\langle P, Q\rangle &= \left\langle \frac{1}{c_1+c_2}(c_2 p, c_1 p), (q, q)\right\rangle \\
&= \frac{1}{c_1+c_2}\langle (c_2 p, c_1 p), (q, q)\rangle \\
&= \frac{1}{c_1+c_2}(c_1\langle p, q\rangle + c_2\langle p, q\rangle) \\
&= \frac{1}{c_1+c_2}(c_1 + c_2)\langle p, q\rangle) \\
&= 1.
\end{aligned}$$

Temos também que

$$JQ = \begin{pmatrix} A + c_1 I_0 & -c_1 I_0 \\ -c_2 I_0 & A + c_2 I_0 \end{pmatrix}\begin{pmatrix} q \\ q \end{pmatrix} = \begin{pmatrix} Aq + c_1 I_0 q - c_1 I_0 q \\ -c_2 I_0 q + Aq + c_2 I_0 q \end{pmatrix}$$

$$= \begin{pmatrix} Aq \\ Aq \end{pmatrix} = \begin{pmatrix} \lambda_1 q \\ \lambda_1 q \end{pmatrix} = \lambda_1 \begin{pmatrix} q \\ q \end{pmatrix} = \lambda_1 Q.$$

Analogamente

$$J^\top P = \begin{pmatrix} A^\top + c_1 I_0 & -c_2 I_0 \\ -c_1 I_0 & A^\top + c_2 I_0 \end{pmatrix} \begin{pmatrix} \frac{1}{c_1+c_2}(c_2 p) \\ \frac{1}{c_1+c_2}(c_1 p) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{c_1+c_2}(A^\top c_2 p + c_1 c_2 I_0 p - c_1 c_2 I_0 p) \\ \frac{1}{c_1+c_2}(-c_1 c_2 I_0 p + A^\top c_1 p + c_1 c_2 I_0 p) \end{pmatrix}$$

$$= \frac{1}{c_1+c_2} \begin{pmatrix} A^\top p c_2 \\ A^\top p c_1 \end{pmatrix} = \frac{1}{c_1+c_2} \begin{pmatrix} \lambda_2 p c_2 \\ \lambda_2 p c_1 \end{pmatrix} = \lambda_2 \frac{1}{c_1+c_2} \begin{pmatrix} c_2 p \\ c_1 p \end{pmatrix} = \lambda_2 P.$$

□

Lema 3.2.2. *Sejam* $u = (u_1, u_2)^\top \in \mathbb{C}^2$ *e* $U = (u, u)^\top \in \mathbb{C}^4$. *Então,* $J^{-1}U = (A^{-1}u, A^{-1}u)^\top$ *e* $(xI_4 - J)^{-1}U = ((xI_2 - A)^{-1}u, (xI_2 - A)^{-1}u)^\top$ *para todo* $x \in \mathbb{C}$ *tal que* $(xI_4 - J)$ *é invertível.*

Demonstração. Multiplicando $(A^{-1}u, A^{-1}u)$ por J pelo lado esquerdo temos

$$J(A^{-1}u, A^{-1}u) = \begin{pmatrix} A + c_1 I_0 & -c_1 I_0 \\ -c_2 I_0 & A + c_2 I_0 \end{pmatrix} \begin{pmatrix} A^{-1}u \\ A^{-1}u \end{pmatrix}$$

$$= \begin{pmatrix} AA^{-1}u + c_1 I_0 A^{-1}u - c_1 I_0 A^{-1}u \\ -c_2 I_0 A^{-1}u + AA^{-1}u + c_2 I_0 A^{-1}u \end{pmatrix} = \begin{pmatrix} u \\ u \end{pmatrix} = U$$

$$\Rightarrow J^{-1}U = (A^{-1}u, A^{-1}u).$$

Do mesmo modo, multiplicando $((xI_2 - A)^{-1}u, (xI_2 - A)^{-1}u)$ por $(x - I_4)$ pelo lado esquerdo temos

$$(x - I_4)((xI_2 - A)^{-1}u, (xI_2 - A)^{-1}u) = \begin{pmatrix} xI_2 - A - c_1 I_0 & c_1 I_0 \\ c_2 I_0 & xI_2 - A - c_2 I_0 \end{pmatrix} \begin{pmatrix} (xI_2 - A)^{-1}u \\ (xI_2 - A)^{-1}u \end{pmatrix}$$

$$= \begin{pmatrix} (xI_2 - A)(xI_2 - A)^{-1}u - c_1 I_0 (xI_2 - A)^{-1}u + c_1 I_0 (xI_2 - A)^{-1}u \\ c_2 I_0 (xI_2 - A)^{-1}u + (xI_2 - A)(xI_2 - A)^{-1}u - c_2 I_0 (xI_2 - A)^{-1}u \end{pmatrix} = \begin{pmatrix} u \\ u \end{pmatrix} = U$$

$$\Rightarrow (xI_4 - J)^{-1}U = ((xI_2 - A)^{-1}u, (xI_2 - A)^{-1}u).$$

□

Usando o método da projeção e as expressões do primeiro e segundo coeficiente de Lyapunov vistos no Capítulo 1 obtemos o seguinte resultado.

Proposição 3.2.1. *Sejam q e p os autovetores do Lema 3.2.1. O primeiro coeficiente de Lyapunov de (3.13) para $\beta = 0$, $c_1 + c_2 \neq 0$ é dado por*

$$l_1 = \frac{1}{2\omega_0} \mathrm{Re}\left[\langle p, C(q,q,\bar{q})\rangle + 2\left\langle p, B(q, A^{-1}B(q,\bar{q})\right\rangle + \left\langle p, B(\bar{q}, (2i\omega_0 I_2 - A)^{-1}B(q,q))\right\rangle\right].$$

Assim, para $a = 1$ teremos $\lambda = i\sqrt{b-1} = i\omega_0$ com $\omega_0 = \sqrt{b-1}$ e os autovetores q e p dados por

$$q = (-1 + i\omega_0, 1), \quad p = \frac{1}{2\omega_0}(1, 1 - i\omega_0).$$

Pela Proposição 3.2.1 temos que o primeiro coeficiente de Lyapunov de (3.13) é

$$l_1 = -\frac{1}{2\omega_0} = -\frac{1}{2\sqrt{b-1}}.$$

Concluindo que, para este caso, acontece uma bifurcação de Hopf. Cruzando o plano $a = 1$ e quando os planos $x = z$ e $y = w$, surgirá uma órbita periódica em torno do equilíbrio repulsor $\mathbf{x}_0$.

Observação 3.2.1. *Levando em conta os Lemas 3.2.1, 3.2.2, e utilizando as fórmulas deduzidas no Capítulo 1, a Proposição 3.2.1 afirma que os primeiros coeficientes de Lyapunov para os sistemas (3.1) e (3.3), que são equivalentes aos sistemas (3.5) e (3.13), respectivamente, no **Caso(i)**, para $\beta = 0$, são idênticos.*

Caso(ii): Neste caso temos $\lambda_3 = -\lambda_4 = i\omega_1$ com $\omega_1 = \sqrt{\det(A) + c_1 + c_2}$. Sendo assim, temos os seguintes lemas.

Lema 3.2.3. *Se $q = (-1+i\omega_1, 1)^\top \neq 0$ então $Q = (c_1 q, -c_2 q)^\top \in \mathbb{C}^4$ é um autovetor de J correspondente a λ_3. Se $p = (1+i\omega_1, \omega_1^2+1)^\top \neq 0$ então $P = \dfrac{1}{(c_1+c_2)\overline{\langle p, q\rangle}}(p, -p)^\top \in \mathbb{C}^4$ é o autovetor de $J^\top$ correspondente a λ_4, normalizado com respeito a Q.*

Demonstração. Escrevendo a matriz J na forma 4×4 e para $a = 1 + c_1 + c_2$, obtemos

$$J = \begin{pmatrix} -1-c_2 & -b(1+c_1+c_2) & -c_1 & 0 \\ 1 & 1 & 0 & 0 \\ -c_2 & 0 & -1-c_1 & -b(1+c_1+c_2) \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{3.16}$$

Se $q = (-1 + i\omega_1, 1)^\top$ e $p = (1 + i\omega_1, \omega^2 + 1)^\top$ então $Q = (ic_1\omega_1 - c_1, c1, -ic_2\omega_1 + c_2, -c_2)^\top$ e $P = \dfrac{1}{(c_1+c_2)\overline{\langle p, q\rangle}}(1 + i\omega_1, \omega_1^2 + 1, -1 - i\omega_1, -\omega_1^2 - 1)^\top$. Lembrando que $\omega_1^2 = b(1 + c_1 + c_2) - 1$.

Assim,

$$JQ = \begin{pmatrix} -1-c_2 & -b(1+c_1+c_2) & -c_1 & 0 \\ 1 & 1 & 0 & 0 \\ -c_2 & 0 & -1-c_1 & -b(1+c_1+c_2) \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} ic_1\omega_1 - c_1 \\ c1 \\ -ic_2\omega_1 + c_2 \\ -c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -ic_1w_1 + c_1 - bc_1(1 + c_1 + c_2) \\ ic_1w_1 \\ ic_2w_1 - c_2 + bc_2(1 + c_1 + c_2 \\ -ic_2w_1 \end{pmatrix} \quad = \quad i\omega_1 \begin{pmatrix} ic_1\omega_1 - c_1 \\ c_1 \\ -ic_2\omega_1 - c_1 \end{pmatrix} = \quad \lambda_3 Q.$$

Analogamente,

$$J^\top P = \begin{pmatrix} -1 - c_2 & 1 & -c_2 & 0 \\ -b(1 + c_1 + c_2) & 1 & 0 & 0 \\ -c_1 & 0 & -1 - c_1 & 1 \\ 0 & 0 & -b(1 + c_1 + c_2) & 1 \end{pmatrix} \frac{1}{(c_1 + c_2)\overline{\langle p, q \rangle}} \begin{pmatrix} 1 + i\omega_1 \\ \omega_1^2 + 1 \\ -1 - i\omega_1 \\ -\omega_1^2 - 1 \end{pmatrix}$$

$$= \frac{1}{(c_1 + c_2)\overline{\langle p, q \rangle}} \begin{pmatrix} -1 - i\omega_1 - c_2 - ic_2\omega_1 + \omega_1^2 + 1 + c_2 + ic_2\omega_1 \\ -b - ib\omega_1 - bc_1 - ibc_1\omega_1 - bc_2 - ibc_2\omega_1 + \omega_1^2 + 1 \\ 1 + i\omega_1 + c_1 + ic_1\omega_1 - \omega_1^2 - 1 - c_1 - ic_1\omega_1 \\ b + ib\omega_1 + bc_1 + ibc_1\omega_1 + bc_2 + ibc_2\omega_1 - \omega_1^2 - 1 \end{pmatrix}$$

$$= \frac{1}{(c_1 + c_2)\overline{\langle p, q \rangle}} \begin{pmatrix} \omega_1(\omega_1 - i) \\ -ib(1 + c_1 + c_2)\omega_1 \\ \omega_1(i - \omega_1) \\ ib(1 + c_1 + c_2)\omega_1 \end{pmatrix} = -i\omega_1 \frac{1}{(c_1 + c_2)\overline{\langle p, q \rangle}} \begin{pmatrix} 1 + i\omega_1 \\ \omega_1^2 + 1 \\ -1 - i\omega_1 \\ -\omega_1^2 - 1 \end{pmatrix} = \lambda_4 P.$$

Agora basta mostrar que $\langle P, Q \rangle = 1$. Temos então que

$$\langle P, Q\rangle = \left\langle \frac{1}{(c_1+c_2)\overline{\langle p,q\rangle}}(p,-p),(c_1q,-c_2q)\right\rangle$$

$$= \frac{1}{(c_1+c_2)\langle p,q\rangle}\langle (p,-p),(c_1q,-c_2q)\rangle$$

$$= \frac{1}{(c_1+c_2)\langle p,q\rangle}(c_1\bar{p}q+c_2\bar{p}q)$$

$$= \frac{1}{(c_1+c_2)\langle p,q\rangle}(c_1+c_2)\langle p,q\rangle$$

$$= 1.$$

□

Lema 3.2.4. *Sejam* $u=(u_1,u_2)^\top \in \mathbb{C}^2$ *e* $U=(c_1^2u, c_2^2u)^\top \in \mathbb{C}^4$. *Considere as matrizes* $A_j(x) = A + \alpha_j(x)I_0$, $j=1,2$, *onde*

$$\alpha_j(x) = \frac{(-1)^{3-j}(c_1^2-c_2^2)(x^2 - xT(A)+\det(A))}{c_jx^2 - x[c_{3-j}^2 + c_j(c_{3-j}+T(A))] + c_j\det(A) + c_{3-j}(c_1+c_2)}. \tag{3.17}$$

Então, $(xI_4 - J)^{-1}U = (c_1^2(xI_2 - A_1(x))^{-1}u, c_2^2(xI_2-A_2(x))^{-1}u)^\top$, *para todo* $x\in\mathbb{C}$ *tal que* $xI_4 - J$ *é invertível.*

Proposição 3.2.2. *Para* (β, c_1, c_2) *tal que* $T(A)+c_1+c_2 = 0$, $T(A)\neq 0$, $c_1+c_2\neq 0$ *e* $\det(A)+c_1+c_2 > 0$, *o primeiro coeficiente de Lyapunov de (3.13) para* (β, c_1, c_2) *tem a forma*

$$l_1(\beta, c_1, c_2) = \frac{1}{2(c_1+c_2)}\mathrm{Re}\left[\frac{1}{\langle p,q\rangle}\sum_{j=1}^{2} c_j^3\langle p, Z_j\rangle\right], \tag{3.18}$$

com

$$Z_j = C(q,q,\bar{q}) - 2B(q,(A_j(0))^{-1}B(q,\bar{q})) + B(\bar{q},(2i\omega_1 I_2 - A_j(2i\omega_1))^{-1}B(q,q)),$$

onde p *e* q *são os autovetores dados no Lema 3.2.3.*

Para o caso particular de $c_1 = c_2 = c$, temos $A_1(x) = A_2(x) = A$ e o primeiro coeficiente de Lyapunov se escreve como

$$l_1(\beta, c) = \frac{c^2}{2}\text{Re}\left[\frac{1}{\langle p, q\rangle}\langle p, Z\rangle\right], \tag{3.19}$$

com

$$Z = C(q, q, \bar{q}) - 2B(q, A^{-1}B(q, \bar{q})) + B(\bar{q}, (2i\omega_1 I_2 - A)^{-1}B(q, q)).$$

Neste caso, seguem os resultados.

Lema 3.2.5. *Sejam* $u = (u_1, u_2)^\top \in \mathbb{C}^2$ *e* $U = (u, -u)^\top \in \mathbb{C}^4$. *Considere a matriz* $A_0 = A + 2cI_0$. *Então* $(xI_4 - J)^{-1}U = ((xI_2 - A_0)^{-1}u, -(xI_2 - A_0)^{-1}u)^\top$, *para todo* $x \in \mathbb{C}$ *tal que* $xI_4 - J$ *é invertível.*

Proposição 3.2.3. *Sejam* $c_1 = c_2 = c$ *e* (β, c) *tal que* $T(A) + 2c = 0$, $T(A) \neq 0$, $c \neq 0$ *e* $\det(A) + 2c > 0$. *Se* $l_1 = 0$, *o segundo coeficiente de Lyapunov de (3.13) para* (β, c), *tem a expressão dada por*

$$l_2(\beta, c) = \frac{1}{12}\text{Re}\left[\frac{1}{\langle p, q\rangle}\langle p, H_{32}\rangle\right], \tag{3.20}$$

onde

$$\begin{aligned} H_{32} = {} & 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) \\ & +2B(\bar{q}, h_{31}) + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) \\ & +6C(q, \bar{q}, h_{21}) + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) \\ & +6D(q, q, \bar{q}, h_{11}) + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}), \end{aligned}$$

p *e* q *são os autovetores do Lema 3.2.3, e os vetores bi-dimensionais* h_{ij} *são dados por*

$$\begin{aligned}
h_{11} &= -A^{-1}(B(q,\bar{q})),\\
h_{20} &= (2i\omega_1 I_2 - A)^{-1}B(q,q),\\
h_{30} &= (3i\omega_1 I_2 - A_0)^{-1}(C(q,q,q) + 3B(q,h_{20})),\\
h_{31} &= (2i\omega_1 I_2 - A)^{-1}(3B(q,h_{21}) + B(\bar{q},h_{30}) + 3B(h_{20},h_{11}) + 3C(q,q,h_{11})\\
&\quad +3C(q,\bar{q},h_{20}) + D(q,q,q,\bar{q}) - 3g_{21}h_{20}),\\
h_{22} &= -A^{-1}(2B(h_{11},h_{11}) + 2B(q,\bar{h}_{21}) + 2B(\bar{q},h_{21}) + B(\bar{h}_{20},h_{20})\\
&\quad +C(q,q,\bar{h}_{20}) + C(\bar{q},\bar{q},h_{20}) + 4C(q,\bar{q},h_{11}) + D(q,q,\bar{q},\bar{q})),
\end{aligned}$$

e do Capítulo 1 temos que h_{21} pode ser encontrado resolvendo o sistema (1.46).

Observação 3.2.2. *As demonstrações dos Lemas 3.2.4, 3.2.5 e das Proposições 3.2.2 e 3.2.3 podem ser encontradas no Apêndice II.*

Sendo assim, considerando então $a = 1 + c_1 + c_2$ teremos $\lambda = i\sqrt{ab-1} = i\omega_1$. Os autovetores q e p escolhidos como no Lema 3.2.3 são

$$q = (-1 + i\omega_1, 1)^\top \quad e \quad p = (1 + i\omega_1, \omega^2 + 1)^\top.$$

Substituindo os valores e fazendo os cálculos para o caso particular de $c_1 = c_2 = c$ da expressão (3.19) temos

$$l_1 = \frac{1-2c}{2} + \frac{\mathrm{Re}\,[(\omega_1 - i)(2i\omega_1 - 1)(\omega_1^2 - 2c + ai\omega_1)(2\omega_1^2 + a - ab - 2ai\omega_1)(4ic\omega_1 + 3\omega_1^2 + 2c)]}{\omega_1(\omega_1^2+1)[-16c^2\omega_1^2 - (3\omega_1^2 + 2c)^2]}. \quad (3.21)$$

Uma vez que apenas o sinal de l_1 é importante, ao em vez de (3.21) usaremos a seguinte expressão, que é a mesma sem o denominador (sempre positivo) e um fator multiplicador (positivo)

$$l_1 = -3(3 + 20c + 12c^2) + 2(9 + 52c + 92c^2 + 48c^3)b - (9 + 44c + 52c^2)b^2.$$

A Figura 3.5 esboça a curva onde l_1 se anula. Já as Figuras 3.6 e 3.7 mostram que as curvas onde l_1 e l_2 se anulam não se interceptam.

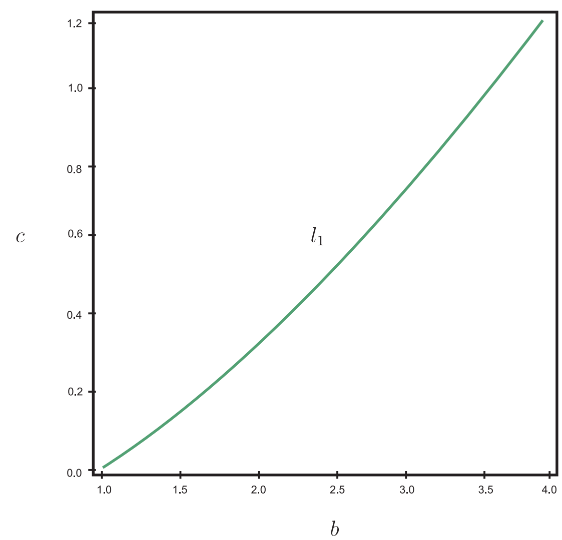

Figura 3.5: Curva de nível zero de l_1 para os parâmetros $c_1 = c_2 = c$.

Note também que

$$\gamma'(a) = \text{Re}\left\langle p, \left.\frac{\partial A}{\partial a}\right|_{a=1+2c} q\right\rangle = -\frac{1}{2} < 0.$$

satisfazendo a condição de transversalidade, que é necessária para ocorrer uma bifurcação de Hopf.

Como conclusão, descobrimos que para $a = c_1 + c_2 + 1$, $c_{1,2} > 0$ e $b > 1$, l_1 pode se anular. Assim, nesse caso, pode ocorrer uma bifurcação de Hopf supercrítica, isto é, acontece o surgimento de uma órbita periódica e uma mudança de estabilidade do foco a partir da perturbação do sistema com o parâmetro β quando $l_1 < 0$, ou uma bifurcação

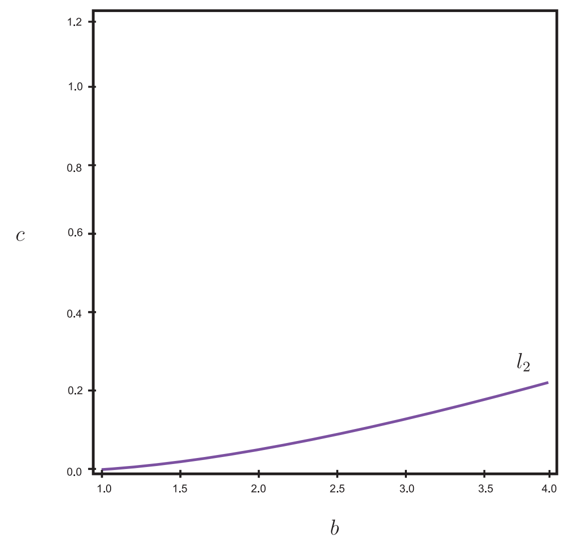

Figura 3.6: Curva de nível zero de l_2 para os parâmetros $c_1 = c_2 = c$.

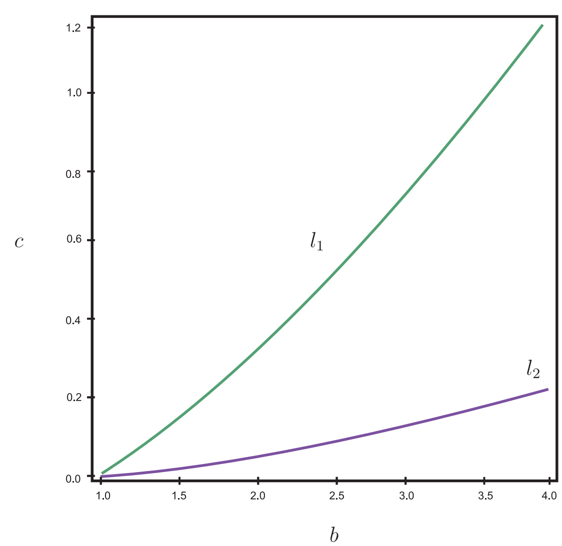

Figura 3.7: Curvas de nível zero de l_1 e l_2 para os parâmetros $c_1 = c_2 = c$.

de Hopf subcrítica que é caracterizada pelo desaparecimento de uma órbita periódica repulsora que ocorre quando passamos pelo valor crítico do parâmetro β quando $l_1 > 0$ ou, finalmente, uma bifurcação de Hopf degenerada quando $l_1 = 0$, veja Figura 1.5.

3.3 Conclusão e significado econômico

Do ponto de vista econômico, é importante determinar os parâmetros para os quais o comportamento futuro das variáveis é estável ou periódica.

Isso significa que devemos que determinar as regiões do espaço de parâmetros para os quais existem atratores ou ciclos limites com propriedades atrativas. Como já mencionamos, a origem é estável quando $a > 1 + c_1 + c_2$. A seguir, analisaremos a natureza dos ciclos limites nascidos como resultados da bifurcação Hopf da origem.

Consideremos em primeiro lugar um ciclo limite que apareceu como resultado da bifurcação de Hopf no caso $a = 1$. Se fixarmos $b = 2$, $c_1 = 1$ e $c_2 = 3$, surgirá um ciclo limite para cada parâmetro a, situados em uma vizinhança de parâmetro da bifurcação $a = 1$, para $a < 1$. Ele envolve o equilíbrio repulsor x_0 e está situado no plano $x = z$, $y = w$. Como neste caso a bifurcação de Hopf é supercrítica, o ciclo limite é atrator. A Figura 3.8 mostra algumas projeções bi-dimensionais do ciclo limite que existe para $a = 0,9$, $b = 2$, $c_1 = 1$ e $c_2 = 3$ e as oscilações correspondentes a x, y, z, w são enfatizadas.

Pode-se ver que as oscilações de x e z são idênticas e que coincidem com as oscilações da variável u do sistema único (3.5) sobre o ciclo limite obtido quando $a = 0,9$, $b = 2$. Da mesma forma, as oscilações de y e z são idênticas aos da variável v. Naturalmente, o ciclo limite do modelo (3.5) depende apenas dos parâmetros a e b. Por conseguinte, não importa quais são os valores dos parâmetros c_1 e c_2, as oscilações de x e z (y e w) num ciclo limite existente no sistema (3.13) para os parâmetros $\{a, b, c_1, c_2\}$ são as mesmas que as oscilações de u (v) para o ciclo limite do modelo único (3.5), existente para os parâmetros $\{a, b\}$. É por isso que no ciclo limite de 4-dimensões há apenas 4 regimes de comportamento, veja Figura 3.8, como no caso do sistema (3.5) (Figura 3.4).

Concluindo, para um ciclo limite que nasce como resultado de uma bifurcação de Hopf em $a = 1$, os períodos de prosperidade, saturação, recessão e recuperação sucedem simultaneamente para os dois produtos. Segue-se então que apenas se as condições iniciais para os sistemas acoplados em (3.13) forem as mesmas (ou seja, $x(0) = z(0)$, $y(0) = w(0)$), as variáveis do sistema (3.13) vão evoluir para os regimes de 4 comportamentos correspondentes a este ciclo.

Considere agora o ciclo limite que apareceu como resultado de uma bifurcação de Hopf supercrítica no caso $a = 1+c_1+c_2$, $c_1 \neq c_2$, quando os osciladores são não-simetricamente acoplados. Neste caso, o equilíbrio é atrator e o ciclo limite é assintoticamente estável. Este ciclo existe para $a < 1+c_1+c_2$, na vizinhança do valor da bifurcação. Na Figura 3.9 estão representadas as oscilações correspondentes ao ciclo limite assintoticamente estável existente para $a = 1,9$, $b = 4$, $c_1 = 0,2$ e $c_2 = 0,8$, e algumas projeções bi-dimensionais deste ciclo. Mesmo se os dois osciladores que foram acoplados no sistema (3.13) forem idênticos, as oscilações de x e z são bem diferentes, como oscilações de y e w. O número máximo dos potenciais compradores e dos utilizadores do segundo produto são maiores do que aqueles para o primeiro produto. Embora existam diferenças na amplitude das oscilações, sobre o ciclo há apenas 8 regimes de comportamento, devido ao fato de que os períodos de prosperidade, saturação, desaceleração e recuperação são bem-sucedidos para os dois produtos com um tempo de atraso. Esses regimes de comportamento são devidos a alterações de monotonia das variáveis de estado como funções do tempo, ou seja:

1. período de recuperação para a primeira marca e de prosperidade para a segunda;
2. continua o período de recuperação da primeira marca e há um período de saturação da segunda;
3. continua o período de recuperação da primeira marca e há um período de declínio da segunda;
4. prosperidade e declínio;
5. prosperidade e recuperação;
6. saturação e recuperação;
7. declínio e recuperação;
8. declínio e prosperidade.

Finalmente, considere o ciclo limite que apareceu como resultado de uma bifurcação de Hopf supercrítica, no caso particular $a = 1+c_1+c_2$, $c_1 = c_2 = c$. Neste caso, os osciladores estão simetricamente acoplados e existe o ciclo limite estável para $a < 2c+1$. Ele envolve uma sela $(2, 2)$ no equilíbrio. Em [14] foram representadas as oscilações correspondentes a tal ciclo limite e também demonstraram que as oscilações de x e z têm a mesma forma, mas há um tempo de atraso entre elas. A mesma conclusão é válida para as oscilações de y e w. Devido a este atraso temporal, no ciclo de lá foram enfatizadas 8 regimes de comportamento.

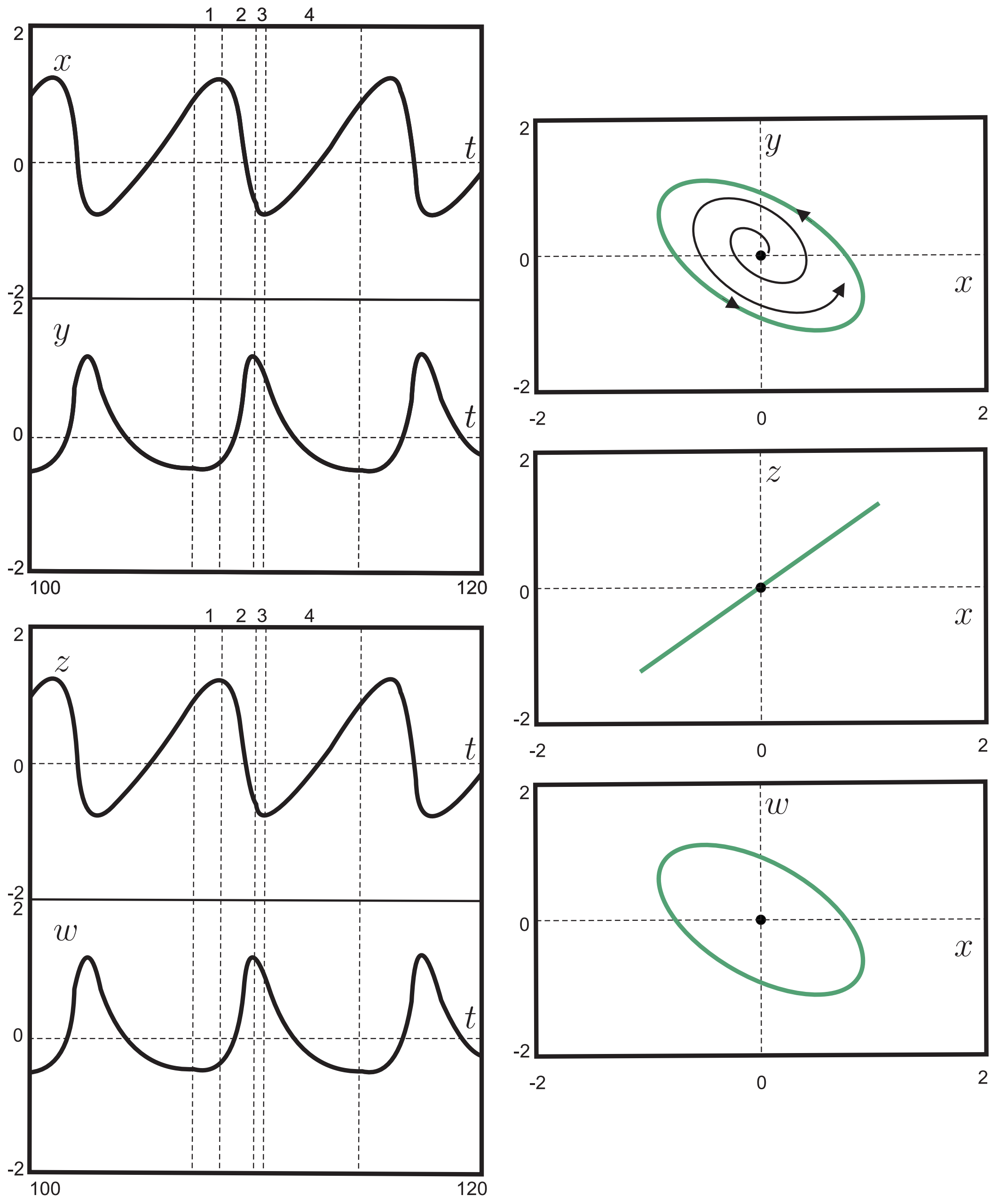

Figura 3.8: Oscilações e e projeções em duas dimensões para o ciclo limite no caso **(i)**.

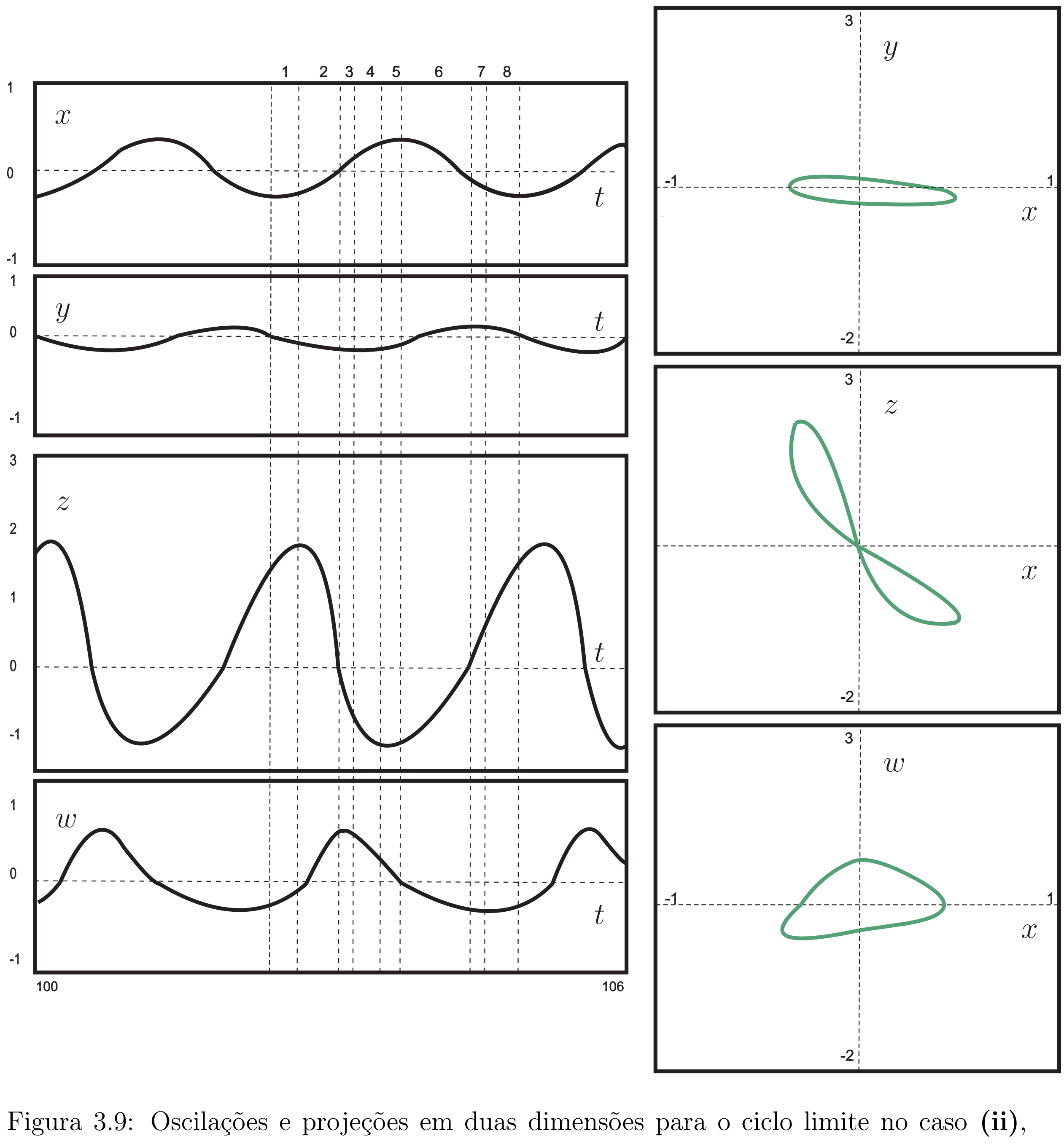

Figura 3.9: Oscilações e projeções em duas dimensões para o ciclo limite no caso **(ii)**, $c_1 \neq c_2$.

Conclusões

O estudo dos teoremas e definições apresentados no Capítulo 1 nos permitiu usar ferramentas da dinâmica não linear para determinar a estabilidade dos equilíbrios, a existência de uma bifurcação de Hopf e a presença de ciclos limites.

No capítulo 2 analisamos um sistema tridimensional analiticamente e numericamente. No qual, uma bifurcação de Hopf foi detectada para um parâmetro apropriado.Também vimos que mais de uma órbita periódica poderá surgir para alguns valores de parâmetros.

No capítulo 3 primeiramente estudamos um sistema em $\mathbb{R}^2$, no qual modela a dinâmica de comportamento entre o número de potenciais compradores e o número de clientes atuais de um produto por meio da publicidade. A partir desses resultados estendemos para um sistema em $\mathbb{R}^4$, no qual modela a mesma dinâmica anterior, porém de duas marcas concorrentes disponíveis no mercado. Podemos perceber que o acoplamento dos sistemas bidimensionais facilitou os cálculos para o sistema em $\mathbb{R}^4$ uma vez que os mesmos que foram usados para o sistema em $\mathbb{R}^2$ foram reutilizados para o sistema em $\mathbb{R}^4$.

Observamos também neste capítulo a existência de uma bifurcação de Hopf e o surgimento ou desaparecimento de órbitas periódicas dependendo dos valores dos parâmetros.

Sendo assim concluímos que a modelagem não linear e o acoplamento dos sistemas além de facilitar os cálculos, foram úteis na analise do comportamento econômico do modelo utilizado.

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Apêndice I

Apêndice I

Modelo Tridimensional

```
(* Componentes do Sistema (2.1) onde η = n e μ = m *)

f1[x_, y_, z_] := n * x - y^2

f2[x_, y_, z_] := m * (z - y)

f3[x_, y_, z_] := a * y - b * z + x * y
```

```
(* Pontos de equilíbrio *)

Solve[n * x - y^2 == 0 && m * (z - y) == 0 && a * y - b * z + x * y == 0, {x, y, z}]
resolve

{{x → 0, y → 0, z → 0}, {x → -a + b, y → -√(-a n + b n), z → -√(-(a - b) n)},
 {x → -a + b, y → √(-a n + b n), z → √(-a n + b n)}}
```

```
(* escolhendo o ponto e* = (b-a, √(n*(b-a)), √(n*(b-a))) *)

P0 := {b - a, √(n * (b - a)), √(n * (b - a))}
```

```
(* Parte linear do campo de vetores *)

Df[{x_, y_, z_}] := {{Derivative[1, 0, 0][f1][x, y, z],
                      derivação
   Derivative[0, 1, 0][f1][x, y, z], Derivative[0, 0, 1][f1][x, y, z]},
   derivação                          derivação
  {Derivative[1, 0, 0][f2][x, y, z], Derivative[0, 1, 0][f2][x, y, z],
   derivação                          derivação
   Derivative[0, 0, 1][f2][x, y, z]}, {Derivative[1, 0, 0][f3][x, y, z],
   derivação                           derivação
   Derivative[0, 1, 0][f3][x, y, z], Derivative[0, 0, 1][f3][x, y, z]}}
   derivação                          derivação
```

```
(* Calculando a matriz jacobiana  J(e* ) = A *)

A := Df[P0]

A

{{n, -2 √((-a + b) n), 0}, {0, -m, m}, {√((-a + b) n), b, -b}}
```

```
(* Polinômio Característico *)

Det[A - λ * IdentityMatrix[3]]
```
determinante matriz identidade

$2\,a\,m\,n - 2\,b\,m\,n + b\,n\,\lambda + m\,n\,\lambda - b\,\lambda^2 - m\,\lambda^2 + n\,\lambda^2 - \lambda^3$

```
pc := λ^3 + (b + m - n) * λ^2 - n * (b + m) * λ + 2 * (b - a) * m * n

FullSimplify[Solve[(b + m - n) * (-n) * (b + m) == 2 * (b - a) * m * n, {a}]]
```
simplifica comple⋯ resolve

$$\left\{\left\{a \to \frac{b^2 + 4\,b\,m + m^2 - (b + m)\,n}{2\,m}\right\}\right\}$$

```
(* Valor da Bifurcação *)
```

$$a := \frac{b^2 + 4\,b\,m + m^2 - (b + m)\,n}{2\,m}$$

```
Solve[pc == 0, {λ}]
```
resolve

$$\left\{\left\{\lambda \to -\sqrt{b + m}\,\sqrt{n}\right\}, \left\{\lambda \to \sqrt{b + m}\,\sqrt{n}\right\}, \left\{\lambda \to -b - m + n\right\}\right\}$$

```
(* Autovalores *)
```

$$\text{Eigenvalues[A] /. } \sqrt{b\,n + m\,n} \to I * \omega_0$$
autovalores unidade

$$\left\{-\sqrt{b + m}\,\sqrt{n},\ \sqrt{b + m}\,\sqrt{n},\ -b - m + n\right\}$$

$$\text{Eigensystem[A] /. } \sqrt{(b + m)\,n} \to I * \omega_0 \text{ /. } \sqrt{\frac{(b + m)\,n\,(-b - m + n)}{m}} \to h$$
autovalores e autovetores unidade imaginária

$$\left\{\left\{-\sqrt{b + m}\,\sqrt{n},\ \sqrt{b + m}\,\sqrt{n},\ -b - m + n\right\},\right.$$

$$\left\{\left\{\frac{2\,m\,\sqrt{n\left(b - \frac{b^2 + 4\,b\,m + m^2 - (b + m)\,n}{2\,m}\right)}}{\left(m - \sqrt{b + m}\,\sqrt{n}\right)\left(\sqrt{b + m}\,\sqrt{n} + n\right)},\ \frac{m}{m - \sqrt{b + m}\,\sqrt{n}},\ 1\right\},\right.$$

$$\left\{\frac{2\,m\,\sqrt{n\left(b - \frac{b^2 + 4\,b\,m + m^2 - (b + m)\,n}{2\,m}\right)}}{\left(m + \sqrt{b + m}\,\sqrt{n}\right)\left(-\sqrt{b + m}\,\sqrt{n} + n\right)},\ \frac{m}{m + \sqrt{b + m}\,\sqrt{n}},\ 1\right\},$$

$$\left.\left.\left\{-\frac{\sqrt{2}\,m\,\sqrt{-\frac{n\,(b^2 + 2\,b\,m + m^2 - b\,n - m\,n)}{m}}}{(b + m)\,(b - n)},\ -\frac{m}{b - n},\ 1\right\}\right\}\right\}$$

$$\text{lb1} := I * \omega_0$$
unidade

```
lb2 := -I * ω0
```
unidade

```
lb3 := -b - m + n
```

```
(* Autovetor q e seu conjugado qb *)
```

$$q := \left\{-\frac{\sqrt{2}\, h\, m}{(m + i\, \omega_0)\,(-n + i\, \omega_0)},\ \frac{m}{m + i\, \omega_0},\ 1\right\}$$

```
qb = FullSimplify[ComplexExpand[Conjugate[q], h > 0 && m > 0 && n < 0]]
```
simplifica comple··· expande funções ··· conjugado

$$\left\{\frac{\sqrt{2}\, h\, m}{(i\, m + \omega_0)\,(-i\, n + \omega_0)},\ \frac{m}{m - i\, \omega_0},\ 1\right\}$$

$$h = \sqrt{\frac{(b + m)\, n\, (-b - m + n)}{m}}$$

$$\sqrt{\frac{(b + m)\, n\, (-b - m + n)}{m}}$$

$$\omega_0 = \sqrt{-(b + m)\, n}$$

$$\sqrt{(-b - m)\, n}$$

```
FullSimplify[A.q - i ω0 q]
```
simplifica completamente

```
{0, 0, 0}
```

```
Clear[h]
```
apaga

```
Unset[ω0]
```
elimina alocação

$$\text{Eigensystem[Transpose[A]] /. } \sqrt{(b + m)\, n} \rightarrow I * \omega_0 \text{ /. } \sqrt{\frac{(b + m)\, n\, (-b - m + n)}{m}} \rightarrow h$$
autovalores e a··· transposição unidade imaginária

$$\left\{\left\{-\sqrt{b+m}\,\sqrt{n},\ \sqrt{b+m}\,\sqrt{n},\ -b-m+n\right\},\right.$$

$$\left\{\left\{-\frac{\sqrt{n\left(b - \frac{b^2+4\,b\,m+m^2-(b+m)\,n}{2\,m}\right)}}{\sqrt{b+m}\,\sqrt{n} + n},\ -\frac{-b+\sqrt{b+m}\,\sqrt{n}}{m},\ 1\right\},\right.$$

$$\left\{-\frac{\sqrt{n\left(b - \frac{b^2+4\,b\,m+m^2-(b+m)\,n}{2\,m}\right)}}{-\sqrt{b+m}\,\sqrt{n} + n},\ -\frac{-b-\sqrt{b+m}\,\sqrt{n}}{m},\ 1\right\},$$

$$\left.\left.\left\{-\frac{\sqrt{-\frac{n\,(b^2+2\,b\,m+m^2-b\,n-m\,n)}{m}}}{\sqrt{2}\,(b+m)},\ -\frac{m-n}{m},\ 1\right\}\right\}\right\}$$

```
(* Autovertor p e seu conjugado pb *)
```

$$p := \left(1\Big/\left(1+\frac{i\,\omega_0}{m+i\,\omega_0}+\frac{b+\frac{h^2 m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}\right)\right) * \left\{-\frac{h}{\sqrt{2}\,(n+i\,\omega_0)},\ -\frac{-b+i\,\omega_0}{m},\ 1\right\}$$

$$pb = \text{FullSimplify}\left[\left(1\Big/\left(1+\frac{i\,\omega_0}{m+i\,\omega_0}+\frac{b+\frac{h^2 m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}\right)\right) * \left\{-\frac{h}{\sqrt{2}\,(n-i\,\omega_0)},\ \frac{b+i\,\omega_0}{m},\ 1\right\},\right.$$

simplifica completamente

$$\left.\omega_0 \in \text{Reals}\ \&\&\ m>0\ \&\&\ n<0\ \&\&\ h>0\ \&\&\ b>0\right]$$

números reais

$$\left\{\frac{h\,(-i\,m+\omega_0)\,(i\,n+\omega_0)}{\sqrt{2}\,\left(h^2 m-(n-i\,\omega_0)^2\,(b+m+2\,i\,\omega_0)\right)},\ \frac{b+i\,\omega_0}{m\left(1+\frac{i\,\omega_0}{m+i\,\omega_0}+\frac{b+\frac{h^2 m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}\right)},\ \frac{1}{1+\frac{i\,\omega_0}{m+i\,\omega_0}+\frac{b+\frac{h^2 m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}}\right\}$$

$$h = \sqrt{\frac{(b+m)\,n\,(-b-m+n)}{m}}$$

$$\sqrt{\frac{(b+m)\,n\,(-b-m+n)}{m}}$$

$$\omega_0 = \sqrt{-(b+m)\,n}$$

$$\sqrt{(-b-m)\,n}$$

$$\text{FullSimplify}[\text{Transpose}[A].pb - i\,\omega_0\,pb]$$

simplifica comple··· transposição

{0, 0, 0}

```
Clear[h]
```

apaga

$$\text{Unset}[\omega_0]$$

elimina alocação

$$\text{FullSimplify}[A * q - lb1 * q]\ /.\ \sqrt{(b+m)\,n} \to I * \omega_0\ /.\ \sqrt{-\frac{(b+m)\,(b+m-n)\,n}{m}} \to h$$

simplifica completamente · unidade imaginária

$$\left\{\left\{\frac{\sqrt{2}\,h\,m}{m+i\,\omega_0},\ \frac{h\,m\,(2\,i\,h-\sqrt{2}\,\omega_0)}{(n-i\,\omega_0)\,(-i\,m+\omega_0)},\ \frac{\sqrt{2}\,h\,m\,\omega_0}{(m+i\,\omega_0)\,(i\,n+\omega_0)}\right\},\right.$$
$$\left.\left\{-\frac{i\,m\,\omega_0}{m+i\,\omega_0},\ -m,\ \frac{m\,(m-i\,\omega_0)}{m+i\,\omega_0}\right\},\ \left\{\frac{h}{\sqrt{2}}-i\,\omega_0,\ b-i\,\omega_0,\ -b-i\,\omega_0\right\}\right\}$$

```
(* Teste Normalização *)
```

```
FullSimplify[pb.q]
```
simplifica completamente

1

```
(* Funções multilineares simétricas B, C, D  e  E *)
```
c⋯ deriv⋯ númer

```
(* Função B *)

bb[{x1_, x2_, x3_}, {y1_, y2_, y3_}] := {-2 x2 y2, 0 , x1 * y2 + x2 * y1}

(* Função C *)
```
consta

```
cc[{x1_, x2_, x3_}, {y1_, y2_, y3_}, {u1_, u2_, u3_}] := {0, 0, 0}

(* Função D *)
```
derivа

```
dd[{x1_, x2_, x3_}, {y1_, y2_, y3_}, {u1_, u2_, u3_}, {v1_, v2_, v3_}] := {0, 0, 0}

(* Função E *)
```
númer

```
ee[{x1_, x2_, x3_}, {y1_, y2_, y3_},
  {u1_, u2_, u3_}, {v1_, v2_, v3_}, {w1_, w2_, w3_}] := {0, 0, 0}

(* Parte linear do campo de vetores *)
```

A = Simplify[Df[P0]] /. $\sqrt{-\frac{(b+m)\ (b+m-n)\ n}{m}} \to h$

simplifica

$$\left\{\{n, -\sqrt{2}\ h, 0\}, \{0, -m, m\}, \left\{\frac{h}{\sqrt{2}}, b, -b\right\}\right\}$$

```
(* Inversa da matriz A *)

AI = FullSimplify[Inverse[A]]
```
simplifica comple⋯ matriz inversa

$$\left\{\left\{0, \frac{\sqrt{2}\ b}{h\ m}, \frac{\sqrt{2}}{h}\right\}, \left\{-\frac{1}{\sqrt{2}\ h}, \frac{b\ n}{h^2\ m}, \frac{n}{h^2}\right\}, \left\{-\frac{1}{\sqrt{2}\ h}, \frac{1+\frac{b\ n}{h^2}}{m}, \frac{n}{h^2}\right\}\right\}$$

(* Matriz D2 = 2$i\omega_0$I *)

unidad

```
D2 = 2 ⅈ ω0 IdentityMatrix[3]
```

matriz identidade

$$\{\{2\,i\,\omega_0,\,0,\,0\},\,\{0,\,2\,i\,\omega_0,\,0\},\,\{0,\,0,\,2\,i\,\omega_0\}\}$$

```
(* Matriz DA = 2ⅈω0I-A *)
```

unidade ir

```
DA = D2 - A
```

$$\left\{\left\{-n+2\,i\,\omega_0,\,\sqrt{2}\,h,\,0\right\},\,\{0,\,m+2\,i\,\omega_0,\,-m\},\,\left\{-\frac{h}{\sqrt{2}},\,-b,\,b+2\,i\,\omega_0\right\}\right\}$$

```
(* Inversa da matriz DA *)
```

```
DAI = FullSimplify[Inverse[DA], ω0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
```

simplifica comple··· matriz inversa números reais

$$\left\{\left\{-\frac{2\,i\,(b+m+2\,i\,\omega_0)\,\omega_0}{-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)},\right.\right.$$
$$\left.-\frac{\sqrt{2}\,h\,(b+2\,i\,\omega_0)}{h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0},\,-\frac{\sqrt{2}\,h\,m}{h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0}\right\},$$
$$\left\{\frac{h\,m}{\sqrt{2}\,\left(h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0\right)},\,-\frac{(n-2\,i\,\omega_0)\,(b+2\,i\,\omega_0)}{h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0},\right.$$
$$\left.\frac{-i\,m\,n-2\,m\,\omega_0}{i\,h^2\,m+2\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0}\right\},\,\left\{\frac{h\,(m+2\,i\,\omega_0)}{\sqrt{2}\,\left(h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0\right)},\right.$$
$$\left.\left.-\frac{h^2+b\,n-2\,i\,b\,\omega_0}{h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0},\,-\frac{(n-2\,i\,\omega_0)\,(m+2\,i\,\omega_0)}{h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0}\right\}\right\}$$

```
(* Calculo do vetor complexo h20 *)
```

```
h20 = FullSimplify[DAI.bb[q, q], ω0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
```

simplifica completamente números reais

$$\left\{\left(4\,m^2\,\left(i\,h^2\,m-(n-i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0\right)\right)\Big/\right.$$
$$\left((m+i\,\omega_0)^2\,(i\,n+\omega_0)\,\left(-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)\right)\right),$$
$$\left(\sqrt{2}\,h\,m^3\,(3\,i\,n+5\,\omega_0)\right)\Big/\left((m+i\,\omega_0)^2\,(i\,n+\omega_0)\,\left(-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)\right)\right),$$
$$\left(\sqrt{2}\,h\,m^2\,(m+2\,i\,\omega_0)\,(3\,i\,n+5\,\omega_0)\right)\Big/$$
$$\left.\left((m+i\,\omega_0)^2\,(i\,n+\omega_0)\,\left(-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)\right)\right)\right\}$$

```
(* Vetor complexo h20b *)
```

```
h20b = Simplify[ComplexExpand[Conjugate[h20]],
        simplifica   expande funções ⋯  conjugado
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
      números reais
```

$$\left\{\left(4\, m^2 \left(i\, h^2\, m + (b+m)\, n\, \omega_0 + i\, (b+m-2\, n)\, \omega_0^2 + 2\, \omega_0^3\right)\right) \Big/ \left(\left(m - i\, \omega_0\right)^2 \left(-i\, n + \omega_0\right) \left(h^2\, m + 2\, i\, (b+m)\, n\, \omega_0 - 4\, (b+m-n)\, \omega_0^2 + 8\, i\, \omega_0^3\right)\right),\right.$$
$$-\left(\left(\sqrt{2}\, h\, m^3 \left(-3\, i\, n + 5\, \omega_0\right)\right) \Big/ \left(\left(m - i\, \omega_0\right)^2 \left(-i\, n + \omega_0\right) \left(h^2\, m + 2\, i\, (b+m)\, n\, \omega_0 - 4\, (b+m-n)\, \omega_0^2 + 8\, i\, \omega_0^3\right)\right)\right),$$
$$\left.-\left(\left(\sqrt{2}\, h\, m^2 \left(m - 2\, i\, \omega_0\right) \left(-3\, i\, n + 5\, \omega_0\right)\right) \Big/ \left(\left(m - i\, \omega_0\right)^2 \left(-i\, n + \omega_0\right) \left(h^2\, m + 2\, i\, (b+m)\, n\, \omega_0 - 4\, (b+m-n)\, \omega_0^2 + 8\, i\, \omega_0^3\right)\right)\right)\right\}$$

```
(* Calculo do vetor complexo h11 *)
```

```
h11 = Simplify[-AI.bb[q, qb], ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
      simplifica                  números reais
```

$$\left\{-\frac{4\, m^2\, n}{\left(m^2 + \omega_0^2\right) \left(n^2 + \omega_0^2\right)},\ -\frac{\sqrt{2}\, m^2 \left(3\, n^2 + \omega_0^2\right)}{h \left(m^2 + \omega_0^2\right) \left(n^2 + \omega_0^2\right)},\ -\frac{\sqrt{2}\, m^2 \left(3\, n^2 + \omega_0^2\right)}{h \left(m^2 + \omega_0^2\right) \left(n^2 + \omega_0^2\right)}\right\}$$

```
(* Cálculo do número complexo G21 *)
```

```
G21 = FullSimplify[pb.(cc[q, q, qb] + 2 bb[q, h11] + bb[qb, h20]),
      simplifica completamente
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
      números reais
```

$$\left(4\, m^3 \left(12\, i\, h^2\, m\, n^2 + \omega_0 \left(7\, h^2\, m\, n + 15\, (b+m)\, n^3 + i \left(3\, h^2\, m + 5\, n^2\, (-7\, (b+m) + 6\, n)\right) \omega_0 - 5\, (b+m-14\, n)\, n\, \omega_0^2 - i\, (7\, (b+m) + 10\, n)\, \omega_0^3 + 14\, \omega_0^4\right)\right)\right) \Big/$$
$$\left(\left(m - i\, \omega_0\right) \left(m + i\, \omega_0\right) \left(-i\, n + \omega_0\right) \left(-h^2\, m + 2 \left(b + m + 2\, i\, \omega_0\right) \omega_0 \left(i\, n + 2\, \omega_0\right)\right) \left(h^2\, m - (b+m)\, n^2 + \omega_0 \left(2\, i\, (b+m-n)\, n + \left(b + m - 4\, n + 2\, i\, \omega_0\right) \omega_0\right)\right)\right)$$

```
(* Cálculo do número complexo G21b *)
```

```
G21b = FullSimplify[ComplexExpand[Conjugate[G21]],
        simplifica comple··· expande funções ··· conjugado
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
        números reais
```

$$\Big(4\,\mathrm{i}\,m^3\,\Big(-12\,\mathrm{i}\,h^2 m n^2+\omega_0\,\big(7 h^2 m n+15\,(b+m)\,n^3+\omega_0\,\big(-\mathrm{i}\,\big(3 h^2 m+5 n^2\,(-7\,(b+m)+6 n)\big)+\omega_0\,\big(-5\,(b+m-14 n)\,n+\omega_0\,\big(7\,\mathrm{i}\,(b+m)+10\,\mathrm{i}\,n+14\,\omega_0\big)\big)\big)\big)\Big)\Big)\Big/\Big(\big(n-\mathrm{i}\,\omega_0\big)\,\big(m^2+\omega_0^2\big)\,\big(h^2 m+2\,\big(n+2\,\mathrm{i}\,\omega_0\big)\,\omega_0\,\big(\mathrm{i}\,(b+m)+2\,\omega_0\big)\big)\,\big(h^2 m-(b+m)\,n^2+\omega_0\,\big(-2\,\mathrm{i}\,(b+m-n)\,n+\big(b+m-4 n-2\,\mathrm{i}\,\omega_0\big)\,\omega_0\big)\big)\Big)$$

```
(* Cálculo da parte real do número complexo G21 *)
```

```
ReG21 =
 FullSimplify[ComplexExpand[Re[G21]], ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
 simplifica comple··· expande funções ···  parte real          números reais
```

$$\Big(8 m^3\,\Big(6 h^4 m^2 n^3\,\big(h^2 m-(b+m)\,n^2\big)+\omega_0^2\,\Big(-2 h^6 m^3 n+6 h^2 m\,(b+m)\,(9\,(b+m)-8 n)\,n^5-15\,(b+m)^3 n^7+h^4 m^2 n^3\,(-41\,(b+m)+34 n)-n\,\Big(-5 h^4 m^2\,(b+m)-20 h^4 m^2 n-128 h^2 m\,(b+m)^2 n^2+48 h^2 m\,(b+m)\,n^3+\big(85 b^3+255 b^2 m+16 h^2 m+255 b m^2+85 m^3\big)\,n^4+20\,(b+m)^2 n^5+60\,(b+m)\,n^6\Big)\,\omega_0^2+\Big(2 h^4 m^2+2 h^2 m n\,\big(13\,(b+m)^2+12\,(b+m)\,n+48 n^2\big)+n^3\,\big(-101\,(b+m)^3-116\,(b+m)^2 n-340\,(b+m)\,n^2-80 n^3\big)\Big)\,\omega_0^4-\Big(-24 h^2 m\,(b+m)+\big(31 b^3+93 b^2 m+80 h^2 m+93 b m^2+31 m^3\big)\,n+124\,(b+m)^2 n^2+404\,(b+m)\,n^3+464 n^4\Big)\,\omega_0^6-4\,\big(7\,(b+m)^2+31\,(b+m)\,n+124 n^2\big)\,\omega_0^8-112\,\omega_0^{10}\Big)\Big)\Big/\Big(\big(m^2+\omega_0^2\big)\,\big(n^2+\omega_0^2\big)\,\Big(\big(h^2 m-(b+m)\,n^2\big)^2+2\,\big(h^2 m\,(b+m)-4 h^2 m n+(b+m)^2 n^2+2 n^4\big)\,\omega_0^2+\big((b+m)^2+8 n^2\big)\,\omega_0^4+4\,\omega_0^6\Big)\,\Big(h^4 m^2+4\,\omega_0^2\,\big(-2 h^2 m\,(b+m)+2 h^2 m n+(b+m)^2 n^2+4\,\omega_0^2\,\big((b+m)^2+n^2+4\,\omega_0^2\big)\big)\Big)\Big)$$

```
(* Cálculo de l1 *)
```

```
l1 = 1/2 FullSimplify[ReG21, ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```
simplifica completamente — números reais

$$\Big(4 m^3 \big(6 h^4 m^2 n^3 (h^2 m - (b+m) n^2) + \omega_0^2 \big(-2 h^6 m^3 n + 6 h^2 m (b+m) (9 (b+m) - 8 n) n^5 - 15 (b+m)^3 n^7 + h^4 m^2 n^3 (-41 (b+m) + 34 n) - n \big(-5 h^4 m^2 (b+m) - 20 h^4 m^2 n - 128 h^2 m (b+m)^2 n^2 + 48 h^2 m (b+m) n^3 + (85 b^3 + 255 b^2 m + 16 h^2 m + 255 b m^2 + 85 m^3) n^4 + 20 (b+m)^2 n^5 + 60 (b+m) n^6\big) \omega_0^2 + \big(2 h^4 m^2 + 2 h^2 m n (13 (b+m)^2 + 12 (b+m) n + 48 n^2) + n^3 (-101 (b+m)^3 - 116 (b+m)^2 n - 340 (b+m) n^2 - 80 n^3)\big) \omega_0^4 - \big(-24 h^2 m (b+m) + (31 b^3 + 93 b^2 m + 80 h^2 m + 93 b m^2 + 31 m^3) n + 124 (b+m)^2 n^2 + 404 (b+m) n^3 + 464 n^4\big) \omega_0^6 - 4 (7 (b+m)^2 + 31 (b+m) n + 124 n^2) \omega_0^8 - 112 \omega_0^{10}\big)\big)\Big) \Big/ \Big((m^2 + \omega_0^2) (n^2 + \omega_0^2) \big((h^2 m - (b+m) n^2)^2 + 2 (h^2 m (b+m) - 4 h^2 m n + (b+m)^2 n^2 + 2 n^4) \omega_0^2 + ((b+m)^2 + 8 n^2) \omega_0^4 + 4 \omega_0^6\big) \big(h^4 m^2 + 4 \omega_0^2 (-2 h^2 m (b+m) + 2 h^2 m n + (b+m)^2 n^2 + 4 \omega_0^2 ((b+m)^2 + n^2 + 4 \omega_0^2))\big)\Big)$$

```
(* Matriz D3 = 3iω0I *)
```
unidad

```
D3 = 3 i ω0 IdentityMatrix[3]
```
matriz identidade

$$\{\{3 i \omega_0, 0, 0\}, \{0, 3 i \omega_0, 0\}, \{0, 0, 3 i \omega_0\}\}$$

```
(* Matriz TA = 3iω0I-A *)
```
unidade ir

```
TA = D3 - A
```

$$\left\{\left\{-n + 3 i \omega_0, \sqrt{2}\, h, 0\right\}, \{0, m + 3 i \omega_0, -m\}, \left\{-\frac{h}{\sqrt{2}}, -b, b + 3 i \omega_0\right\}\right\}$$

```
(* Matriz inversa da matriz TA *)
```

```
TAI = FullSimplify[Inverse[TA], ω0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
```
simplifica comple··· matriz inversa números reais

$$\left\{\left\{-\frac{3\,i\,(b+m+3\,i\,\omega_0)\,\omega_0}{-h^2\,m+3\,(b+m+3\,i\,\omega_0)\,\omega_0\,(i\,n+3\,\omega_0)},\right.\right.$$
$$\left.-\frac{\sqrt{2}\,h\,(b+3\,i\,\omega_0)}{h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0},\ -\frac{\sqrt{2}\,h\,m}{h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0}\right\},$$
$$\left\{\frac{h\,m}{\sqrt{2}\,\left(h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0\right)},\ -\frac{(n-3\,i\,\omega_0)\,(b+3\,i\,\omega_0)}{h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0},\right.$$
$$\left.\frac{-i\,m\,n-3\,m\,\omega_0}{i\,h^2\,m+3\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0}\right\},\ \left\{\frac{h\,(m+3\,i\,\omega_0)}{\sqrt{2}\,\left(h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0\right)},\right.$$
$$\left.\left.-\frac{h^2+b\,n-3\,i\,b\,\omega_0}{h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0},\ -\frac{(n-3\,i\,\omega_0)\,(m+3\,i\,\omega_0)}{h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0}\right\}\right\}$$

(* Cálculo do vetor complexo h30 *)

```
h30 = FullSimplify[TAI.(3 bb[q, h20] + cc[q, q, q]),
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
```
simplifica completamente

números reais

$$\Big\{\Big(6\sqrt{2}\,h\,m^4\,\big(h^2\,m\,(-5\,n+7\,i\,\omega_0)-(n-i\,\omega_0)\,\omega_0\,(11\,i\,(b+m)\,n+(17\,(b+m)-31\,n+49\,i\,\omega_0)\,\omega_0)\big)\Big)\Big/$$
$$\Big((m+i\,\omega_0)^3\,(i\,n+\omega_0)^2\,(-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0))\,(-h^2\,m+3\,(b+m+3\,i\,\omega_0)\,\omega_0\,(i\,n+3\,\omega_0))\Big),$$
$$\Big(12\,m^4\,\big(-4\,h^2\,m\,n^2+\omega_0\,(15\,i\,h^2\,m\,n-i\,(b+m)\,n^3+\omega_0\,(13\,h^2\,m+n^2\,(-5\,(b+m)+2\,n)+\omega_0\,(i\,(7\,(b+m)-10\,n)\,n+(3\,(b+m)-14\,n+6\,i\,\omega_0)\,\omega_0)))\big)\Big)\Big/$$
$$\Big((m+i\,\omega_0)^3\,(i\,n+\omega_0)^2\,(-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0))\,(-h^2\,m+3\,(b+m+3\,i\,\omega_0)\,\omega_0\,(i\,n+3\,\omega_0))\Big),\ \Big(12\,m^3\,(m+3\,i\,\omega_0)$$
$$\big(-4\,h^2\,m\,n^2+\omega_0\,(15\,i\,h^2\,m\,n-i\,(b+m)\,n^3+\omega_0\,(13\,h^2\,m+n^2\,(-5\,(b+m)+2\,n)+\omega_0\,(i\,(7\,(b+m)-10\,n)\,n+(3\,(b+m)-14\,n+6\,i\,\omega_0)\,\omega_0)))\big)\Big)\Big/$$
$$\Big((m+i\,\omega_0)^3\,(i\,n+\omega_0)^2\,(-h^2\,m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0))\,(-h^2\,m+3\,(b+m+3\,i\,\omega_0)\,\omega_0\,(i\,n+3\,\omega_0))\Big)\Big\}$$

(* Cálculo do vetor complexo h30b *)

```
h30b = FullSimplify[ComplexExpand[Conjugate[h30]],
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```

simplifica comple··· expande funções ··· conjugado

números reais

$$\Big\{\Big(6\sqrt{2}\,h\,m^4\,\big(-h^2 m\,(5n+7\,i\,\omega_0)+(n+i\,\omega_0)\,\omega_0\,(11\,i\,(b+m)\,n+(-17\,(b+m)+31\,n+49\,i\,\omega_0)\,\omega_0)\big)\Big)\Big/\Big((m-i\,\omega_0)^3\,(-i\,n+\omega_0)^2\,(-h^2 m+2\,(b+m-2\,i\,\omega_0)\,\omega_0\,(-i\,n+2\,\omega_0))\,(-h^2 m+3\,(b+m-3\,i\,\omega_0)\,\omega_0\,(-i\,n+3\,\omega_0))\Big),$$

$$\Big(12\,m^4\,\big(-4\,h^2\,m\,n^2+\omega_0\,(-15\,i\,h^2\,m\,n+i\,(b+m)\,n^3+\omega_0\,(13\,h^2\,m+n^2\,(-5\,(b+m)+2\,n)+\omega_0\,(-i\,(7\,(b+m)-10\,n)\,n+(3\,(b+m)-14\,n-6\,i\,\omega_0)\,\omega_0)))\big)\Big)\Big/\Big((m-i\,\omega_0)^3\,(-i\,n+\omega_0)^2\,(-h^2 m+2\,(b+m-2\,i\,\omega_0)\,\omega_0\,(-i\,n+2\,\omega_0))\,(-h^2 m+3\,(b+m-3\,i\,\omega_0)\,\omega_0\,(-i\,n+3\,\omega_0))\Big),$$

$$\Big(12\,m^3\,(m-3\,i\,\omega_0)\,\big(-4\,h^2\,m\,n^2+\omega_0\,(-15\,i\,h^2\,m\,n+i\,(b+m)\,n^3+\omega_0\,(13\,h^2\,m+n^2\,(-5\,(b+m)+2\,n)+\omega_0\,(-i\,(7\,(b+m)-10\,n)\,n+(3\,(b+m)-14\,n-6\,i\,\omega_0)\,\omega_0)))\big)\Big)\Big/\Big((m-i\,\omega_0)^3\,(-i\,n+\omega_0)^2\,(-h^2 m+2\,(b+m-2\,i\,\omega_0)\,\omega_0\,(-i\,n+2\,\omega_0))\,(-h^2 m+3\,(b+m-3\,i\,\omega_0)\,\omega_0\,(-i\,n+3\,\omega_0))\Big)\Big\}$$

```
(* Matriz D1 = iω0I *)
```

unidad

```
D1 = i ω0 IdentityMatrix[3]
```

matriz identidade

$$\{\{i\,\omega_0,\,0,\,0\},\,\{0,\,i\,\omega_0,\,0\},\,\{0,\,0,\,i\,\omega_0\}\}$$

```
(* Matriz L = iω0I-A *)
```

unidade ir

```
L = D1 - A
```

$$\Big\{\{-n+i\,\omega_0,\,\sqrt{2}\,h,\,0\},\,\{0,\,m+i\,\omega_0,\,-m\},\,\Big\{-\frac{h}{\sqrt{2}},\,-b,\,b+i\,\omega_0\Big\}\Big\}$$

```
q
```

$$\Big\{-\frac{\sqrt{2}\,h\,m}{(m+i\,\omega_0)\,(-n+i\,\omega_0)},\,\frac{m}{m+i\,\omega_0},\,1\Big\}$$

```
pb
```

$$\Big\{\frac{h\,(-i\,m+\omega_0)\,(i\,n+\omega_0)}{\sqrt{2}\,\big(h^2\,m-(n-i\,\omega_0)^2\,(b+m+2\,i\,\omega_0)\big)},\,\frac{b+i\,\omega_0}{m\left(1+\frac{i\,\omega_0}{m+i\,\omega_0}+\frac{b+\frac{h^2\,m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}\right)},\,\frac{1}{1+\frac{i\,\omega_0}{m+i\,\omega_0}+\frac{b+\frac{h^2\,m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}}\Big\}$$

```
i ω0 IdentityMatrix[3] - A
```

matriz identidade

$$\Big\{\{-n+i\,\omega_0,\,\sqrt{2}\,h,\,0\},\,\{0,\,m+i\,\omega_0,\,-m\},\,\Big\{-\frac{h}{\sqrt{2}},\,-b,\,b+i\,\omega_0\Big\}\Big\}$$

$$\text{(* Matriz L21 = } \left(\begin{pmatrix} \mathrm{i}\omega_0 \text{IdentityMatrix[3]-A} & q \\ pb & 0 \end{pmatrix}\right) \text{ *)}$$

$$\text{L21} = \left\{\left\{-n + \mathrm{i}\,\omega_0,\ \sqrt{2}\,h,\ 0,\ -\frac{\sqrt{2}\,h\,m}{(m + \mathrm{i}\,\omega_0)\,(-n + \mathrm{i}\,\omega_0)}\right\},\right.$$
$$\left\{0,\ m + \mathrm{i}\,\omega_0,\ -m,\ \frac{m}{m + \mathrm{i}\,\omega_0}\right\},\ \left\{-\frac{h}{\sqrt{2}},\ -b,\ b + \mathrm{i}\,\omega_0,\ 1\right\},$$
$$\left.\left\{\frac{h\,(-\mathrm{i}\,m + \omega_0)\,(\mathrm{i}\,n + \omega_0)}{\sqrt{2}\,\left(h^2 m - (n - \mathrm{i}\,\omega_0)^2\,(b + m + 2\,\mathrm{i}\,\omega_0)\right)},\ \frac{b + \mathrm{i}\,\omega_0}{m\left(1 + \frac{\mathrm{i}\,\omega_0}{m+\mathrm{i}\,\omega_0} + \frac{b+\frac{h^2 m}{(\mathrm{i}\,n+\omega_0)^2}}{m+\mathrm{i}\,\omega_0}\right)},\ \frac{1}{1 + \frac{\mathrm{i}\,\omega_0}{m+\mathrm{i}\,\omega_0} + \frac{b+\frac{h^2 m}{(\mathrm{i}\,n+\omega_0)^2}}{m+\mathrm{i}\,\omega_0}},\ 0\right\}\right\}$$

$$\left\{\left\{-n + \mathrm{i}\,\omega_0,\ \sqrt{2}\,h,\ 0,\ -\frac{\sqrt{2}\,h\,m}{(m + \mathrm{i}\,\omega_0)\,(-n + \mathrm{i}\,\omega_0)}\right\},\right.$$
$$\left\{0,\ m + \mathrm{i}\,\omega_0,\ -m,\ \frac{m}{m + \mathrm{i}\,\omega_0}\right\},\ \left\{-\frac{h}{\sqrt{2}},\ -b,\ b + \mathrm{i}\,\omega_0,\ 1\right\},$$
$$\left.\left\{\frac{h\,(-\mathrm{i}\,m + \omega_0)\,(\mathrm{i}\,n + \omega_0)}{\sqrt{2}\,\left(h^2 m - (n - \mathrm{i}\,\omega_0)^2\,(b + m + 2\,\mathrm{i}\,\omega_0)\right)},\ \frac{b + \mathrm{i}\,\omega_0}{m\left(1 + \frac{\mathrm{i}\,\omega_0}{m+\mathrm{i}\,\omega_0} + \frac{b+\frac{h^2 m}{(\mathrm{i}\,n+\omega_0)^2}}{m+\mathrm{i}\,\omega_0}\right)},\ \frac{1}{1 + \frac{\mathrm{i}\,\omega_0}{m+\mathrm{i}\,\omega_0} + \frac{b+\frac{h^2 m}{(\mathrm{i}\,n+\omega_0)^2}}{m+\mathrm{i}\,\omega_0}},\ 0\right\}\right\}$$

```
(* Inversa da matriz L21 *)

L21I = Simplify[Inverse[L21]];
       simplifica  matriz inversa

(* Cálculo de R21 *)

{b11, b22, b33} = Simplify[cc[q, q, qb] + bb[qb, h20] + 2 bb[q, h11] - G21 q,
                  simplifica
   ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0];
      números reais

(* Cálculo de H21 *)

H21 = {b11, b22, b33, 0};

(* Cálculo de h21 *)

{r21, r22, r23, S} =
  FullSimplify[L21I.H21, ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0];
  simplifica completamente    números reais

(* Cálculo do vetor complexo h21 *)

h21 = FullSimplify[{r21, r22, r23}, ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
      simplifica completamente          números reais
```

$$\left\{\left(2\sqrt{2}\,m^3\right.\right.$$
$$\left(3\,\mathrm{i}\,h^2\,m\,n^2\left(5\,h^2\,m\,(b+m) - 5\,h^2\,m\,n + 3\,(b+m)^2\,n^2\right) + \omega_0\left(n\left(12\,h^4\,m^2\,(b+m) - 57\,h^4\,m^2\,n + \right.\right.\right.$$
$$\left.38\,h^2\,m\,(b+m)^2\,n^2 - 45\,h^2\,m\,(b+m)\,n^3 + 12\,(b+m)^3\,n^4\right) +$$

$$\omega_0 \left(-i \left(h^4 m^2 (b+m) - 37 h^4 m^2 n + 52 h^2 m (b+m)^2 n^2 - 196 h^2 m (b+m) n^3 + 3\right.\right.$$
$$\left(16 b^3 + 48 b^2 m + 21 h^2 m + 48 b m^2 + 16 m^3\right) n^4 - 60 (b+m)^2 n^5\Big) +$$
$$\omega_0 \left(3 h^4 m^2 - 2 h^2 m (b+m-130 n) (b+m-n) n + 4 (b+m) n^3 \left(-14 (b+m)^2 +\right.\right.$$
$$60 (b+m) n - 27 n^2\Big) + \omega_0 \left(-i \left(13 h^2 m (b+m)^2 + 28 h^2 m (b+m) n -\right.\right.$$
$$2 \left(4 b^3 + 12 b^2 m + 161 h^2 m + 12 b m^2 + 4 m^3\right) n^2 + 280 (b+m)^2 n^3 -$$
$$432 (b+m) n^4 + 72 n^5\Big) + \omega_0 \left(51 h^2 m (b+m) - 4 \left(5 b^3 + 15 b^2 m - 9 h^2\right.\right.$$
$$m + 15 b m^2 + 5 m^3\Big) n - 40 (b+m)^2 n^2 + 504 (b+m) n^3 - 288 n^4 +$$
$$\omega_0 \left(i \left(8 b^3 + 24 b^2 m + 57 h^2 m + 24 b m^2 + 8 m^3 - 100 (b+m)^2 n -\right.\right.$$
$$72 (b+m) n^2 + 336 n^3\Big) + 4 \omega_0 \left(-10 (b+m)^2 + 45 (b+m) n +\right.$$
$$12 n^2 + 6 \omega_0 \left(-3 i (b+m) + 5 i n + 2 \omega_0\right)))))))))) \Big/$$
$$\Big(h \left(m - i \omega_0\right) \left(m + i \omega_0\right)^2 \left(-i n + \omega_0\right) \left(i n + \omega_0\right) \left(h^2 m - 2 i \left(n - 2 i \omega_0\right)\right.$$
$$\left(b + m + 2 i \omega_0\right)$$
$$\omega_0\Big)$$
$$\Big(i n \left(2 h^2 m (b+m) - h^2 m n - (b+m)^2 n^2\right) +$$
$$\omega_0$$
$$\Big(3 h^2 m (b+m) - 6 h^2 m n - 3 (b+m)^2 n^2 + 3 (b+m) n^3 +$$
$$\omega_0 \left(3 i \left(2 h^2 m + n \left((b+m)^2 - 3 (b+m) n + n^2\right)\right) +\right.$$
$$\omega_0 \left((b+m)^2 - 9 (b+m) n + 9 n^2 + 3 i \left(b + m - 3 n + i \omega_0\right) \omega_0\right)))))\Big),$$
$$\Big(2 m^4 \left(3 h^2 m n^2 \left(5 h^2 m + (3 (b+m) - 8 n) n^2\right) +\right.$$
$$\omega_0$$
$$\Big(-2 i n \left(6 h^4 m^2 + h^2 m (19 (b+m) - 52 n) n^2 + 3 (b+m) (2 (b+m) - 5 n) n^4\right) +$$
$$\omega_0 \left(-h^4 m^2 - 4 h^2 m (13 (b+m) - 46 n) n^2 + 4 n^4 \left(-12 (b+m)^2 + 52 (b+m) n - 15 n^2\right) +\right.$$
$$\omega_0 \left(2 i n \left(h^2 m (b+m) - 76 h^2 m n + 28 (b+m)^2 n^2 - 251 (b+m) n^3 + 184 n^4\right) + \omega_0\right.$$
$$\Big(-13 h^2 m (b+m) + 8 (b+m)^2 n^2 - 480 (b+m) n^3 + 812 n^4 +$$
$$2 i \left(-16 h^2 m + n \left(10 (b+m)^2 + 45 (b+m) n - 368 n^2\right)\right) \omega_0 + 4 \left(2 (b+m)^2 -\right.$$
$$28 (b+m) n - 37 n^2\Big) \omega_0^2 + 2 i (23 (b+m) - 72 n) \omega_0^3 - 60 \omega_0^4)))))) \Big/$$
$$\Big(\left(m - i \omega_0\right) \left(m + i \omega_0\right)^2 \left(-i n + \omega_0\right) \left(i n + \omega_0\right)^2 \left(-h^2 m + 2 \left(b + m + 2 i \omega_0\right)\right.$$
$$\omega_0$$
$$\left(i n + 2 \omega_0\right)\Big)\Big)$$
$$\Big(i n \left(2 h^2 m (b+m) - h^2 m n - (b+m)^2 n^2\right) +$$
$$\omega_0$$
$$\Big(3 h^2 m (b+m) - 6 h^2 m n - 3 (b+m)^2 n^2 + 3 (b+m) n^3 +$$
$$\omega_0 \left(3 i \left(2 h^2 m + n \left((b+m)^2 - 3 (b+m) n + n^2\right)\right) +\right.$$
$$\omega_0 \left((b+m)^2 - 9 (b+m) n + 9 n^2 + 3 i \left(b + m - 3 n + i \omega_0\right) \omega_0\right)))))\Big),$$
$$\Big(2 m^3 \left(3 h^2 m n^2 \left(5 h^2 m (m - n) + n^2 (3 m (b+m) + 8 b n)\right) +\right.$$
$$\omega_0 \left(-2 i n \left(3 h^4 m^2 (2 m - 7 n) + 3 (b+m) n^4 (2 m (b+m) + 5 b n) +\right.\right.$$
$$h^2 m n^2 \left(19 m (b+m) + 2 (17 b - 9 m) n - 12 n^2\right)\Big) +$$
$$\omega_0 \left(-h^4 m^2 (m - 25 n) + h^2 m n^2 \left(-52 m (b+m) + 2 (-13 b + 79 m) n + 41 n^2\right) +\right.$$
$$2 n^4 \left(-24 m (b+m)^2 + 8 (b+m) (-10 b + 3 m) n + 15 (3 b + m) n^2\right) +$$
$$\omega_0 \left(-2 i \left(h^4 m^2 - h^2 m n \left(m^2 + b (m - 29 n) - 105 m n + 38 n^2\right) + n^3 \left(-28 m (b+m)^2 +\right.\right.\right.$$
$$(b+m) (-155 b + 96 m) n + 2 (117 b + 25 m) n^2 - 30 n^3\Big)\Big) + \omega_0$$
$$\Big(-13 h^2 m^2 (b+m) - 26 h^2 m (b+m) n + 2 m \left(85 h^2 + 4 (b+m)^2\right) n^2 +$$
$$32 (8 b - 7 m) (b+m) n^3 - 14 (63 b + 5 m) n^4 + 296 n^5 +$$
$$\omega_0 \left(-2 i \left(h^2 m (3 b + 19 m) - 2 m \left(-9 h^2 + 5 (b+m)^2\right) n +\right.\right.$$
$$(29 b - 16 m) (b+m) n^2 + 4 (-89 b + 3 m) n^3 + 262 n^4\Big) +$$

$$\omega_0 \left(m \left(25 h^2 + 8 (b+m)^2\right) + 16 (2 b - 5 m) (b+m) n + 2 (83 b + 9 m)\right.$$
$$n^2 - 400 n^3 + 2 i \omega_0 \left(-7 b^2 + 9 b m + 16 m^2 + 38 b n - 34 m n +\right.$$
$$\left.50 n^2 + \omega_0 \left(-17 i b + 13 i m + 12 i n + 6 \omega_0\right)\right)))))))))\Big/$$
$$\Big(\left(m - i \omega_0\right) \left(m + i \omega_0\right)^2 \left(-i n + \omega_0\right) \left(i n + \omega_0\right)^2 \left(-h^2 m + 2 \left(b + m + 2 i \omega_0\right)\right.$$
$$\omega_0$$
$$\left.\left(i n + 2 \omega_0\right)\right)$$
$$\left(i n \left(2 h^2 m (b+m) - h^2 m n - (b+m)^2 n^2\right) +\right.$$
$$\omega_0$$
$$\left(3 h^2 m (b+m) - 6 h^2 m n - 3 (b+m)^2 n^2 + 3 (b+m) n^3 +\right.$$
$$\omega_0 \left(3 i \left(2 h^2 m + n \left((b+m)^2 - 3 (b+m) n + n^2\right)\right) +\right.$$
$$\omega_0 \left((b+m)^2 - 9 (b+m) n + 9 n^2 + 3 i \left(b + m - 3 n + i \omega_0\right) \omega_0\right)))))\}$$

(* Cálculo do vetor complexo h21b *)

```
h21b = FullSimplify[ComplexExpand[Conjugate[h21]],
       simplifica comple··· expande funções ··· conjugado
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
     números reais
```

$$\Big\{\Big(2 \sqrt{2}\, m^3$$
$$\left(-3 i h^2 m n^2 \left(5 h^2 m (b+m) - 5 h^2 m n + 3 (b+m)^2 n^2\right) + \omega_0 \left(n \left(12 h^4 m^2 (b+m) - 57 h^4 m^2 n +\right.\right.\right.$$
$$\left.38 h^2 m (b+m)^2 n^2 - 45 h^2 m (b+m) n^3 + 12 (b+m)^3 n^4\right) +$$
$$\omega_0 \left(i \left(h^4 m^2 (b+m) - 37 h^4 m^2 n + 52 h^2 m (b+m)^2 n^2 - 196 h^2 m (b+m) n^3 + 3\right.\right.$$
$$\left.\left(16 b^3 + 48 b^2 m + 21 h^2 m + 48 b m^2 + 16 m^3\right) n^4 - 60 (b+m)^2 n^5\right) +$$
$$\omega_0 \left(3 h^4 m^2 - 2 h^2 m (b + m - 130 n) (b + m - n) n + 4 (b+m) n^3 \left(-14 (b+m)^2 +\right.\right.$$
$$\left.60 (b+m) n - 27 n^2\right) + \omega_0 \left(i \left(13 h^2 m (b+m)^2 + 28 h^2 m (b+m) n -\right.\right.$$
$$2 \left(4 b^3 + 12 b^2 m + 161 h^2 m + 12 b m^2 + 4 m^3\right) n^2 + 280 (b+m)^2 n^3 -$$
$$\left.432 (b+m) n^4 + 72 n^5\right) + \omega_0 \left(51 h^2 m (b+m) - 4 \left(5 b^3 + 15 b^2 m - 9 h^2\right.\right.$$
$$\left.m + 15 b m^2 + 5 m^3\right) n - 40 (b+m)^2 n^2 + 504 (b+m) n^3 - 288 n^4 +$$
$$\omega_0 \left(-i \left(8 b^3 + 24 b^2 m + 57 h^2 m + 24 b m^2 + 8 m^3 - 100 (b+m)^2 n -\right.\right.$$
$$\left.72 (b+m) n^2 + 336 n^3\right) + 4 \omega_0 \left(-10 (b+m)^2 + 45 (b+m) n +\right.$$
$$\left.12 n^2 + 6 \omega_0 \left(3 i (b+m) - 5 i n + 2 \omega_0\right)\right)))))))))\Big/$$
$$\Big(h \left(m - i \omega_0\right)^2 \left(m + i \omega_0\right) \left(n^2 + \omega_0^2\right) \left(h^2 m + 2 \left(n + 2 i \omega_0\right) \omega_0 \left(i (b+m) + 2 \omega_0\right)\right)$$
$$\left(-i\right.$$
$$n$$
$$\left(2 h^2 m (b+m) - h^2 m n - (b+m)^2 n^2\right) +$$
$$\omega_0 \left(3 h^2 m (b+m) - 6 h^2 m n - 3 (b+m)^2 n^2 + 3 (b+m) n^3 +\right.$$
$$\omega_0 \left(-3 i \left(2 h^2 m + n \left((b+m)^2 - 3 (b+m) n + n^2\right)\right) +\right.$$
$$\omega_0 \left((b+m)^2 - 9 (b+m) n + 9 n^2 - 3 i (b + m - 3 n) \omega_0 - 3 \omega_0^2\right)))))\Big),$$
$$\Big(2 m^4 \left(3 h^2 m n^2 \left(5 h^2 m + (3 (b+m) - 8 n) n^2\right) +\right.$$
$$\omega_0$$
$$\left(2 i n \left(6 h^4 m^2 + h^2 m (19 (b+m) - 52 n) n^2 + 3 (b+m) (2 (b+m) - 5 n) n^4\right) +\right.$$
$$\omega_0 \left(-h^4 m^2 - 4 h^2 m (13 (b+m) - 46 n) n^2 + 4 n^4 \left(-12 (b+m)^2 + 52 (b+m) n - 15 n^2\right) +\right.$$
$$\omega_0 \left(-2 i n \left(h^2 m (b+m) - 76 h^2 m n + 28 (b+m)^2 n^2 - 251 (b+m) n^3 + 184 n^4\right) + \omega_0\right.$$
$$\left(-13 h^2 m (b+m) + 8 (b+m)^2 n^2 - 480 (b+m) n^3 + 812 n^4 -\right.$$
$$2 i \left(-16 h^2 m + n \left(10 (b+m)^2 + 45 (b+m) n - 368 n^2\right)\right) \omega_0 + 4 \left(2 (b+m)^2 -\right.$$
$$\left.28 (b+m) n - 37 n^2\right) \omega_0^2 - 2 i (23 (b+m) - 72 n) \omega_0^3 - 60 \omega_0^4))))))\Big/$$

```
((m - ⅈ ω0)^2 (m + ⅈ ω0) (-ⅈ n + ω0)^2 (ⅈ n + ω0) (-h^2 m + 2 (b + m - 2 ⅈ ω0)
      ω0
      (-ⅈ n + 2 ω0))
   (-ⅈ n (2 h^2 m (b + m) - h^2 m n - (b + m)^2 n^2) +
     ω0
      (3 h^2 m (b + m) - 6 h^2 m n - 3 (b + m)^2 n^2 + 3 (b + m) n^3 +
        ω0 (-3 ⅈ (2 h^2 m + n ((b + m)^2 - 3 (b + m) n + n^2)) +
           ω0 ((b + m)^2 - 9 (b + m) n + 9 n^2 - 3 ⅈ (b + m - 3 n) ω0 - 3 ω0^2))))),
 (2 m^3 (3 h^2 m n^2 (5 h^2 m (m - n) + n^2 (3 m (b + m) + 8 b n)) +
      ω0 (2 ⅈ n (3 h^4 m^2 (2 m - 7 n) + 3 (b + m) n^4 (2 m (b + m) + 5 b n) +
              h^2 m n^2 (19 m (b + m) + 2 (17 b - 9 m) n - 12 n^2)) +
         ω0 (-h^4 m^2 (m - 25 n) + h^2 m n^2 (-52 m (b + m) + 2 (-13 b + 79 m) n + 41 n^2) +
              2 n^4 (-24 m (b + m)^2 + 8 (b + m) (-10 b + 3 m) n + 15 (3 b + m) n^2) +
              ω0 (2 ⅈ (h^4 m^2 - h^2 m n (m^2 + b (m - 29 n) - 105 m n + 38 n^2) + n^3 (-28 m (b + m)^2 +
                          (b + m) (-155 b + 96 m) n + 2 (117 b + 25 m) n^2 - 30 n^3)) + ω0
                     (-13 h^2 m^2 (b + m) - 26 h^2 m (b + m) n + 2 m (85 h^2 + 4 (b + m)^2) n^2 +
                        32 (8 b - 7 m) (b + m) n^3 - 14 (63 b + 5 m) n^4 + 296 n^5 +
                        ω0 (2 ⅈ (h^2 m (3 b + 19 m) - 2 m (-9 h^2 + 5 (b + m)^2) n +
                                   (29 b - 16 m) (b + m) n^2 + 4 (-89 b + 3 m) n^3 + 262 n^4) + ω0
                              (m (25 h^2 + 8 (b + m)^2) + 16 (2 b - 5 m) (b + m) n + 2 (83 b + 9 m) n^2 -
                                 400 n^3 + 2 ω0 (ⅈ (7 b^2 - 9 b m - 16 m^2 - 38 b n + 34 m n - 50 n^2) +
                                       (17 b - 13 m - 12 n - 6 ⅈ ω0) ω0))))))))) /
  ((m - ⅈ ω0)^2 (m + ⅈ ω0) (-ⅈ n + ω0)^2 (ⅈ n + ω0) (-h^2 m + 2 (b + m - 2 ⅈ ω0)
      ω0
      (-ⅈ n + 2 ω0))
   (-ⅈ n (2 h^2 m (b + m) - h^2 m n - (b + m)^2 n^2) +
     ω0
      (3 h^2 m (b + m) - 6 h^2 m n - 3 (b + m)^2 n^2 + 3 (b + m) n^3 +
        ω0 (-3 ⅈ (2 h^2 m + n ((b + m)^2 - 3 (b + m) n + n^2)) +
           ω0 ((b + m)^2 - 9 (b + m) n + 9 n^2 - 3 ⅈ (b + m - 3 n) ω0 - 3 ω0^2)))))}
```

(* Matriz 4ⅈω0I *)

unidad

```
D4 = Simplify[4 ⅈ ω0 IdentityMatrix[3]]
```

simplifica — matriz identidade

```
{{4 ⅈ ω0, 0, 0}, {0, 4 ⅈ ω0, 0}, {0, 0, 4 ⅈ ω0}}
```

(* Matriz QA = 4ⅈω0I-A *)

unidade ir

```
QA = Simplify[D4 - A]
```

simplifica

$$\left\{\left\{-n+4\,i\,\omega_0,\ \sqrt{2}\,h,\ 0\right\},\ \{0,\ m+4\,i\,\omega_0,\ -m\},\ \left\{-\frac{h}{\sqrt{2}},\ -b,\ b+4\,i\,\omega_0\right\}\right\}$$

(* Inversa da matriz QA *)

```
QAI = FullSimplify[Inverse[QA], ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```
simplifica comple··· matriz inversa números reais

$$\left\{\left\{-\frac{4\,i\,(b+m+4\,i\,\omega_0)\,\omega_0}{-h^2\,m+4\,(b+m+4\,i\,\omega_0)\,\omega_0\,(i\,n+4\,\omega_0)},\right.\right.$$
$$\left.-\frac{\sqrt{2}\,h\,(b+4\,i\,\omega_0)}{h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0},-\frac{\sqrt{2}\,h\,m}{h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0}\right\},$$
$$\left\{\frac{h\,m}{\sqrt{2}\,(h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0)},-\frac{(n-4\,i\,\omega_0)\,(b+4\,i\,\omega_0)}{h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0},\right.$$
$$\left.\frac{-i\,m\,n-4\,m\,\omega_0}{i\,h^2\,m+4\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0}\right\},\left\{\frac{h\,(m+4\,i\,\omega_0)}{\sqrt{2}\,(h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0)},\right.$$
$$\left.\left.-\frac{h^2+b\,n-4\,i\,b\,\omega_0}{h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0},-\frac{(n-4\,i\,\omega_0)\,(m+4\,i\,\omega_0)}{h^2\,m-4\,i\,(n-4\,i\,\omega_0)\,(b+m+4\,i\,\omega_0)\,\omega_0}\right\}\right\}$$

```
(* Vetor complexo h40 *)

h40 = FullSimplify[ComplexExpand[
    QAI.(3 bb[h20, h20] + 4 bb[q, h30] + 6 cc[q, q, h20] + dd[q, q, q, q])],
   ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```
simplifica comple··· expande funções complexas números reais

$$\{(96\,m^5\,(-8\,h^6\,m^3\,n^2+\omega_0\,(i\,h^4\,m^2\,n\,(25\,h^2\,m-11\,(b+m)\,n^2)+\omega_0\,(h^2\,m\,(19\,h^4\,m^2-59\,(b+m)^2\,n^4+h^2\,m\,n^2\,(-32\,(b+m)+63\,n))+\omega_0\,(i\,n\,(15\,h^4\,m^2\,(b+m)-246\,h^4\,m^2\,n+396\,h^2\,m\,(b+m)^2\,n^2-346\,h^2\,m\,(b+m)\,n^3-8\,(b+m)^3\,n^4)+\omega_0\,(-10\,h^4\,m^2\,(b+m)-285\,h^4\,m^2\,n+964\,h^2\,m\,(b+m)^2\,n^2-2346\,h^2\,m\,(b+m)\,n^3-(64\,b^3+192\,b^2\,m-483\,h^2\,m+192\,b\,m^2+64\,m^3)\,n^4+64\,(b+m)^2\,n^5+\omega_0\,(2\,i\,(47\,h^4\,m^2+h^2\,m\,n\,(-502\,(b+m)^2+2882\,(b+m)\,n-1653\,n^2)+16\,(b+m)\,n^3\,(6\,(b+m)^2-16\,(b+m)\,n+5\,n^2))+\omega_0\,(-377\,h^2\,m\,(b+m)^2+6054\,h^2\,m\,(b+m)\,n+4\,(68\,b^3+204\,b^2\,m-2047\,h^2\,m+204\,b\,m^2+68\,m^3)\,n^2-1536\,(b+m)^2\,n^3+1280\,(b+m)\,n^4-128\,n^5+\omega_0\,(-2\,i\,(1145\,h^2\,m\,(b+m)+(92\,b^3+276\,b^2\,m-4331\,h^2\,m+276\,b\,m^2+92\,m^3)\,n-1088\,(b+m)^2\,n^2+1920\,(b+m)\,n^3-512\,n^4)+\omega_0\,(-3\,(16\,b^3+48\,b^2\,m-1099\,h^2\,m+48\,b\,m^2+16\,m^3)+1472\,(b+m)^2\,n-5440\,(b+m)\,n^2+3072\,n^3+32\,\omega_0\,(-i\,(12\,(b+m)^2-115\,(b+m)\,n+136\,n^2)+2\,(15\,(b+m)-46\,n+12\,i\,\omega_0)\,\omega_0))))))))))\,/$$
$$((n-i\,\omega_0)^3\,(m+i\,\omega_0)^4\,(h^2\,m-2\,i\,(n-2\,i\,\omega_0)\,(b+m+2\,i\,\omega_0)\,\omega_0)^2\,(h^2\,m-3\,i\,(n-3\,i\,\omega_0)\,(b+m+3\,i\,\omega_0)\,\omega_0)$$

$$\left(h^2 m - 4 i \left(n - 4 i \omega_0\right) \left(b + m + 4 i \omega_0\right) \omega_0\right)\Big),$$

$$-\Big(6\sqrt{2}\, h\, m^6 \Big(105 h^4 m^2 n^3 + \omega_0 \Big(- i h^2 m n^2 \left(647 h^2 m + 159 (b+m) n^2\right) + \omega_0 \Big(-1231 h^4 m^2 n + 156 (b+m)^2 n^5 + h^2 m n^3 (-1394 (b+m) + 345 n) + \omega_0 \Big(i \left(737 h^4 m^2 + 2 h^2 m (2252 (b+m) - 1529 n) n^2 + 4 (b+m) n^4 (-382 (b+m) + 183 n)\right) + \omega_0 \Big(2 n \left(3087 h^2 m (b+m) - 5002 h^2 m n - 2776 (b+m)^2 n^2 + 3596 (b+m) n^3 - 420 n^4\right) + \omega_0 \Big(- i \left(3001 h^2 m (b+m) - 13806 h^2 m n - 9432 (b+m)^2 n^2 + 26192 (b+m) n^3 - 8272 n^4\right) + \omega_0 \Big(6707 h^2 m + 4 n \left(1885 (b+m)^2 - 11142 (b+m) n + 7544 n^2\right) + 4 \omega_0 \Big(- i \left(572 (b+m)^2 - 8917 (b+m) n + 12852 n^2\right) + 2 \left(1354 (b+m) - 5147 n + 1564 i \omega_0\right) \omega_0\Big)\Big)\Big)\Big)\Big)\Big)\Big)\Big)\Big) \Big/$$

$$\Big(\left(n - i \omega_0\right)^3 \left(m + i \omega_0\right)^4 \left(h^2 m - 2 i \left(n - 2 i \omega_0\right) \left(b + m + 2 i \omega_0\right) \omega_0\right)^2 \left(h^2 m - 3 i \left(n - 3 i \omega_0\right) \left(b + m + 3 i \omega_0\right) \omega_0\right) \left(h^2 m - 4 i \left(n - 4 i \omega_0\right) \left(b + m + 4 i \omega_0\right) \omega_0\right)\Big),$$

$$-\Big(6\sqrt{2}\, h\, m^5 \left(- i m + 4 \omega_0\right) \Big(105 i h^4 m^2 n^3 + \omega_0 \Big(h^2 m n^2 \left(647 h^2 m + 159 (b+m) n^2\right) + \omega_0 \Big(- i n \left(1231 h^4 m^2 + h^2 m (1394 (b+m) - 345 n) n^2 - 156 (b+m)^2 n^4\right) + \omega_0 \Big(-737 h^4 m^2 + 4 (b+m) (382 (b+m) - 183 n) n^4 + 2 h^2 m n^2 (-2252 (b+m) + 1529 n) + \omega_0 \Big(4 i n \left(\frac{3087}{2} h^2 m (b+m) - 2501 h^2 m n - 1388 (b+m)^2 n^2 + 1798 (b+m) n^3 - 210 n^4\right) + \omega_0 \Big(3001 h^2 m (b+m) - 13806 h^2 m n - 9432 (b+m)^2 n^2 + 26192 (b+m) n^3 - 8272 n^4 + \omega_0 \Big(i \left(6707 h^2 m + 4 n \left(1885 (b+m)^2 - 11142 (b+m) n + 7544 n^2\right)\right) + 4 \omega_0 \Big(572 (b+m)^2 - 8917 (b+m) n + 12852 n^2 + 2 i \left(1354 (b+m) - 5147 n + 1564 i \omega_0\right) \omega_0\Big)\Big)\Big)\Big)\Big)\Big)\Big)\Big)\Big) \Big/$$

$$\Big(\left(n - i \omega_0\right)^3 \left(m + i \omega_0\right)^4 \left(h^2 m - 2 i \left(n - 2 i \omega_0\right) \left(b + m + 2 i \omega_0\right) \omega_0\right)^2 \Big(h^2 m - 3 i \left(n - 3 i \omega_0\right) \left(b + m + 3 i \omega_0\right)$$

```
ω0) (h2
m - 4
ⅈ
(n - 4 ⅈ ω0)
(b + m + 4 ⅈ ω0)
ω0)) }
```

(* Vetor complexo h40b *)

```
h40b = FullSimplify[ComplexExpand[Conjugate[h40]],
       simplifica comple··· expande funções ···  conjugado
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
       números reais
```

```
{(96 m^5
    (-8 ⅈ h^6 m^3 n^2 + ω0 (h^4 m^2 n (25 h^2 m - 11 (b + m) n^2) + ω0 (ⅈ h^2 m (19 h^4 m^2 - 59 (b + m)^2 n^4 +
              h^2 m n^2 (-32 (b + m) + 63 n)) + ω0 (n (15 h^4 m^2 (b + m) - 246 h^4 m^2 n +
                 396 h^2 m (b + m)^2 n^2 - 346 h^2 m (b + m) n^3 - 8 (b + m)^3 n^4) + ω0
              (-ⅈ (10 h^4 m^2 (b + m) + 285 h^4 m^2 n - 964 h^2 m (b + m)^2 n^2 + 2346 h^2
                    m (b + m) n^3 + (64 b^3 + 192 b^2 m - 483 h^2 m + 192 b m^2 + 64 m^3) n^4 -
                   64 (b + m)^2 n^5) + ω0 (94 h^4 m^2 + 2 h^2 m n (-502 (b + m)^2 + 2882 (b + m)
                       n - 1653 n^2) + 32 (b + m) n^3 (6 (b + m)^2 - 16 (b + m) n + 5 n^2) +
                   ω0 (-ⅈ (377 h^2 m (b + m)^2 - 6054 h^2 m (b + m) n -
                         4 (68 b^3 + 204 b^2 m - 2047 h^2 m + 204 b m^2 + 68 m^3) n^2 +
                         1536 (b + m)^2 n^3 - 1280 (b + m) n^4 + 128 n^5) + ω0 (-2290 h^2 m
                          (b + m) - 2 (92 b^3 + 276 b^2 m - 4331 h^2 m + 276 b m^2 + 92 m^3)
                          n + 2176 (b + m)^2 n^2 - 3840 (b + m) n^3 + 1024 n^4 +
                         ω0 (-ⅈ (3 (16 b^3 + 48 b^2 m - 1099 h^2 m + 48 b m^2 + 16 m^3) -
                               1472 (b + m)^2 n + 5440 (b + m) n^2 - 3072 n^3) +
                            32 ω0 (-12 (b + m)^2 + 115 (b + m) n - 136 n^2 +
                               2 ⅈ (15 (b + m) - 46 n) ω0 + 24 ω0^2)))))))))) /
   ((ⅈ m + ω0)^4 (-ⅈ n + ω0)^3 (h^2 m + 2 ⅈ (b + m - 2 ⅈ ω0) (n + 2 ⅈ ω0) ω0)^2 (-h^2
         m +
        3
         (b +
           m -
           3 ⅈ ω0) ω0 (-ⅈ n +
           3 ω0))
     (-h^2 m + 4 (b + m - 4 ⅈ ω0) ω0 (-ⅈ n + 4 ω0))),
 -(6
      √2
      h
      m^6
      (105
         ⅈ
         h^4
```

m^2

$n^3 +$

$\omega_0 \Big(h^2 m n^2 \big(-647 h^2 m - 159 (b+m) n^2\big) +$

$\omega_0 \Big(-i n \big(1231 h^4 m^2 + h^2 m (1394 (b+m) - 345 n) n^2 - 156 (b+m)^2 n^4\big) +$

$\omega_0 \Big(737 h^4 m^2 + 2 h^2 m (2252 (b+m) - 1529 n) n^2 + 4 (b+m) n^4$

$(-382 (b+m) + 183 n) + \omega_0 \Big(4 i n \Big(\frac{3087}{2} h^2 m (b+m) - 2501 h^2 m n - 1388$

$(b+m)^2 n^2 + 1798 (b+m) n^3 - 210 n^4\Big) + \omega_0 \big(-3001 h^2 m (b+m) +$

$13\,806 h^2 m n + 9432 (b+m)^2 n^2 - 26\,192 (b+m) n^3 + 8272 n^4 + \omega_0$

$\big(i \big(6707 h^2 m + 4 n \big(1885 (b+m)^2 - 11\,142 (b+m) n + 7544 n^2\big)\big) +$

$4 \omega_0 \big(-572 (b+m)^2 + 8917 (b+m) n - 12\,852 n^2 +$

$2 i (1354 (b+m) - 5147 n) \omega_0 + 3128 \omega_0^2)))\big)\big)\Big)\Big)\Big)\Big) \Big/$

$\Big(\big(i m + \omega_0\big)^4 \big(-i n + \omega_0\big)^3 \big(h^2 m + 2 i \big(b + m - 2 i \omega_0\big) \big(n + 2 i \omega_0\big) \omega_0\big)^2$

$\big(-h^2$

$m + 3$

$\big(b + m - 3 i \omega_0\big)$

ω_0

$\big(-i n + 3 \omega_0\big)\big)$

$\big(-h^2 m + 4 \big(b + m - 4 i \omega_0\big) \omega_0 \big(-i n + 4 \omega_0\big)\big)\Big)$, $-\Big(6$

$\sqrt{2}$

h

m^5

$\big(m -$

$4 i \omega_0\big)$

$\Big(105 i h^4 m^2 n^3 + \omega_0 \Big(h^2 m n^2 \big(-647 h^2 m - 159 (b+m) n^2\big) +$

$\omega_0 \Big(-i n \big(1231 h^4 m^2 + h^2 m (1394 (b+m) - 345 n) n^2 - 156 (b+m)^2 n^4\big) +$

$\omega_0 \Big(737 h^4 m^2 + 2 h^2 m (2252 (b+m) - 1529 n) n^2 + 4 (b+m) n^4$

$(-382 (b+m) + 183 n) + \omega_0 \Big(4 i n \Big(\frac{3087}{2} h^2 m (b+m) - 2501 h^2 m n - 1388$

$(b+m)^2 n^2 + 1798 (b+m) n^3 - 210 n^4\Big) + \omega_0 \big(-3001 h^2 m (b+m) +$

$13\,806 h^2 m n + 9432 (b+m)^2 n^2 - 26\,192 (b+m) n^3 + 8272 n^4 + \omega_0$

$\big(i \big(6707 h^2 m + 4 n \big(1885 (b+m)^2 - 11\,142 (b+m) n + 7544 n^2\big)\big) +$

$4 \omega_0 \big(-572 (b+m)^2 + 8917 (b+m) n - 12\,852 n^2 +$

$2 i (1354 (b+m) - 5147 n) \omega_0 + 3128 \omega_0^2)))\big)\big)\Big)\Big)\Big)\Big) \Big/$

$\Big(\big(i m + \omega_0\big)^4 \big(-i n + \omega_0\big)^3 \big(h^2 m + 2 i \big(b + m - 2 i \omega_0\big) \big(n + 2 i \omega_0\big) \omega_0\big)^2$

$\big(-h^2$

$m + 3$

$\big(b + m - 3 i \omega_0\big)$

$\omega_0 \big(-i n + 3 \omega_0\big)\big)$

$\big(-h^2 m + 4 \big(b + m - 4 i \omega_0\big) \omega_0 \big(-i n + 4 \omega_0\big)\big)\Big)\Big]$

```
(* Cálculo do vetor complexo h31 *)
(* h31=FullSimplify[DAI.(3 bb[h20,h11]+bb[qb,h30]+3 bb[q,h21]-3 G21 h20),
    simplifica completamente
  ω0∈Reals && m>0 && n<0 && h>0 &&b>0] *)
  números reais
```

```
h311 = FullSimplify[DAI.(3 bb[h20, h11]),
    simplifica completamente
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
    números reais
```

$$\{(24 m^5 (-6 i h^2 m n^2 + \omega_0 (-5 h^2 m n + 12 (b+m) n^3 + \omega_0 (-i (h^2 m + 6 (3 (b+m) - 4 n) n^2) + 2 \omega_0 (2 n (b+m+9 n) + \omega_0 (-3 i (b+m) + 4 i n + 6 \omega_0))))))\Big/$$
$$((m - i \omega_0) (m + i \omega_0)^3 (-i n + \omega_0) (i n + \omega_0)^2 (h^2 m - 2 (b + m + 2 i \omega_0) \omega_0 (i n + 2 \omega_0))^2),$$
$$(6 \sqrt{2} m^5 (-21 i h^2 m n^3 + \omega_0 (-49 h^2 m n^2 + 6 (b+m) n^4 + \omega_0 (3 i (5 h^2 m n - 6 (b+m) n^3 + 4 n^4) + \omega_0 (-9 h^2 m + 2 n^2 (-5 (b+m) + 18 n) - 2 i n (3 (b+m) + 10 n) \omega_0 - 4 (b + m - 3 n) \omega_0^2 - 8 i \omega_0^3)))))\Big/$$
$$(h (m - i \omega_0) (m + i \omega_0)^3 (-i n + \omega_0) (i n + \omega_0)^2 (h^2 m - 2 i (n - 2 i \omega_0) (b + m + 2 i \omega_0) \omega_0)^2),$$
$$-(6 \sqrt{2} m^4 (m + 2 i \omega_0) (21 i h^2 m n^3 + \omega_0 (49 h^2 m n^2 - 6 (b+m) n^4 + \omega_0 (3 i (-5 h^2 m n + 6 (b+m) n^3 - 4 n^4) + \omega_0 (9 h^2 m + 2 (5 (b+m) - 18 n) n^2 + 2 i n (3 (b+m) + 10 n) \omega_0 + 4 (b + m - 3 n) \omega_0^2 + 8 i \omega_0^3)))))\Big/$$
$$(h (m - i \omega_0) (m + i \omega_0)^3 (-i n + \omega_0) (i n + \omega_0)^2 (h^2 m - 2 i (n - 2 i \omega_0) (b + m + 2 i \omega_0) \omega_0)^2)\}$$

```
h312 = FullSimplify[DAI.(bb[qb, h30]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
        simplifica completamente                  números reais

{(12 m^5 (13 i h^4 m^2 n^2 + ω0 (h^2 m n (32 h^2 m - 29 (b + m) n^2) +
          ω0 (-i (19 h^4 m^2 + 4 (b + m)^2 n^4 + h^2 m n^2 (-71 (b + m) + 67 n)) +
            ω0 (n (-5 h^2 m (b + m) - 157 h^2 m n - 16 (b + m)^2 n^2 + 16 (b + m) n^3) + ω0
                (i (63 h^2 m (b + m) - 19 h^2 m n + 8 (b + m)^2 n^2 - 64 (b + m) n^3 + 16 n^4) +
                  ω0 (-141 h^2 m + 16 n (-(b + m)^2 - 2 (b + m) n + 4 n^2) +
                    4 ω0 (i (3 (b + m)^2 - 16 (b + m) n - 8 n^2) -
                      4 (3 (b + m) - 4 n + 3 i ω0) ω0)))))))) /
  ((m - i ω0) (m + i ω0)^3 (-i n + ω0) (i n + ω0)^2 (h^2 m - 2 (b + m + 2 i ω0) ω0 (i n + 2 ω0))^2
    (-h^2
       m + 3 (b + m + 3 i ω0)
       ω0 (i n + 3 ω0))),
 (6 √2 h m^6 (21 i h^2 m n^3 + ω0 (80 h^2 m n^2 - 15 (b + m) n^4 +
          ω0 (-i n (79 h^2 m + n^2 (-61 (b + m) + 39 n)) + ω0 (-12 h^2 m + (61 (b + m) - 155 n)
                 n^2 + ω0 (13 i n (b + m + 11 n) + (28 (b + m) - 59 n + 86 i ω0) ω0)))))) /
  ((m - i ω0) (m + i ω0)^3 (-i n + ω0) (i n + ω0)^2 (h^2 m - 2 (b + m + 2 i ω0) ω0 (i n + 2 ω0))^2
    (-h^2 m +
      3 (b + m + 3 i ω0) ω0 (i n + 3 ω0))),
 (6 √2 h m^5 (m + 2 i ω0) (21 i h^2 m n^3 + ω0 (80 h^2 m n^2 - 15 (b + m) n^4 +
          ω0 (-i n (79 h^2 m + n^2 (-61 (b + m) + 39 n)) + ω0 (-12 h^2 m + (61 (b + m) - 155 n)
                 n^2 + ω0 (13 i n (b + m + 11 n) + (28 (b + m) - 59 n + 86 i ω0) ω0)))))) /
  ((m - i ω0) (m + i ω0)^3 (-i n + ω0) (i n + ω0)^2 (h^2 m - 2 (b + m + 2 i ω0) ω0 (i n + 2 ω0))^2
    (-h^2 m + 3 (b + m + 3 i ω0) ω0 (i n + 3 ω0)))}

h313 = Simplify[DAI.(3 bb[q, h21]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
        simplifica                          números reais

{(12 m^5 (3 i h^2 m n^2 (5 h^4 m^2 + h^2 m (8 b + 8 m - 13 n) n^2 + 3 (b + m)^2 n^4) +
        (12 h^6 m^3 n + h^4 m^2 (50 b + 50 m - 191 n) n^3 + h^2 m (b + m) (50 b + 50 m - 27 n) n^5 +
           12 (b + m)^3 n^7) ω0 - i (h^6 m^3 + 2 h^4 m^2 (19 b + 19 m - 145 n) n^2 + 48 (b + m)^3 n^6 +
           h^2 m n^4 (91 b^2 + 182 b m + 91 m^2 - 202 b n - 202 m n + 27 n^2)) ω0^2 -
        n (44 b^3 n^4 + 4 b^2 (5 h^2 m n^2 + n^4 (33 m + 41 n)) + b (-10 h^4 m^2 + h^2 m (40 m - 437 n) n^2 +
              4 n^4 (33 m^2 + 82 m n - 33 n^2)) + m (-2 h^4 m (5 m + 89 n) +
              4 n^4 (11 m^2 + 41 m n - 33 n^2) + h^2 n^2 (20 m^2 - 437 m n + 242 n^2))) ω0^3 -
        i (h^4 m^2 (14 m - 9 n) + 40 b^3 n^4 + h^2 m n^2 (57 m^2 + 196 m n - 565 n^2) +
           8 n^4 (5 m^3 - 105 m^2 n + 145 m n^2 - 21 n^3) + 3 b^2 (19 h^2 m n^2 + 40 (m - 7 n) n^4) +
           2 b (7 h^4 m^2 + h^2 m n^2 (57 m + 98 n) + 20 n^4 (3 m^2 - 42 m n + 29 n^2))) ω0^4 +
        (33 h^4 m^2 + h^2 m n (-22 b^2 - 44 b m - 22 m^2 + 163 b n + 163 m n + 332 n^2) -
           4 n^3 (19 b^3 + 19 m^3 + b^2 (57 m - 385 n) - 385 m^2 n +
              931 m n^2 - 320 n^3 + b (57 m^2 - 770 m n + 931 n^2))) ω0^5 +
        i (16 b^3 n^2 + h^2 m (-5 m^2 - 78 m n + 123 n^2) + b^2 (-5 h^2 m + 16 (3 m - 71 n) n^2) +
           8 n^2 (2 m^3 - 142 m^2 n + 689 m n^2 - 467 n^3) -
           2 b (h^2 m (5 m + 39 n) - 4 n^2 (6 m^2 - 284 m n + 689 n^2))) ω0^6 -
        (20 b^3 n + 12 b^2 n (5 m + 9 n) - h^2 m (19 m + 94 n) +
```

$$4 n (5 m^3 + 27 m^2 n - 895 m n^2 + 1296 n^3) + b (-19 h^2 m + 4 n (15 m^2 + 54 m n - 895 n^2)))$$
$$\omega_0^7 + i (8 b^3 + 11 h^2 m + 8 b^2 (3 m - 31 n) + 8 b (3 m^2 - 62 m n - 35 n^2) +$$
$$8 (m^3 - 31 m^2 n - 35 m n^2 + 403 n^3)) \omega_0^8 -$$
$$4 (21 b^2 + 42 b m + 21 m^2 - 179 b n - 179 m n - 64 n^2) \omega_0^9 -$$
$$8 i (29 b + 29 m - 75 n) \omega_0^{10} + 192 \omega_0^{11})) /$$
$$((m - i \omega_0) (m + i \omega_0)^3 (-i n + \omega_0) (i n + \omega_0)^3$$
$$(-h^2 m + 2 i (b + m) n \omega_0 + 4 (b + m - n) \omega_0^2 + 8 i \omega_0^3)^2$$
$$(-i n (b^2 n^2 + 2 b m (-h^2 + n^2) + m (m n^2 + h^2 (-2 m + n))) -$$
$$3 (b^2 n^2 - b (h^2 m + n^2 (-2 m + n)) - m (h^2 (m - 2 n) + n^2 (-m + n))) \omega_0 +$$
$$3 i (2 h^2 m + n (b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2)) \omega_0^2 +$$
$$(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2) \omega_0^3 + 3 i (b + m - 3 n) \omega_0^4 - 3 \omega_0^5)),$$
$$(6 \sqrt{2} m^5 (3 i h^2 m n^3 (10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b + m)^2 n^4) +$$
$$(69 h^6 m^3 n^2 + h^4 m^2 (175 b + 175 m - 397 n) n^4 +$$
$$7 h^2 m (b + m) (14 b + 14 m - 15 n) n^6 + 12 (b + m)^3 n^8) \omega_0 -$$
$$i (38 h^6 m^3 n + 6 h^4 m^2 (57 b + 57 m - 170 n) n^3 + 12 (b + m)^2 (8 b + 8 m - 5 n) n^7 +$$
$$3 h^2 m n^5 (127 b^2 + 254 b m + 127 m^2 - 294 b n - 294 m n + 61 n^2)) \omega_0^2 -$$
$$(3 h^6 m^3 + 2 h^4 m^2 (127 b + 127 m - 661 n) n^2 + 4 (b + m) n^6$$
$$(77 b^2 + 77 m^2 + 2 b (77 m - 60 n) - 120 m n + 27 n^2) +$$
$$h^2 m n^4 (674 b^2 + 1348 b m + 674 m^2 - 2899 b n - 2899 m n + 1428 n^2)) \omega_0^3 +$$
$$i n (h^4 m^2 (9 m - 767 n) + 496 b^3 n^4 + h^2 m n^2 (515 m^2 - 4612 m n + 4405 n^2) +$$
$$4 n^4 (124 m^3 - 385 m^2 n + 216 m n^2 - 18 n^3) + b^2 (515 h^2 m n^2 + 4 (372 m - 385 n) n^4) +$$
$$b (9 h^4 m^2 + 2 h^2 m (515 m - 2306 n) n^2 + 8 n^4 (186 m^2 - 385 m n + 108 n^2))) \omega_0^4 +$$
$$(388 b^3 n^4 - h^4 m^2 (37 m + 25 n) + h^2 m n^2 (46 m^2 - 3383 m n + 6658 n^2) +$$
$$4 n^4 (97 m^3 - 620 m^2 n + 693 m n^2 - 144 n^3) + 2 b^2 (23 h^2 m n^2 + 2 (291 m - 620 n) n^4) +$$
$$b (-37 h^4 m^2 + h^2 m (92 m - 3383 n) n^2 + 4 n^4 (291 m^2 - 1240 m n + 693 n^2))) \omega_0^5 -$$
$$i (90 h^4 m^2 - h^2 m n (137 b^2 + 274 b m + 137 m^2 + 506 b n + 506 m n - 4721 n^2) +$$
$$4 n^3 (16 b^3 + 16 m^3 + b^2 (48 m - 485 n) - 485 m^2 n + 1116 m n^2 - 462 n^3 +$$
$$2 b (24 m^2 - 485 m n + 558 n^2))) \omega_0^6 + (116 b^3 n^2 + h^2 m (50 m^2 - 627 m n - 752 n^2) +$$
$$4 n^2 (29 m^3 + 80 m^2 n - 873 m n^2 + 744 n^3) + b^2 (50 h^2 m + 4 n^2 (87 m + 80 n)) +$$
$$b (h^2 m (100 m - 627 n) + 4 n^2 (87 m^2 + 160 m n - 873 n^2))) \omega_0^7 -$$
$$i (80 b^3 n + 20 b^2 (12 m - 29 n) n + 15 h^2 m (-16 m + 51 n) + 4 n$$
$$(20 m^3 - 145 m^2 n - 144 m n^2 + 582 n^3) - 8 b (30 h^2 m + n (-30 m^2 + 145 m n + 72 n^2)))$$
$$\omega_0^8 - 2 (8 b^3 + 147 h^2 m + 8 m^3 + 8 b^2 (3 m - 25 n) - 200 m^2 n + 522 m n^2 +$$
$$192 n^3 + b (24 m^2 - 400 m n + 522 n^2)) \omega_0^9 -$$
$$8 i (10 b^2 + 20 b m + 10 m^2 - 90 b n - 90 m n + 87 n^2) \omega_0^{10} +$$
$$48 (3 b + 3 m - 10 n) \omega_0^{11} + 96 i \omega_0^{12})) /$$
$$(h (m - i \omega_0) (m + i \omega_0)^3 (-i n + \omega_0) (i n + \omega_0)^3$$
$$(-h^2 m + 2 i (b + m) n \omega_0 + 4 (b + m - n) \omega_0^2 + 8 i \omega_0^3)^2$$
$$(-i n (b^2 n^2 + 2 b m (-h^2 + n^2) + m (m n^2 + h^2 (-2 m + n))) -$$
$$3 (b^2 n^2 - b (h^2 m + n^2 (-2 m + n)) - m (h^2 (m - 2 n) + n^2 (-m + n))) \omega_0 +$$
$$3 i (2 h^2 m + n (b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2)) \omega_0^2 +$$
$$(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2) \omega_0^3 + 3 i (b + m - 3 n) \omega_0^4 - 3 \omega_0^5)),$$
$$-(6 \sqrt{2} m^4 (m + 2 i \omega_0) (-3 i h^2 m n^3 (10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b + m)^2 n^4) -$$
$$(69 h^6 m^3 n^2 + h^4 m^2 (175 b + 175 m - 397 n) n^4 +$$
$$7 h^2 m (b + m) (14 b + 14 m - 15 n) n^6 + 12 (b + m)^3 n^8) \omega_0 +$$

$$i\left(38 h^6 m^3 n + 6 h^4 m^2 (57 b + 57 m - 170 n) n^3 + 12 (b + m)^2 (8 b + 8 m - 5 n) n^7 + 3 h^2 m n^5 \left(127 b^2 + 254 b m + 127 m^2 - 294 b n - 294 m n + 61 n^2\right)\right) \omega_0^2 +$$

$$\left(3 h^6 m^3 + 2 h^4 m^2 (127 b + 127 m - 661 n) n^2 + 4 (b + m) n^6 \left(77 b^2 + 77 m^2 + 2 b (77 m - 60 n) - 120 m n + 27 n^2\right) + h^2 m n^4 \left(674 b^2 + 1348 b m + 674 m^2 - 2899 b n - 2899 m n + 1428 n^2\right)\right) \omega_0^3 -$$

$$i n \left(h^4 m^2 (9 m - 767 n) + 496 b^3 n^4 + h^2 m n^2 \left(515 m^2 - 4612 m n + 4405 n^2\right) + 4 n^4 \left(124 m^3 - 385 m^2 n + 216 m n^2 - 18 n^3\right) + b^2 \left(515 h^2 m n^2 + 4 (372 m - 385 n) n^4\right) + b \left(9 h^4 m^2 + 2 h^2 m (515 m - 2306 n) n^2 + 8 n^4 \left(186 m^2 - 385 m n + 108 n^2\right)\right)\right) \omega_0^4 +$$

$$\left(-388 b^3 n^4 + h^4 m^2 (37 m + 25 n) + h^2 m n^2 \left(-46 m^2 + 3383 m n - 6658 n^2\right) + 4 n^4 \left(-97 m^3 + 620 m^2 n - 693 m n^2 + 144 n^3\right) - 2 b^2 \left(23 h^2 m n^2 + 2 (291 m - 620 n) n^4\right) + b \left(37 h^4 m^2 + h^2 m n^2 (-92 m + 3383 n) - 4 n^4 \left(291 m^2 - 1240 m n + 693 n^2\right)\right)\right) \omega_0^5 +$$

$$i \left(90 h^4 m^2 - h^2 m n \left(137 b^2 + 274 b m + 137 m^2 + 506 b n + 506 m n - 4721 n^2\right) + 4 n^3 \left(16 b^3 + 16 m^3 + b^2 (48 m - 485 n) - 485 m^2 n + 1116 m n^2 - 462 n^3 + 2 b \left(24 m^2 - 485 m n + 558 n^2\right)\right)\right) \omega_0^6 - \left(116 b^3 n^2 + h^2 m \left(50 m^2 - 627 m n - 752 n^2\right) + 4 n^2 \left(29 m^3 + 80 m^2 n - 873 m n^2 + 744 n^3\right) + b^2 \left(50 h^2 m + 4 n^2 (87 m + 80 n)\right) + b \left(h^2 m (100 m - 627 n) + 4 n^2 \left(87 m^2 + 160 m n - 873 n^2\right)\right)\right) \omega_0^7 +$$

$$i \left(80 b^3 n + 20 b^2 (12 m - 29 n) n + 15 h^2 m (-16 m + 51 n) + 4 n \left(20 m^3 - 145 m^2 n - 144 m n^2 + 582 n^3\right) - 8 b \left(30 h^2 m + n \left(-30 m^2 + 145 m n + 72 n^2\right)\right)\right) \omega_0^8 +$$

$$2 \left(8 b^3 + 147 h^2 m + 8 m^3 + 8 b^2 (3 m - 25 n) - 200 m^2 n + 522 m n^2 + 192 n^3 + b \left(24 m^2 - 400 m n + 522 n^2\right)\right) \omega_0^9 +$$

$$8 i \left(10 b^2 + 20 b m + 10 m^2 - 90 b n - 90 m n + 87 n^2\right) \omega_0^{10} - 48 (3 b + 3 m - 10 n) \omega_0^{11} - 96 i \omega_0^{12}\Big)\Big) \Big/$$

$$\Big(h (m - i \omega_0) (m + i \omega_0)^3 (-i n + \omega_0) (i n + \omega_0)^3 \left(-h^2 m + 2 i (b + m) n \omega_0 + 4 (b + m - n) \omega_0^2 + 8 i \omega_0^3\right)^2$$

$$\Big(-i n \left(b^2 n^2 + 2 b m \left(-h^2 + n^2\right) + m \left(m n^2 + h^2 (-2 m + n)\right)\right) - 3 \left(b^2 n^2 - b \left(h^2 m + n^2 (-2 m + n)\right) - m \left(h^2 (m - 2 n) + n^2 (-m + n)\right)\right) \omega_0 + 3 i \left(2 h^2 m + n \left(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2\right)\right) \omega_0^2 + \left(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2\right) \omega_0^3 + 3 i (b + m - 3 n) \omega_0^4 - 3 \omega_0^5\Big)\Big)\Big\}$$

```
h314 = FullSimplify[DAI.(-3 G21 h20), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
       simplifica completamente                  números reais
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$$\Big\{\Big(\big(24\, m^5 \big(3\, h^2\, m\, (b+m)\, n + \omega_0 \big(-3\, \mathrm{i}\, h^2\, m\, (3\, (b+m) - 4\, n) +$$
$$4\, \omega_0 \big(7\, h^2\, m + (b+m)^2\, n + \omega_0 \big(-\mathrm{i}\, (b+m)\, (b+m-4\, n) + 4\, \big(b+m-n+\mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\big)\big)$$
$$\big(12\, h^2\, m\, n^2 + \omega_0 \big(-\mathrm{i}\, n \big(7\, h^2\, m + 15\, (b+m)\, n^2\big) + \omega_0 \big(3\, h^2\, m + 5\, n^2\, (-7\, (b+m) + 6\, n) +$$
$$\omega_0 \big(5\, \mathrm{i}\, (b+m-14\, n)\, n - \big(7\, (b+m) + 10\, n + 14\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\big)\big)\big) \Big/$$
$$\Big(\big(m - \mathrm{i}\, \omega_0\big)\, \big(m + \mathrm{i}\, \omega_0\big)^3\, \big(-\mathrm{i}\, n + \omega_0\big)\, \big(\mathrm{i}\, n + \omega_0\big)\, \big(-h^2\, m + 2\, \big(b+m+2\, \mathrm{i}\, \omega_0\big)\, \omega_0\, \big(\mathrm{i}\, n + 2\, \omega_0\big)\big)^3$$
$$\big(h^2\, m - (b+m)\, n^2 + \omega_0 \big(2\, \mathrm{i}\, (b+m-n)\, n + \big(b+m-4\, n+2\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\Big)\Big),$$
$$-\Big(12\, \sqrt{2}\, h\, m^6 \big(2\, h^2\, m - 3\, (b+m)\, n^2 + \mathrm{i}\, (13\, (b+m) - 12\, n)\, n\, \omega_0 + 12\, (b+m-4\, n)\, \omega_0^2 + 44\, \mathrm{i}\, \omega_0^3\big)$$
$$\big(12\, h^2\, m\, n^2 + \omega_0 \big(-\mathrm{i}\, n \big(7\, h^2\, m + 15\, (b+m)\, n^2\big) + \omega_0 \big(3\, h^2\, m + 5\, n^2\, (-7\, (b+m) + 6\, n) +$$
$$\omega_0 \big(5\, \mathrm{i}\, (b+m-14\, n)\, n - \big(7\, (b+m) + 10\, n + 14\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\big)\big)\Big) \Big/$$
$$\Big(\big(m - \mathrm{i}\, \omega_0\big)\, \big(m + \mathrm{i}\, \omega_0\big)^3\, \big(-\mathrm{i}\, n + \omega_0\big)\, \big(\mathrm{i}\, n + \omega_0\big)\, \big(-h^2\, m + 2\, \big(b+m+2\, \mathrm{i}\, \omega_0\big)\, \omega_0\, \big(\mathrm{i}\, n + 2\, \omega_0\big)\big)^3$$
$$\big(h^2\, m - (b+m)\, n^2 + \omega_0 \big(2\, \mathrm{i}\, (b+m-n)\, n + \big(b+m-4\, n+2\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\Big),$$
$$-\Big(12\, \sqrt{2}\, h\, m^5 \big(m \big(h^2\, (2\, m - 3\, n) - 3\, (b+m)\, n^2\big) + \omega_0 \big(\mathrm{i}\, m \big(9\, h^2 + (13\, (b+m) - 12\, n)\, n\big) +$$
$$4\, \omega_0 \big(3\, m\, (b+m) - (b+13\, m)\, n + 3\, n^2 + \mathrm{i}\, \big(b+12\, m-13\, n+12\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\big)$$
$$\big(12\, h^2\, m\, n^2 + \omega_0 \big(-\mathrm{i}\, n \big(7\, h^2\, m + 15\, (b+m)\, n^2\big) + \omega_0 \big(3\, h^2\, m + 5\, n^2\, (-7\, (b+m) + 6\, n) +$$
$$\omega_0 \big(5\, \mathrm{i}\, (b+m-14\, n)\, n - \big(7\, (b+m) + 10\, n + 14\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\big)\big)\Big) \Big/$$
$$\Big(\big(m - \mathrm{i}\, \omega_0\big)\, \big(m + \mathrm{i}\, \omega_0\big)^3\, \big(-\mathrm{i}\, n + \omega_0\big)\, \big(\mathrm{i}\, n + \omega_0\big)\, \big(-h^2\, m + 2\, \big(b+m+2\, \mathrm{i}\, \omega_0\big)\, \omega_0\, \big(\mathrm{i}\, n + 2\, \omega_0\big)\big)^3$$
$$\big(h^2\, m - (b+m)\, n^2 + \omega_0 \big(2\, \mathrm{i}\, (b+m-n)\, n + \big(b+m-4\, n+2\, \mathrm{i}\, \omega_0\big)\, \omega_0\big)\big)\Big)\Big\}$$

h31 = Simplify[h311 + h312 + h313 + h314, ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]

simplifica — números reais

$$\Big\{\frac{1}{\big(m - \mathrm{i}\, \omega_0\big)\, \big(m + \mathrm{i}\, \omega_0\big)^3\, \big(-\mathrm{i}\, n + \omega_0\big)\, \big(\mathrm{i}\, n + \omega_0\big)^3}$$
$$12\, m^5 \Big(\big(3\, \mathrm{i}\, h^2\, m\, n^2 \big(5\, h^4\, m^2 + h^2\, m\, (8\, b + 8\, m - 13\, n)\, n^2 + 3\, (b+m)^2\, n^4\big) +$$
$$\big(12\, h^6\, m^3\, n + h^4\, m^2\, (50\, b + 50\, m - 191\, n)\, n^3 + h^2\, m\, (b+m)\, (50\, b + 50\, m - 27\, n)\, n^5 +$$
$$12\, (b+m)^3\, n^7\big)\, \omega_0 - \mathrm{i}\, \big(h^6\, m^3 + 2\, h^4\, m^2\, (19\, b + 19\, m - 145\, n)\, n^2 + 48\, (b+m)^3\, n^6 +$$
$$h^2\, m\, n^4 \big(91\, b^2 + 182\, b\, m + 91\, m^2 - 202\, b\, n - 202\, m\, n + 27\, n^2\big)\big)\, \omega_0^2 -$$
$$n \big(44\, b^3\, n^4 + 4\, b^2 \big(5\, h^2\, m\, n^2 + n^4\, (33\, m + 41\, n)\big) + b \big(-10\, h^4\, m^2 +$$
$$h^2\, m\, (40\, m - 437\, n)\, n^2 + 4\, n^4 \big(33\, m^2 + 82\, m\, n - 33\, n^2\big)\big) + m \big(-2\, h^4\, m\, (5\, m + 89\, n) +$$
$$4\, n^4 \big(11\, m^2 + 41\, m\, n - 33\, n^2\big) + h^2\, n^2 \big(20\, m^2 - 437\, m\, n + 242\, n^2\big)\big)\big)\, \omega_0^3 -$$
$$\mathrm{i}\, \big(h^4\, m^2\, (14\, m - 9\, n) + 40\, b^3\, n^4 + h^2\, m\, n^2 \big(57\, m^2 + 196\, m\, n - 565\, n^2\big) +$$
$$8\, n^4 \big(5\, m^3 - 105\, m^2\, n + 145\, m\, n^2 - 21\, n^3\big) + 3\, b^2 \big(19\, h^2\, m\, n^2 + 40\, (m - 7\, n)\, n^4\big) +$$
$$2\, b \big(7\, h^4\, m^2 + h^2\, m\, n^2\, (57\, m + 98\, n) + 20\, n^4 \big(3\, m^2 - 42\, m\, n + 29\, n^2\big)\big)\big)\, \omega_0^4 +$$
$$\big(33\, h^4\, m^2 + h^2\, m\, n \big(-22\, b^2 - 44\, b\, m - 22\, m^2 + 163\, b\, n + 163\, m\, n + 332\, n^2\big) -$$
$$4\, n^3 \big(19\, b^3 + 19\, m^3 + b^2\, (57\, m - 385\, n) - 385\, m^2\, n +$$
$$931\, m\, n^2 - 320\, n^3 + b \big(57\, m^2 - 770\, m\, n + 931\, n^2\big)\big)\big)\, \omega_0^5 +$$
$$\mathrm{i}\, \big(16\, b^3\, n^2 + h^2\, m \big(-5\, m^2 - 78\, m\, n + 123\, n^2\big) + b^2 \big(-5\, h^2\, m + 16\, (3\, m - 71\, n)\, n^2\big) +$$
$$8\, n^2 \big(2\, m^3 - 142\, m^2\, n + 689\, m\, n^2 - 467\, n^3\big) -$$
$$2\, b \big(h^2\, m\, (5\, m + 39\, n) - 4\, n^2 \big(6\, m^2 - 284\, m\, n + 689\, n^2\big)\big)\big)\, \omega_0^6 -$$
$$\big(20\, b^3\, n + 12\, b^2\, n\, (5\, m + 9\, n) - h^2\, m\, (19\, m + 94\, n) + 4\, n \big(5\, m^3 + 27\, m^2\, n -$$
$$895\, m\, n^2 + 1296\, n^3\big) + b \big(-19\, h^2\, m + 4\, n \big(15\, m^2 + 54\, m\, n - 895\, n^2\big)\big)\big)\, \omega_0^7 +$$
$$\mathrm{i}\, \big(8\, b^3 + 11\, h^2\, m + 8\, b^2\, (3\, m - 31\, n) + 8\, b \big(3\, m^2 - 62\, m\, n - 35\, n^2\big) +$$
$$8 \big(m^3 - 31\, m^2\, n - 35\, m\, n^2 + 403\, n^3\big)\big)\, \omega_0^8 -$$
$$4 \big(21\, b^2 + 42\, b\, m + 21\, m^2 - 179\, b\, n - 179\, m\, n - 64\, n^2\big)\, \omega_0^9 -$$
$$8\, \mathrm{i}\, (29\, b + 29\, m - 75\, n)\, \omega_0^{10} + 192\, \omega_0^{11}\big) \Big/$$

$$\left(\left(-h^2 m+2 i (b+m) n \omega_0+4 (b+m-n) \omega_0^2+8 i \omega_0^3\right)^2\right.$$
$$\left(-i n \left(b^2 n^2+2 b m \left(-h^2+n^2\right)+m \left(m n^2+h^2 (-2 m+n)\right)\right)-\right.$$
$$3 \left(b^2 n^2-b \left(h^2 m+n^2 (-2 m+n)\right)-m \left(h^2 (m-2 n)+n^2 (-m+n)\right)\right) \omega_0+$$
$$3 i \left(2 h^2 m+n \left(b^2+2 b m+m^2-3 b n-3 m n+n^2\right)\right) \omega_0^2+$$
$$\left.\left(b^2+2 b m+m^2-9 b n-9 m n+9 n^2\right) \omega_0^3+3 i (b+m-3 n) \omega_0^4-3 \omega_0^5\right)+$$
$$\left(2 (i n+\omega_0)^2 \left(3 h^2 m (b+m) n+\omega_0 \left(-3 i h^2 m (3 (b+m)-4 n)+4 \omega_0\right.\right.\right.$$
$$\left.\left.\left.\left(7 h^2 m+(b+m)^2 n+\omega_0 \left(-i (b+m) (b+m-4 n)+4 \left(b+m-n+i \omega_0\right) \omega_0\right)\right)\right)\right)$$
$$\left(12 h^2 m n^2+\omega_0 \left(-i n \left(7 h^2 m+15 (b+m) n^2\right)+\omega_0 \left(3 h^2 m+5 n^2 (-7 (b+m)+6 n)+\right.\right.\right.$$
$$\left.\left.\left.\left.\omega_0 \left(5 i (b+m-14 n) n-\left(7 (b+m)+10 n+14 i \omega_0\right) \omega_0\right)\right)\right)\right)\right) \Big/$$
$$\left(\left(-h^2 m+2 \left(b+m+2 i \omega_0\right) \omega_0 \left(i n+2 \omega_0\right)\right)^3 \left(h^2 m-(b+m) n^2+\right.\right.$$
$$\left.\left.\omega_0 \left(2 i (b+m-n) n+\left(b+m-4 n+2 i \omega_0\right) \omega_0\right)\right)\right)+\left(2 \left(i n+\omega_0\right)\right.$$
$$\left(-6 i h^2 m n^2+\omega_0 \left(-5 h^2 m n+12 (b+m) n^3+\omega_0 \left(-i \left(h^2 m+6 (3 (b+m)-4 n) n^2\right)+\right.\right.\right.$$
$$\left.\left.\left.\left.2 \omega_0 \left(2 n (b+m+9 n)+\omega_0 \left(-3 i (b+m)+4 i n+6 \omega_0\right)\right)\right)\right)\right)\right) \Big/$$
$$\left(h^2 m-2 \left(b+m+2 i \omega_0\right) \omega_0 \left(i n+2 \omega_0\right)\right)^2+\left(\left(i n+\omega_0\right)\right.$$
$$\left(13 i h^4 m^2 n^2+\omega_0 \left(h^2 m n \left(32 h^2 m-29 (b+m) n^2\right)+\right.\right.$$
$$\omega_0 \left(-i \left(19 h^4 m^2+4 (b+m)^2 n^4+h^2 m n^2 (-71 (b+m)+67 n)\right)+\right.$$
$$\omega_0 \left(n \left(-5 h^2 m (b+m)-157 h^2 m n-16 (b+m)^2 n^2+16 (b+m) n^3\right)+\right.$$
$$\omega_0 \left(i \left(63 h^2 m (b+m)-19 h^2 m n+8 (b+m)^2 n^2-64 (b+m) n^3+16 n^4\right)+\right.$$
$$\omega_0 \left(-141 h^2 m+16 n \left(-(b+m)^2-2 (b+m) n+4 n^2\right)+\right.$$
$$4 \omega_0 \left(i \left(3 (b+m)^2-16 (b+m) n-8 n^2\right)-\right.$$
$$\left.\left.\left.\left.\left.\left.\left.\left.4 \left(3 (b+m)-4 n+3 i \omega_0\right) \omega_0\right)\right)\right)\right)\right)\right)\right)\right) \Big/$$
$$\left.\left(\left(h^2 m-2 \left(b+m+2 i \omega_0\right) \omega_0 \left(i n+2 \omega_0\right)\right)^2 \left(-h^2 m+3 \left(b+m+3 i \omega_0\right) \omega_0 \left(i n+3 \omega_0\right)\right)\right)\right),$$

$$\frac{1}{h \left(m-i \omega_0\right) \left(m+i \omega_0\right)^3 \left(-i n+\omega_0\right) \left(i n+\omega_0\right)^3}$$
$$6$$
$$\sqrt{2}$$
$$m^5$$
$$\left(\left(3 h^2 m n^3 \left(10 h^4 m^2+h^2 m (11 b+11 m-21 n) n^2+3 (b+m)^2 n^4\right)-\right.\right.$$
$$i \left(69 h^6 m^3 n^2+h^4 m^2 (175 b+175 m-397 n) n^4+\right.$$
$$\left.7 h^2 m (b+m) (14 b+14 m-15 n) n^6+12 (b+m)^3 n^8\right) \omega_0-$$
$$\left(38 h^6 m^3 n+6 h^4 m^2 (57 b+57 m-170 n) n^3+12 (b+m)^2 (8 b+8 m-5 n) n^7+\right.$$
$$\left.3 h^2 m n^5 \left(127 b^2+254 b m+127 m^2-294 b n-294 m n+61 n^2\right)\right) \omega_0^2+$$
$$i \left(3 h^6 m^3+2 h^4 m^2 (127 b+127 m-661 n) n^2+4 (b+m) n^6\right.$$
$$\left(77 b^2+77 m^2+2 b (77 m-60 n)-120 m n+27 n^2\right)+$$
$$\left.h^2 m n^4 \left(674 b^2+1348 b m+674 m^2-2899 b n-2899 m n+1428 n^2\right)\right) \omega_0^3+$$
$$n \left(h^4 m^2 (9 m-767 n)+496 b^3 n^4+h^2 m n^2 \left(515 m^2-4612 m n+4405 n^2\right)+4\right.$$
$$n^4 \left(124 m^3-385 m^2 n+216 m n^2-18 n^3\right)+b^2 \left(515 h^2 m n^2+4 (372 m-385 n) n^4\right)+$$
$$\left.b \left(9 h^4 m^2+2 h^2 m (515 m-2306 n) n^2+8 n^4 \left(186 m^2-385 m n+108 n^2\right)\right)\right) \omega_0^4-$$
$$i \left(388 b^3 n^4-h^4 m^2 (37 m+25 n)+h^2 m n^2 \left(46 m^2-3383 m n+6658 n^2\right)+4 n^4\right.$$
$$\left(97 m^3-620 m^2 n+693 m n^2-144 n^3\right)+2 b^2 \left(23 h^2 m n^2+2 (291 m-620 n) n^4\right)+$$
$$\left.b \left(-37 h^4 m^2+h^2 m (92 m-3383 n) n^2+4 n^4 \left(291 m^2-1240 m n+693 n^2\right)\right)\right) \omega_0^5+$$
$$\left(-90 h^4 m^2+h^2 m n \left(137 b^2+274 b m+137 m^2+506 b n+506 m n-4721 n^2\right)-\right.$$
$$4 n^3 \left(16 b^3+16 m^3+b^2 (48 m-485 n)-485 m^2 n+1116 m n^2-462 n^3+2 b \left(24 m^2-\right.\right.$$
$$\left.\left.\left.485 m n+558 n^2\right)\right)\right) \omega_0^6-i \left(116 b^3 n^2+h^2 m \left(50 m^2-627 m n-752 n^2\right)+\right.$$

$$4\,n^2\left(29\,m^3+80\,m^2\,n-873\,m\,n^2+744\,n^3\right)+b^2\left(50\,h^2\,m+4\,n^2\,(87\,m+80\,n)\right)+$$
$$b\left(h^2\,m\,(100\,m-627\,n)+4\,n^2\left(87\,m^2+160\,m\,n-873\,n^2\right)\right)\Big)\,\omega_0^7+$$
$$\Big(15\,h^2\,m\,(16\,m-51\,n)-80\,b^3\,n+20\,b^2\,n\,(-12\,m+29\,n)+4\,n\left(-20\,m^3+145\,m^2\,n+\right.$$
$$\left.144\,m\,n^2-582\,n^3\right)+8\,b\left(30\,h^2\,m+n\left(-30\,m^2+145\,m\,n+72\,n^2\right)\right)\Big)\,\omega_0^8+$$
$$2\,\mathbb{i}\,\Big(8\,b^3+147\,h^2\,m+8\,m^3+8\,b^2\,(3\,m-25\,n)-200\,m^2\,n+522\,m\,n^2+$$
$$192\,n^3+b\left(24\,m^2-400\,m\,n+522\,n^2\right)\Big)\,\omega_0^9-$$
$$8\left(10\,b^2+20\,b\,m+10\,m^2-90\,b\,n-90\,m\,n+87\,n^2\right)\omega_0^{10}-$$
$$48\,\mathbb{i}\,(3\,b+3\,m-10\,n)\,\omega_0^{11}+96\,\omega_0^{12}\Big)\Big/$$
$$\Big(\left(-h^2\,m+2\,\mathbb{i}\,(b+m)\,n\,\omega_0+4\,(b+m-n)\,\omega_0^2+8\,\mathbb{i}\,\omega_0^3\right)^2$$
$$\Big(-n\left(b^2\,n^2+2\,b\,m\left(-h^2+n^2\right)+m\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)+$$
$$3\,\mathbb{i}\left(b^2\,n^2-b\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)\omega_0+$$
$$3\left(2\,h^2\,m+n\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\omega_0^2-$$
$$\mathbb{i}\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\omega_0^3+3\,(b+m-3\,n)\,\omega_0^4+3\,\mathbb{i}\,\omega_0^5\Big)\Big)-$$
$$\Big(2\,h^2\,m\left(\mathbb{i}\,n+\omega_0\right)^2\Big(2\,h^2\,m-3\,(b+m)\,n^2+\mathbb{i}\,(13\,(b+m)-12\,n)\,n\,\omega_0+$$
$$12\,(b+m-4\,n)\,\omega_0^2+44\,\mathbb{i}\,\omega_0^3\Big)$$
$$\Big(12\,h^2\,m\,n^2+\omega_0\Big(-\mathbb{i}\,n\left(7\,h^2\,m+15\,(b+m)\,n^2\right)+\omega_0\Big(3\,h^2\,m+5\,n^2\,(-7\,(b+m)+6\,n)+$$
$$\omega_0\left(5\,\mathbb{i}\,(b+m-14\,n)\,n-\left(7\,(b+m)+10\,n+14\,\mathbb{i}\,\omega_0\right)\omega_0\right)\Big)\Big)\Big)\Big)\Big/$$
$$\Big(\left(-h^2\,m+2\left(b+m+2\,\mathbb{i}\,\omega_0\right)\omega_0\left(\mathbb{i}\,n+2\,\omega_0\right)\right)^3\Big(h^2\,m-(b+m)\,n^2+$$
$$\omega_0\left(2\,\mathbb{i}\,(b+m-n)\,n+\left(b+m-4\,n+2\,\mathbb{i}\,\omega_0\right)\omega_0\right)\Big)\Big)+$$
$$1\Big/\left(h^2\,m-2\,\mathbb{i}\left(n-2\,\mathbb{i}\,\omega_0\right)\left(b+m+2\,\mathbb{i}\,\omega_0\right)\omega_0\right)^2\left(\mathbb{i}\,n+\omega_0\right)$$
$$\Big(-21\,\mathbb{i}\,h^2\,m\,n^3+\omega_0\Big(-49\,h^2\,m\,n^2+6\,(b+m)\,n^4+$$
$$\omega_0\Big(3\,\mathbb{i}\left(5\,h^2\,m\,n-6\,(b+m)\,n^3+4\,n^4\right)+\omega_0\Big(-9\,h^2\,m+2\,n^2\,(-5\,(b+m)+18\,n)-$$
$$2\,\mathbb{i}\,n\,(3\,(b+m)+10\,n)\,\omega_0-4\,(b+m-3\,n)\,\omega_0^2-8\,\mathbb{i}\,\omega_0^3\Big)\Big)\Big)+$$
$$\Big(h^2\,m\left(\mathbb{i}\,n+\omega_0\right)\Big(21\,\mathbb{i}\,h^2\,m\,n^3+\omega_0\Big(80\,h^2\,m\,n^2-15\,(b+m)\,n^4+\omega_0$$
$$\Big(-\mathbb{i}\,n\left(79\,h^2\,m+n^2\,(-61\,(b+m)+39\,n)\right)+\omega_0\Big(-12\,h^2\,m+(61\,(b+m)-155\,n)$$
$$n^2+\omega_0\left(13\,\mathbb{i}\,n\,(b+m+11\,n)+\left(28\,(b+m)-59\,n+86\,\mathbb{i}\,\omega_0\right)\omega_0\right)\Big)\Big)\Big)\Big)\Big)\Big/$$
$$\Big(\left(h^2\,m-2\left(b+m+2\,\mathbb{i}\,\omega_0\right)\omega_0\left(\mathbb{i}\,n+2\,\omega_0\right)\right)^2\left(-h^2\,m+3\left(b+m+3\,\mathbb{i}\,\omega_0\right)\omega_0\left(\mathbb{i}\,n+3\,\omega_0\right)\right)\Big)\Big),$$

$$\frac{1}{h\left(m-\mathbb{i}\,\omega_0\right)\left(m+\mathbb{i}\,\omega_0\right)^3\left(-\mathbb{i}\,n+\omega_0\right)\left(\mathbb{i}\,n+\omega_0\right)^3}$$

6

$\sqrt{2}$

m^4

$$\Big(-\Big(\Big(m+2\,\mathbb{i}\,\omega_0\Big)\Big(-3\,\mathbb{i}\,h^2\,m\,n^3\left(10\,h^4\,m^2+h^2\,m\,(11\,b+11\,m-21\,n)\,n^2+3\,(b+m)^2\,n^4\right)-$$
$$\Big(69\,h^6\,m^3\,n^2+h^4\,m^2\,(175\,b+175\,m-397\,n)\,n^4+$$
$$7\,h^2\,m\,(b+m)\,(14\,b+14\,m-15\,n)\,n^6+12\,(b+m)^3\,n^8\Big)\,\omega_0+$$
$$\mathbb{i}\,\Big(38\,h^6\,m^3\,n+6\,h^4\,m^2\,(57\,b+57\,m-170\,n)\,n^3+12\,(b+m)^2\,(8\,b+8\,m-5\,n)\,n^7+$$
$$3\,h^2\,m\,n^5\left(127\,b^2+254\,b\,m+127\,m^2-294\,b\,n-294\,m\,n+61\,n^2\right)\Big)\,\omega_0^2+$$
$$\Big(3\,h^6\,m^3+2\,h^4\,m^2\,(127\,b+127\,m-661\,n)\,n^2+4\,(b+m)\,n^6$$
$$\left(77\,b^2+77\,m^2+2\,b\,(77\,m-60\,n)-120\,m\,n+27\,n^2\right)+$$
$$h^2\,m\,n^4\left(674\,b^2+1348\,b\,m+674\,m^2-2899\,b\,n-2899\,m\,n+1428\,n^2\right)\Big)\,\omega_0^3-\mathbb{i}\,n$$
$$\Big(h^4\,m^2\,(9\,m-767\,n)+496\,b^3\,n^4+h^2\,m\,n^2\left(515\,m^2-4612\,m\,n+4405\,n^2\right)+4\,n^4\,\Big(124$$
$$m^3-385\,m^2\,n+216\,m\,n^2-18\,n^3\Big)+b^2\left(515\,h^2\,m\,n^2+4\,(372\,m-385\,n)\,n^4\right)+$$
$$b\left(9\,h^4\,m^2+2\,h^2\,m\,(515\,m-2306\,n)\,n^2+8\,n^4\left(186\,m^2-385\,m\,n+108\,n^2\right)\right)\Big)\,\omega_0^4+$$
$$\Big(-388\,b^3\,n^4+h^4\,m^2\,(37\,m+25\,n)+h^2\,m\,n^2\left(-46\,m^2+3383\,m\,n-6658\,n^2\right)+$$

$$4\,n^4\,\left(-97\,m^3+620\,m^2\,n-693\,m\,n^2+144\,n^3\right)-$$
$$2\,b^2\,\left(23\,h^2\,m\,n^2+2\,(291\,m-620\,n)\,n^4\right)+$$
$$b\,\left(37\,h^4\,m^2+h^2\,m\,n^2\,(-92\,m+3383\,n)-4\,n^4\,\left(291\,m^2-1240\,m\,n+693\,n^2\right)\right)\Big)\,\omega_0^5+$$
$$i\,\left(90\,h^4\,m^2-h^2\,m\,n\,\left(137\,b^2+274\,b\,m+137\,m^2+506\,b\,n+506\,m\,n-4721\,n^2\right)+\right.$$
$$4\,n^3\,\left(16\,b^3+16\,m^3+b^2\,(48\,m-485\,n)-485\,m^2\,n+\right.$$
$$\left.\left.1116\,m\,n^2-462\,n^3+2\,b\,\left(24\,m^2-485\,m\,n+558\,n^2\right)\right)\right)\,\omega_0^6-$$
$$\left(116\,b^3\,n^2+h^2\,m\,\left(50\,m^2-627\,m\,n-752\,n^2\right)+4\,n^2\,\left(29\,m^3+80\,m^2\,n-\right.\right.$$
$$\left.873\,m\,n^2+744\,n^3\right)+b^2\,\left(50\,h^2\,m+4\,n^2\,(87\,m+80\,n)\right)+$$
$$\left.b\,\left(h^2\,m\,(100\,m-627\,n)+4\,n^2\,\left(87\,m^2+160\,m\,n-873\,n^2\right)\right)\right)\,\omega_0^7+$$
$$i\,\left(80\,b^3\,n+20\,b^2\,(12\,m-29\,n)\,n+15\,h^2\,m\,(-16\,m+51\,n)+4\,n\,\left(20\,m^3-145\,m^2\,n-\right.\right.$$
$$\left.\left.144\,m\,n^2+582\,n^3\right)-8\,b\,\left(30\,h^2\,m+n\,\left(-30\,m^2+145\,m\,n+72\,n^2\right)\right)\right)\,\omega_0^8+$$
$$2\,\left(8\,b^3+147\,h^2\,m+8\,m^3+8\,b^2\,(3\,m-25\,n)-200\,m^2\,n+522\,m\,n^2+192\,n^3+\right.$$
$$\left.b\,\left(24\,m^2-400\,m\,n+522\,n^2\right)\right)\,\omega_0^9+8\,i\,\left(10\,b^2+20\,b\,m+10\,m^2-\right.$$
$$\left.90\,b\,n-90\,m\,n+87\,n^2\right)\,\omega_0^{10}-48\,(3\,b+3\,m-10\,n)\,\omega_0^{11}-96\,i\,\omega_0^{12}\Big)\Big)\Big/$$
$$\left(\left(-h^2\,m+2\,i\,(b+m)\,n\,\omega_0+4\,(b+m-n)\,\omega_0^2+8\,i\,\omega_0^3\right)^2\right.$$
$$\left(-i\,n\,\left(b^2\,n^2+2\,b\,m\,\left(-h^2+n^2\right)+m\,\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)-\right.$$
$$3\,\left(b^2\,n^2-b\,\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\,\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)\,\omega_0+$$
$$3\,i\,\left(2\,h^2\,m+n\,\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\,\omega_0^2+$$
$$\left.\left.\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\,\omega_0^3+3\,i\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5\right)\right)-$$
$$\left(2\,h^2\,m\,\left(i\,n+\omega_0\right)^2\,\left(m\,\left(h^2\,(2\,m-3\,n)-3\,(b+m)\,n^2\right)+\right.\right.$$
$$\omega_0\,\left(i\,m\,\left(9\,h^2+(13\,(b+m)-12\,n)\,n\right)+\right.$$
$$\left.\left.4\,\omega_0\,\left(3\,m\,(b+m)-(b+13\,m)\,n+3\,n^2+i\,\left(b+12\,m-13\,n+12\,i\,\omega_0\right)\,\omega_0\right)\right)\right)$$
$$\left(12\,h^2\,m\,n^2+\omega_0\,\left(-i\,n\,\left(7\,h^2\,m+15\,(b+m)\,n^2\right)+\omega_0\,\left(3\,h^2\,m+5\,n^2\,(-7\,(b+m)+6\,n)+\right.\right.\right.$$
$$\left.\left.\left.\left.\omega_0\,\left(5\,i\,(b+m-14\,n)\,n-\left(7\,(b+m)+10\,n+14\,i\,\omega_0\right)\,\omega_0\right)\right)\right)\right)\right)\Big/$$
$$\left(\left(-h^2\,m+2\,\left(b+m+2\,i\,\omega_0\right)\,\omega_0\,\left(i\,n+2\,\omega_0\right)\right)^3\,\left(h^2\,m-(b+m)\,n^2+\right.\right.$$
$$\left.\left.\omega_0\,\left(2\,i\,(b+m-n)\,n+\left(b+m-4\,n+2\,i\,\omega_0\right)\,\omega_0\right)\right)\right)-$$
$$1\Big/\left(h^2\,m-2\,i\,\left(n-2\,i\,\omega_0\right)\,\left(b+m+2\,i\,\omega_0\right)\,\omega_0\right)^2\,\left(m+2\,i\,\omega_0\right)\,\left(i\,n+\omega_0\right)$$
$$\left(21\,i\,h^2\,m\,n^3+\omega_0\,\left(49\,h^2\,m\,n^2-6\,(b+m)\,n^4+\right.\right.$$
$$\omega_0\,\left(3\,i\,\left(-5\,h^2\,m\,n+6\,(b+m)\,n^3-4\,n^4\right)+\omega_0\,\left(9\,h^2\,m+2\,(5\,(b+m)-18\,n)\,n^2+\right.\right.$$
$$\left.\left.\left.\left.2\,i\,n\,(3\,(b+m)+10\,n)\,\omega_0+4\,(b+m-3\,n)\,\omega_0^2+8\,i\,\omega_0^3\right)\right)\right)\right)+$$
$$\left(h^2\,m\,\left(m+2\,i\,\omega_0\right)\,\left(i\,n+\omega_0\right)\,\left(21\,i\,h^2\,m\,n^3+\omega_0\,\left(80\,h^2\,m\,n^2-15\,(b+m)\,n^4+\omega_0\right.\right.\right.$$
$$\left(-i\,n\,\left(79\,h^2\,m+n^2\,(-61\,(b+m)+39\,n)\right)+\omega_0\,\left(-12\,h^2\,m+(61\,(b+m)-155\,n)\right.\right.$$
$$\left.\left.\left.\left.\left.\left.n^2+\omega_0\,\left(13\,i\,n\,(b+m+11\,n)+\left(28\,(b+m)-59\,n+86\,i\,\omega_0\right)\,\omega_0\right)\right)\right)\right)\right)\right)\Big/$$
$$\left(\left(h^2\,m-2\,\left(b+m+2\,i\,\omega_0\right)\,\omega_0\,\left(i\,n+2\,\omega_0\right)\right)^2\,\left(-h^2\,m+3\,\left(b+m+3\,i\,\omega_0\right)\,\omega_0\,\left(i\,n+3\,\omega_0\right)\right)\right)\Big)\Big\}$$

(* Cálculo do vetor complexo h22 *)

```
(* h22=FullSimplify[
        simplifica completamente
  -AI.( bb[h11,h11]+2 bb[q,h21b]+2 bb[qb,h21]+bb[h20b,h20]-4 h11 l1),
  ω0∈Reals && m>0 && n<0 && h>0 &&b>0] *)
      números reais
```

h221 = FullSimplify[-AI.(bb[h11, h11]), ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
simplifica completamente | números reais

$$\left\{-\frac{16\, m^4\, n\, (3\, n^2+\omega_0^2)}{h^2\, (m^2+\omega_0^2)^2\, (n^2+\omega_0^2)^2},\ -\frac{2\sqrt{2}\, m^4\, (3\, n^2+\omega_0^2)\, (7\, n^2+\omega_0^2)}{h^3\, (m^2+\omega_0^2)^2\, (n^2+\omega_0^2)^2},\ -\frac{2\sqrt{2}\, m^4\, (3\, n^2+\omega_0^2)\, (7\, n^2+\omega_0^2)}{h^3\, (m^2+\omega_0^2)^2\, (n^2+\omega_0^2)^2}\right\}$$

h222 = Simplify[-AI.(2 bb[q, h21b]), ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
simplifica | números reais

$$\{-(8 m^4 (-3 \mathrm{i} h^2 m n^2 (5 h^4 m^2 + h^2 m (8 b + 8 m - 13 n) n^2 + 3 (b+m)^2 n^4) + (12 h^6 m^3 n + h^4 m^2$$
$$(50 b + 50 m - 161 n) n^3 + 25 h^2 m (b+m) (2 b + 2 m - 3 n) n^5 + 12 (b+m)^3 n^7) \omega_0 +$$
$$\mathrm{i} (h^6 m^3 + 2 h^4 m^2 (19 b + 19 m - 103 n) n^2 + 12 (b+m)^2 (4 b + 4 m - 5 n) n^6 +$$
$$h^2 m n^4 (91 b^2 + 182 b m + 91 m^2 - 404 b n - 404 m n + 123 n^2)) \omega_0^2 -$$
$$n (44 b^3 n^4 + 4 b^2 (5 h^2 m n^2 + 33 m n^4 - 60 n^5) + b (-10 h^4 m^2 + h^2 m (40 m - 719 n) n^2 +$$
$$12 n^4 (11 m^2 - 40 m n + 9 n^2)) + m (-2 h^4 m (5 m + 49 n) +$$
$$4 n^4 (11 m^2 - 60 m n + 27 n^2) + h^2 n^2 (20 m^2 - 719 m n + 628 n^2))) \omega_0^3 +$$
$$\mathrm{i} (h^4 m^2 (14 m - 37 n) + 40 b^3 n^4 + 3 h^2 m n^2 (19 m^2 + 104 m n - 357 n^2) +$$
$$4 n^4 (10 m^3 + 55 m^2 n - 108 m n^2 + 18 n^3) + b^2 (57 h^2 m n^2 + 20 n^4 (6 m + 11 n)) +$$
$$2 b (7 h^4 m^2 + 3 h^2 m n^2 (19 m + 52 n) + 4 n^4 (15 m^2 + 55 m n - 54 n^2))) \omega_0^4 +$$
$$(35 h^4 m^2 + h^2 m n (-22 b^2 - 44 b m - 22 m^2 + 223 b n + 223 m n + 512 n^2) - 4 n^3 (19 b^3 +$$
$$19 m^3 + b^2 (57 m - 50 n) - 50 m^2 n - 99 m n^2 + 72 n^3 + b (57 m^2 - 100 m n - 99 n^2)))$$
$$\omega_0^5 - \mathrm{i} (16 b^3 n^2 + h^2 m (-5 m^2 - 140 m n + 231 n^2) + b^2 (-5 h^2 m + 4 (12 m - 95 n) n^2) +$$
$$4 n^2 (4 m^3 - 95 m^2 n + 90 m n^2 + 66 n^3) -$$
$$2 b (5 h^2 m (m + 14 n) - 4 n^2 (6 m^2 - 95 m n + 45 n^2))) \omega_0^6 -$$
$$(20 b^3 n + 20 b^2 n (3 m + 4 n) - 5 h^2 m (m + 36 n) + 4 n (5 m^3 + 20 m^2 n - 171 m n^2 + 60 n^3) +$$
$$b (-5 h^2 m + 4 n (15 m^2 + 40 m n - 171 n^2))) \omega_0^7 -$$
$$\mathrm{i} (8 b^3 - 3 h^2 m + 8 m^3 + 4 b^2 (6 m - 25 n) - 100 m^2 n - 144 m n^2 +$$
$$456 n^3 + 8 b (3 m^2 - 25 m n - 18 n^2)) \omega_0^8 -$$
$$4 (10 b^2 + 10 m^2 + 5 b (4 m - 9 n) - 45 m n - 24 n^2) \omega_0^9 +$$
$$24 \mathrm{i} (3 b + 3 m - 5 n) \omega_0^{10} + 48 \omega_0^{11})) /$$
$$(h^2 (m^2 + \omega_0^2)^2 (n^2 + \omega_0^2)^2 (h^2 m + 2 \mathrm{i} (b+m) n \omega_0 - 4 (b + m - n) \omega_0^2 + 8 \mathrm{i} \omega_0^3)$$
$$(\mathrm{i} n (b^2 n^2 + 2 b m (-h^2 + n^2) + m (m n^2 + h^2 (-2 m + n))) -$$
$$3 (b^2 n^2 - b (h^2 m + n^2 (-2 m + n)) - m (h^2 (m - 2 n) + n^2 (-m + n))) \omega_0 -$$
$$3 \mathrm{i} (2 h^2 m + n (b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2)) \omega_0^2 +$$
$$(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2) \omega_0^3 - 3 \mathrm{i} (b + m - 3 n) \omega_0^4 - 3 \omega_0^5)),$$
$$-(4 \sqrt{2} m^4 (-3 \mathrm{i} h^2 m n^3 (10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b+m)^2 n^4) +$$
$$(9 h^6 m^3 n^2 + h^4 m^2 (79 b + 79 m - 241 n) n^4 +$$
$$h^2 m (b+m) (62 b + 62 m - 105 n) n^6 + 12 (b+m)^3 n^8) \omega_0 -$$
$$\mathrm{i} (10 h^6 m^3 n - 26 h^4 m^2 (2 b + 2 m - 11 n) n^3 - 12 (b+m)^2 (4 b + 4 m - 5 n) n^7 -$$
$$h^2 m n^5 (127 b^2 + 254 b m + 127 m^2 - 582 b n - 582 m n + 183 n^2)) \omega_0^2 +$$
$$(h^6 m^3 + 6 h^4 m^2 n^2 (10 b + 10 m + 11 n) - 4 (b+m) n^6 (11 b^2 + 22 b m + 11 m^2 - 60 b n -$$
$$60 m n + 27 n^2) - h^2 m n^4 (28 b^2 + 56 b m + 28 m^2 - 1013 b n - 1013 m n + 936 n^2))$$
$$\omega_0^3 + \mathrm{i} n (h^4 m^2 (29 m - 189 n) + 40 b^3 n^4 + 5 h^2 m n^2 (21 m^2 + 58 m n - 303 n^2) +$$
$$4 n^4 (10 m^3 + 55 m^2 n - 108 m n^2 + 18 n^3) + 5 b^2 (21 h^2 m n^2 + 4 n^4 (6 m + 11 n)) +$$
$$b (29 h^4 m^2 + 10 h^2 m n^2 (21 m + 29 n) + 8 n^4 (15 m^2 + 55 m n - 54 n^2))) \omega_0^4 +$$

$$(-76\, b^3\, n^4 + h^4\, m^2\, (13\, m + 67\, n) + h^2\, m\, n^2\, (-50\, m^2 + 613\, m\, n + 436\, n^2) +$$
$$4\, n^4\, (-19\, m^3 + 50\, m^2\, n + 99\, m\, n^2 - 72\, n^3) + b^2\, (-50\, h^2\, m\, n^2 + 4\, n^4\, (-57\, m + 50\, n)) +$$
$$b\, (13\, h^4\, m^2 + h^2\, m\, n^2\, (-100\, m + 613\, n) + 4\, n^4\, (-57\, m^2 + 100\, m\, n + 99\, n^2)))\, \omega_0^5 -$$
$$i\, (32\, h^4\, m^2 - h^2\, m\, n\, (17\, b^2 + 34\, b\, m + 17\, m^2 + 342\, b\, n + 342\, m\, n - 819\, n^2) + 4\, n^3\, (4\, b^3 +$$
$$4\, m^3 + b^2\, (12\, m - 95\, n) - 95\, m^2\, n + 90\, m\, n^2 + 66\, n^3 + 2\, b\, (6\, m^2 - 95\, m\, n + 45\, n^2)))$$
$$\omega_0^6 - (20\, b^3\, n^2 + h^2\, m\, (8\, m^2 - 71\, m\, n - 472\, n^2) + 4\, n^2\, (5\, m^3 + 20\, m^2\, n - 171\, m\, n^2 + 60\, n^3) +$$
$$b^2\, (8\, h^2\, m + 20\, n^2\, (3\, m + 4\, n)) +$$
$$b\, (h^2\, m\, (16\, m - 71\, n) + 4\, n^2\, (15\, m^2 + 40\, m\, n - 171\, n^2)))\, \omega_0^7 -$$
$$i\, (8\, b^3\, n + 4\, b^2\, (6\, m - 25\, n)\, n + h^2\, m\, (-46\, m + 81\, n) + 4\, n$$
$$(2\, m^3 - 25\, m^2\, n - 36\, m\, n^2 + 114\, n^3) - 2\, b\, (23\, h^2\, m + 4\, n\, (-3\, m^2 + 25\, m\, n + 18\, n^2)))\, \omega_0^8 +$$
$$4\, (15\, h^2\, m + n\, (-10\, b^2 - 20\, b\, m - 10\, m^2 + 45\, b\, n + 45\, m\, n + 24\, n^2))\, \omega_0^9 +$$
$$24\, i\, (3\, b + 3\, m - 5\, n)\, n\, \omega_0^{10} + 48\, n\, \omega_0^{11}))\, /$$
$$(h^3\, (m^2 + \omega_0^2)^2\, (n^2 + \omega_0^2)^2\, (h^2\, m + 2\, i\, (b + m)\, n\, \omega_0 - 4\, (b + m - n)\, \omega_0^2 + 8\, i\, \omega_0^3)$$
$$(i\, n\, (b^2\, n^2 + 2\, b\, m\, (-h^2 + n^2) + m\, (m\, n^2 + h^2\, (-2\, m + n))) -$$
$$3\, (b^2\, n^2 - b\, (h^2\, m + n^2\, (-2\, m + n)) - m\, (h^2\, (m - 2\, n) + n^2\, (-m + n)))\, \omega_0 -$$
$$3\, i\, (2\, h^2\, m + n\, (b^2 + 2\, b\, m + m^2 - 3\, b\, n - 3\, m\, n + n^2))\, \omega_0^2 +$$
$$(b^2 + 2\, b\, m + m^2 - 9\, b\, n - 9\, m\, n + 9\, n^2)\, \omega_0^3 - 3\, i\, (b + m - 3\, n)\, \omega_0^4 - 3\, \omega_0^5)),$$
$$-(4\, \sqrt{2}\, m^4\, (-3\, i\, h^2\, m\, n^3\, (10\, h^4\, m^2 + h^2\, m\, (11\, b + 11\, m - 21\, n)\, n^2 + 3\, (b + m)^2\, n^4) +$$
$$(9\, h^6\, m^3\, n^2 + h^4\, m^2\, (79\, b + 79\, m - 241\, n)\, n^4 +$$
$$h^2\, m\, (b + m)\, (62\, b + 62\, m - 105\, n)\, n^6 + 12\, (b + m)^3\, n^8)\, \omega_0 -$$
$$i\, (10\, h^6\, m^3\, n - 26\, h^4\, m^2\, (2\, b + 2\, m - 11\, n)\, n^3 - 12\, (b + m)^2\, (4\, b + 4\, m - 5\, n)\, n^7 -$$
$$h^2\, m\, n^5\, (127\, b^2 + 254\, b\, m + 127\, m^2 - 582\, b\, n - 582\, m\, n + 183\, n^2))\, \omega_0^2 +$$
$$(h^6\, m^3 + 6\, h^4\, m^2\, n^2\, (10\, b + 10\, m + 11\, n) - 4\, (b + m)\, n^6\, (11\, b^2 + 22\, b\, m + 11\, m^2 - 60\, b\, n -$$
$$60\, m\, n + 27\, n^2) - h^2\, m\, n^4\, (28\, b^2 + 56\, b\, m + 28\, m^2 - 1013\, b\, n - 1013\, m\, n + 936\, n^2))$$
$$\omega_0^3 + i\, n\, (h^4\, m^2\, (29\, m - 189\, n) + 40\, b^3\, n^4 + 5\, h^2\, m\, n^2\, (21\, m^2 + 58\, m\, n - 303\, n^2) +$$
$$4\, n^4\, (10\, m^3 + 55\, m^2\, n - 108\, m\, n^2 + 18\, n^3) + 5\, b^2\, (21\, h^2\, m\, n^2 + 4\, n^4\, (6\, m + 11\, n)) +$$
$$b\, (29\, h^4\, m^2 + 10\, h^2\, m\, n^2\, (21\, m + 29\, n) + 8\, n^4\, (15\, m^2 + 55\, m\, n - 54\, n^2)))\, \omega_0^4 +$$
$$(-76\, b^3\, n^4 + h^4\, m^2\, (13\, m + 67\, n) + h^2\, m\, n^2\, (-50\, m^2 + 613\, m\, n + 436\, n^2) +$$
$$4\, n^4\, (-19\, m^3 + 50\, m^2\, n + 99\, m\, n^2 - 72\, n^3) + b^2\, (-50\, h^2\, m\, n^2 + 4\, n^4\, (-57\, m + 50\, n)) +$$
$$b\, (13\, h^4\, m^2 + h^2\, m\, n^2\, (-100\, m + 613\, n) + 4\, n^4\, (-57\, m^2 + 100\, m\, n + 99\, n^2)))\, \omega_0^5 -$$
$$i\, (32\, h^4\, m^2 - h^2\, m\, n\, (17\, b^2 + 34\, b\, m + 17\, m^2 + 342\, b\, n + 342\, m\, n - 819\, n^2) + 4\, n^3\, (4\, b^3 +$$
$$4\, m^3 + b^2\, (12\, m - 95\, n) - 95\, m^2\, n + 90\, m\, n^2 + 66\, n^3 + 2\, b\, (6\, m^2 - 95\, m\, n + 45\, n^2)))$$
$$\omega_0^6 - (20\, b^3\, n^2 + h^2\, m\, (8\, m^2 - 71\, m\, n - 472\, n^2) + 4\, n^2\, (5\, m^3 + 20\, m^2\, n - 171\, m\, n^2 + 60\, n^3) +$$
$$b^2\, (8\, h^2\, m + 20\, n^2\, (3\, m + 4\, n)) +$$
$$b\, (h^2\, m\, (16\, m - 71\, n) + 4\, n^2\, (15\, m^2 + 40\, m\, n - 171\, n^2)))\, \omega_0^7 -$$
$$i\, (8\, b^3\, n + 4\, b^2\, (6\, m - 25\, n)\, n + h^2\, m\, (-46\, m + 81\, n) + 4\, n$$
$$(2\, m^3 - 25\, m^2\, n - 36\, m\, n^2 + 114\, n^3) - 2\, b\, (23\, h^2\, m + 4\, n\, (-3\, m^2 + 25\, m\, n + 18\, n^2)))\, \omega_0^8 +$$
$$4\, (15\, h^2\, m + n\, (-10\, b^2 - 20\, b\, m - 10\, m^2 + 45\, b\, n + 45\, m\, n + 24\, n^2))\, \omega_0^9 +$$
$$24\, i\, (3\, b + 3\, m - 5\, n)\, n\, \omega_0^{10} + 48\, n\, \omega_0^{11}))\, /$$
$$(h^3\, (m^2 + \omega_0^2)^2\, (n^2 + \omega_0^2)^2\, (h^2\, m + 2\, i\, (b + m)\, n\, \omega_0 - 4\, (b + m - n)\, \omega_0^2 + 8\, i\, \omega_0^3)$$
$$(i\, n\, (b^2\, n^2 + 2\, b\, m\, (-h^2 + n^2) + m\, (m\, n^2 + h^2\, (-2\, m + n))) -$$
$$3\, (b^2\, n^2 - b\, (h^2\, m + n^2\, (-2\, m + n)) - m\, (h^2\, (m - 2\, n) + n^2\, (-m + n)))\, \omega_0 -$$
$$3\, i\, (2\, h^2\, m + n\, (b^2 + 2\, b\, m + m^2 - 3\, b\, n - 3\, m\, n + n^2))\, \omega_0^2 +$$
$$(b^2 + 2\, b\, m + m^2 - 9\, b\, n - 9\, m\, n + 9\, n^2)\, \omega_0^3 - 3\, i\, (b + m - 3\, n)\, \omega_0^4 - 3\, \omega_0^5))\}$$

h223 = Simplify[-AI.(2 bb[qb, h21]), ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]

simplifica — números reais

$$\{(8\,m^4\,(-3\,\mathrm{i}\,h^2\,m\,n^2\,(5\,h^4\,m^2+h^2\,m\,(8\,b+8\,m-13\,n)\,n^2+3\,(b+m)^2\,n^4)\,-$$
$$(12\,h^6\,m^3\,n+h^4\,m^2\,(50\,b+50\,m-161\,n)\,n^3\,+$$
$$25\,h^2\,m\,(b+m)\,(2\,b+2\,m-3\,n)\,n^5+12\,(b+m)^3\,n^7)\,\omega_0\,+$$
$$\mathrm{i}\,(h^6\,m^3+2\,h^4\,m^2\,(19\,b+19\,m-103\,n)\,n^2+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^6\,+$$
$$h^2\,m\,n^4\,(91\,b^2+182\,b\,m+91\,m^2-404\,b\,n-404\,m\,n+123\,n^2))\,\omega_0^2\,+$$
$$n\,(44\,b^3\,n^4+4\,b^2\,(5\,h^2\,m\,n^2+33\,m\,n^4-60\,n^5)+b\,(-10\,h^4\,m^2+h^2\,m\,(40\,m-719\,n)\,n^2\,+$$
$$12\,n^4\,(11\,m^2-40\,m\,n+9\,n^2))+m\,(-2\,h^4\,m\,(5\,m+49\,n)\,+$$
$$4\,n^4\,(11\,m^2-60\,m\,n+27\,n^2)+h^2\,n^2\,(20\,m^2-719\,m\,n+628\,n^2)))\,\omega_0^3\,+$$
$$\mathrm{i}\,(h^4\,m^2\,(14\,m-37\,n)+40\,b^3\,n^4+3\,h^2\,m\,n^2\,(19\,m^2+104\,m\,n-357\,n^2)\,+$$
$$4\,n^4\,(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3)+b^2\,(57\,h^2\,m\,n^2+20\,n^4\,(6\,m+11\,n))\,+$$
$$2\,b\,(7\,h^4\,m^2+3\,h^2\,m\,n^2\,(19\,m+52\,n)+4\,n^4\,(15\,m^2+55\,m\,n-54\,n^2)))\,\omega_0^4\,+$$
$$(-35\,h^4\,m^2+h^2\,m\,n\,(22\,b^2+44\,b\,m+22\,m^2-223\,b\,n-223\,m\,n-512\,n^2)+4\,n^3\,(19\,b^3\,+$$
$$19\,m^3+b^2\,(57\,m-50\,n)-50\,m^2\,n-99\,m\,n^2+72\,n^3+b\,(57\,m^2-100\,m\,n-99\,n^2)))$$
$$\omega_0^5-\mathrm{i}\,(16\,b^3\,n^2+h^2\,m\,(-5\,m^2-140\,m\,n+231\,n^2)+b^2\,(-5\,h^2\,m+4\,(12\,m-95\,n)\,n^2)\,+$$
$$4\,n^2\,(4\,m^3-95\,m^2\,n+90\,m\,n^2+66\,n^3)\,-$$
$$2\,b\,(5\,h^2\,m\,(m+14\,n)-4\,n^2\,(6\,m^2-95\,m\,n+45\,n^2)))\,\omega_0^6\,+$$
$$(20\,b^3\,n+20\,b^2\,n\,(3\,m+4\,n)-5\,h^2\,m\,(m+36\,n)+4\,n\,(5\,m^3+20\,m^2\,n-171\,m\,n^2+60\,n^3)\,+$$
$$b\,(-5\,h^2\,m+4\,n\,(15\,m^2+40\,m\,n-171\,n^2)))\,\omega_0^7\,-$$
$$\mathrm{i}\,(8\,b^3-3\,h^2\,m+8\,m^3+4\,b^2\,(6\,m-25\,n)-100\,m^2\,n-144\,m\,n^2\,+$$
$$456\,n^3+8\,b\,(3\,m^2-25\,m\,n-18\,n^2))\,\omega_0^8\,+$$
$$4\,(10\,b^2+10\,m^2+5\,b\,(4\,m-9\,n)-45\,m\,n-24\,n^2)\,\omega_0^9+24\,\mathrm{i}\,(3\,b+3\,m-5\,n)\,\omega_0^{10}-48\,\omega_0^{11}))\,/$$
$$(h^2\,(m^2+\omega_0^2)^2\,(n^2+\omega_0^2)^2\,(h^2\,m-2\,\mathrm{i}\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2-8\,\mathrm{i}\,\omega_0^3)$$
$$(-\mathrm{i}\,n\,(b^2\,n^2+2\,b\,m\,(-h^2+n^2)+m\,(m\,n^2+h^2\,(-2\,m+n)))\,-$$
$$3\,(b^2\,n^2-b\,(h^2\,m+n^2\,(-2\,m+n))-m\,(h^2\,(m-2\,n)+n^2\,(-m+n)))\,\omega_0\,+$$
$$3\,\mathrm{i}\,(2\,h^2\,m+n\,(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2))\,\omega_0^2\,+$$
$$(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2)\,\omega_0^3+3\,\mathrm{i}\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5)),$$
$$(4\,\sqrt{2}\,m^4\,(-3\,\mathrm{i}\,h^2\,m\,n^3\,(10\,h^4\,m^2+h^2\,m\,(11\,b+11\,m-21\,n)\,n^2+3\,(b+m)^2\,n^4)\,-$$
$$(9\,h^6\,m^3\,n^2+h^4\,m^2\,(79\,b+79\,m-241\,n)\,n^4\,+$$
$$h^2\,m\,(b+m)\,(62\,b+62\,m-105\,n)\,n^6+12\,(b+m)^3\,n^8)\,\omega_0\,-$$
$$\mathrm{i}\,(10\,h^6\,m^3\,n-26\,h^4\,m^2\,(2\,b+2\,m-11\,n)\,n^3-12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^7\,-$$
$$h^2\,m\,n^5\,(127\,b^2+254\,b\,m+127\,m^2-582\,b\,n-582\,m\,n+183\,n^2))\,\omega_0^2\,+$$
$$(-h^6\,m^3-6\,h^4\,m^2\,n^2\,(10\,b+10\,m+11\,n)+4\,(b+m)\,n^6\,(11\,b^2+22\,b\,m+11\,m^2-60\,b\,n\,-$$
$$60\,m\,n+27\,n^2)+h^2\,m\,n^4\,(28\,b^2+56\,b\,m+28\,m^2-1013\,b\,n-1013\,m\,n+936\,n^2))$$
$$\omega_0^3+\mathrm{i}\,n\,(h^4\,m^2\,(29\,m-189\,n)+40\,b^3\,n^4+5\,h^2\,m\,n^2\,(21\,m^2+58\,m\,n-303\,n^2)\,+$$
$$4\,n^4\,(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3)+5\,b^2\,(21\,h^2\,m\,n^2+4\,n^4\,(6\,m+11\,n))\,+$$
$$b\,(29\,h^4\,m^2+10\,h^2\,m\,n^2\,(21\,m+29\,n)+8\,n^4\,(15\,m^2+55\,m\,n-54\,n^2)))\,\omega_0^4\,+$$
$$(76\,b^3\,n^4-h^4\,m^2\,(13\,m+67\,n)+h^2\,m\,n^2\,(50\,m^2-613\,m\,n-436\,n^2)\,+$$
$$4\,n^4\,(19\,m^3-50\,m^2\,n-99\,m\,n^2+72\,n^3)+b^2\,(50\,h^2\,m\,n^2+4\,(57\,m-50\,n)\,n^4)\,-$$
$$b\,(13\,h^4\,m^2+h^2\,m\,n^2\,(-100\,m+613\,n)+4\,n^4\,(-57\,m^2+100\,m\,n+99\,n^2)))\,\omega_0^5\,-$$
$$\mathrm{i}\,(32\,h^4\,m^2-h^2\,m\,n\,(17\,b^2+34\,b\,m+17\,m^2+342\,b\,n+342\,m\,n-819\,n^2)+4\,n^3\,(4\,b^3\,+$$
$$4\,m^3+b^2\,(12\,m-95\,n)-95\,m^2\,n+90\,m\,n^2+66\,n^3+2\,b\,(6\,m^2-95\,m\,n+45\,n^2)))\,\omega_0^6\,+$$
$$(20\,b^3\,n^2+h^2\,m\,(8\,m^2-71\,m\,n-472\,n^2)+4\,n^2\,(5\,m^3+20\,m^2\,n-171\,m\,n^2+60\,n^3)+b^2$$
$$(8\,h^2\,m+20\,n^2\,(3\,m+4\,n))+b\,(h^2\,m\,(16\,m-71\,n)+4\,n^2\,(15\,m^2+40\,m\,n-171\,n^2)))$$

$$\omega_0^7 - i \left(8 b^3 n + 4 b^2 (6 m - 25 n) n + h^2 m (-46 m + 81 n) + 4 n\right.$$
$$\left(2 m^3 - 25 m^2 n - 36 m n^2 + 114 n^3\right) - 2 b \left(23 h^2 m + 4 n \left(-3 m^2 + 25 m n + 18 n^2\right)\right)\big) \omega_0^8 -$$
$$4 \left(15 h^2 m + n \left(-10 b^2 - 20 b m - 10 m^2 + 45 b n + 45 m n + 24 n^2\right)\right) \omega_0^9 +$$
$$24 i (3 b + 3 m - 5 n) n \omega_0^{10} - 48 n \omega_0^{11}\big)\big) \Big/$$
$$\Big(h^3 \left(m^2 + \omega_0^2\right)^2 \left(n^2 + \omega_0^2\right)^2 \left(h^2 m - 2 i (b + m) n \omega_0 - 4 (b + m - n) \omega_0^2 - 8 i \omega_0^3\right)$$
$$\Big(-i n \left(b^2 n^2 + 2 b m \left(-h^2 + n^2\right) + m \left(m n^2 + h^2 (-2 m + n)\right)\right) -$$
$$3 \left(b^2 n^2 - b \left(h^2 m + n^2 (-2 m + n)\right) - m \left(h^2 (m - 2 n) + n^2 (-m + n)\right)\right) \omega_0 +$$
$$3 i \left(2 h^2 m + n \left(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2\right)\right) \omega_0^2 +$$
$$\left(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2\right) \omega_0^3 + 3 i (b + m - 3 n) \omega_0^4 - 3 \omega_0^5\Big)\Big),$$

$$\Big(4 \sqrt{2} m^4 \Big(-3 i h^2 m n^3 \left(10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b + m)^2 n^4\right) -$$
$$\Big(9 h^6 m^3 n^2 + h^4 m^2 (79 b + 79 m - 241 n) n^4 +$$
$$h^2 m (b + m) (62 b + 62 m - 105 n) n^6 + 12 (b + m)^3 n^8\Big) \omega_0 -$$
$$i \Big(10 h^6 m^3 n - 26 h^4 m^2 (2 b + 2 m - 11 n) n^3 - 12 (b + m)^2 (4 b + 4 m - 5 n) n^7 -$$
$$h^2 m n^5 \left(127 b^2 + 254 b m + 127 m^2 - 582 b n - 582 m n + 183 n^2\right)\Big) \omega_0^2 +$$
$$\Big(-h^6 m^3 - 6 h^4 m^2 n^2 (10 b + 10 m + 11 n) + 4 (b + m) n^6 \Big(11 b^2 + 22 b m + 11 m^2 - 60 b n - 60$$
$$m n + 27 n^2\Big) + h^2 m n^4 \left(28 b^2 + 56 b m + 28 m^2 - 1013 b n - 1013 m n + 936 n^2\right)\Big) \omega_0^3 +$$
$$i n \Big(h^4 m^2 (29 m - 189 n) + 40 b^3 n^4 + 5 h^2 m n^2 \left(21 m^2 + 58 m n - 303 n^2\right) +$$
$$4 n^4 \left(10 m^3 + 55 m^2 n - 108 m n^2 + 18 n^3\right) + 5 b^2 \left(21 h^2 m n^2 + 4 n^4 (6 m + 11 n)\right) +$$
$$b \left(29 h^4 m^2 + 10 h^2 m n^2 (21 m + 29 n) + 8 n^4 \left(15 m^2 + 55 m n - 54 n^2\right)\right)\Big) \omega_0^4 +$$
$$\Big(76 b^3 n^4 - h^4 m^2 (13 m + 67 n) + h^2 m n^2 \left(50 m^2 - 613 m n - 436 n^2\right) +$$
$$4 n^4 \left(19 m^3 - 50 m^2 n - 99 m n^2 + 72 n^3\right) + b^2 \left(50 h^2 m n^2 + 4 (57 m - 50 n) n^4\right) -$$
$$b \left(13 h^4 m^2 + h^2 m n^2 (-100 m + 613 n) + 4 n^4 \left(-57 m^2 + 100 m n + 99 n^2\right)\right)\Big) \omega_0^5 -$$
$$i \Big(32 h^4 m^2 - h^2 m n \left(17 b^2 + 34 b m + 17 m^2 + 342 b n + 342 m n - 819 n^2\right) + 4 n^3 \Big(4 b^3 +$$
$$4 m^3 + b^2 (12 m - 95 n) - 95 m^2 n + 90 m n^2 + 66 n^3 + 2 b \left(6 m^2 - 95 m n + 45 n^2\right)\Big)\Big) \omega_0^6 +$$
$$\Big(20 b^3 n^2 + h^2 m \left(8 m^2 - 71 m n - 472 n^2\right) + 4 n^2 \left(5 m^3 + 20 m^2 n - 171 m n^2 + 60 n^3\right) + b^2$$
$$\left(8 h^2 m + 20 n^2 (3 m + 4 n)\right) + b \left(h^2 m (16 m - 71 n) + 4 n^2 \left(15 m^2 + 40 m n - 171 n^2\right)\right)\Big)$$
$$\omega_0^7 - i \Big(8 b^3 n + 4 b^2 (6 m - 25 n) n + h^2 m (-46 m + 81 n) + 4 n$$
$$\left(2 m^3 - 25 m^2 n - 36 m n^2 + 114 n^3\right) - 2 b \left(23 h^2 m + 4 n \left(-3 m^2 + 25 m n + 18 n^2\right)\right)\Big) \omega_0^8 -$$
$$4 \left(15 h^2 m + n \left(-10 b^2 - 20 b m - 10 m^2 + 45 b n + 45 m n + 24 n^2\right)\right) \omega_0^9 +$$
$$24 i (3 b + 3 m - 5 n) n \omega_0^{10} - 48 n \omega_0^{11}\Big)\Big) \Big/$$
$$\Big(h^3 \left(m^2 + \omega_0^2\right)^2 \left(n^2 + \omega_0^2\right)^2 \left(h^2 m - 2 i (b + m) n \omega_0 - 4 (b + m - n) \omega_0^2 - 8 i \omega_0^3\right)$$
$$\Big(-i n \left(b^2 n^2 + 2 b m \left(-h^2 + n^2\right) + m \left(m n^2 + h^2 (-2 m + n)\right)\right) -$$
$$3 \left(b^2 n^2 - b \left(h^2 m + n^2 (-2 m + n)\right) - m \left(h^2 (m - 2 n) + n^2 (-m + n)\right)\right) \omega_0 +$$
$$3 i \left(2 h^2 m + n \left(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2\right)\right) \omega_0^2 +$$
$$\left(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2\right) \omega_0^3 + 3 i (b + m - 3 n) \omega_0^4 - 3 \omega_0^5\Big)\Big)\}$$

```
h224 =
 FullSimplify[-AI.(bb[h20b, h20]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```

simplifica completamente   números reais

$$\Big\{\Big(16\,m^5\,\big(-3\,h^2\,m\,n+2\,n\,(b+m+3\,n)\,\omega_0^2+10\,\omega_0^4\big)\Big)\Big/\Big(\big(n-i\,\omega_0\big)\,\big(n+i\,\omega_0\big)\,\big(h^2\,m-2\,i\,\big(n-2\,i\,\omega_0\big)\,\big(b+m+2\,i\,\omega_0\big)\,\omega_0\big)\,\big(m^2+\omega_0^2\big)^2\,\big(h^2\,m+2\,\big(n+2\,i\,\omega_0\big)\,\omega_0\,\big(i\,(b+m)+2\,\omega_0\big)\big)\Big),$$

$$\Big(2\,\sqrt{2}\,m^5\,\big(-21\,h^2\,m\,n^2+\big(-25\,h^2\,m+8\,n^2\,(b+m+3\,n)\big)\,\omega_0^2+40\,n\,\omega_0^4\big)\Big)\Big/\Big(h\,\big(n-i\,\omega_0\big)\,\big(n+i\,\omega_0\big)\,\big(h^2\,m-2\,i\,\big(n-2\,i\,\omega_0\big)\,\big(b+m+2\,i\,\omega_0\big)\,\omega_0\big)\,\big(m^2+\omega_0^2\big)^2\,\big(h^2\,m+2\,\big(n+2\,i\,\omega_0\big)\,\omega_0\,\big(i\,(b+m)+2\,\omega_0\big)\big)\Big),$$

$$\Big(2\,\sqrt{2}\,m^5\,\big(-21\,h^2\,m\,n^2+\big(-25\,h^2\,m+8\,n^2\,(b+m+3\,n)\big)\,\omega_0^2+40\,n\,\omega_0^4\big)\Big)\Big/\Big(h\,\big(n-i\,\omega_0\big)\,\big(n+i\,\omega_0\big)\,\big(h^2\,m-2\,i\,\big(n-2\,i\,\omega_0\big)\,\big(b+m+2\,i\,\omega_0\big)\,\omega_0\big)\,\big(m^2+\omega_0^2\big)^2\,\big(h^2\,m+2\,\big(n+2\,i\,\omega_0\big)\,\omega_0\,\big(i\,(b+m)+2\,\omega_0\big)\big)\Big)\Big\}$$

h225 = FullSimplify[-AI.(-4 h11 l1), ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
simplifica completamente — números reais

$$
\begin{aligned}
&\{ (32\ (-b-m)\ m^4\ (3\ n^2+\omega_0^2) \\
&\quad (6\ h^4\ m^2\ n^3\ (h^2\ m-(b+m)\ n^2)+\omega_0^2\ (-2\ h^6\ m^3\ n+6\ h^2\ m\ (b+m)\ (9\ (b+m)-8\ n)\ n^5- \\
&\qquad 15\ (b+m)^3\ n^7+h^4\ m^2\ n^3\ (-41\ (b+m)+34\ n)- \\
&\qquad n\ (-5\ h^4\ m^2\ (b+m)-20\ h^4\ m^2\ n-128\ h^2\ m\ (b+m)^2\ n^2+48\ h^2\ m\ (b+m)\ n^3+ \\
&\qquad\quad (85\ b^3+255\ b^2\ m+16\ h^2\ m+255\ b\ m^2+85\ m^3)\ n^4+20\ (b+m)^2\ n^5+60\ (b+m)\ n^6) \\
&\qquad \omega_0^2+(2\ h^4\ m^2+2\ h^2\ m\ n\ (13\ (b+m)^2+12\ (b+m)\ n+48\ n^2)+ \\
&\qquad\quad n^3\ (-101\ (b+m)^3-116\ (b+m)^2\ n-340\ (b+m)\ n^2-80\ n^3))\ \omega_0^4- \\
&\qquad (-24\ h^2\ m\ (b+m)+(31\ b^3+93\ b^2\ m+80\ h^2\ m+93\ b\ m^2+31\ m^3)\ n+ \\
&\qquad\quad 124\ (b+m)^2\ n^2+404\ (b+m)\ n^3+464\ n^4)\ \omega_0^6- \\
&\qquad 4\ (7\ (b+m)^2+31\ (b+m)\ n+124\ n^2)\ \omega_0^8-112\ \omega_0^{10})))\ / \\
&\ (h^2\ (m^2+\omega_0^2)^2\ (n^2+\omega_0^2)^2\ ((h^2\ m-(b+m)\ n^2)^2+2\ (h^2\ m\ (b+m)-4\ h^2\ m\ n+(b+m)^2\ n^2+2\ n^4) \\
&\qquad \omega_0^2+((b+m)^2+8\ n^2)\ \omega_0^4+4\ \omega_0^6) \\
&\quad (h^4\ m^2+4\ \omega_0^2\ (-2\ h^2\ m\ (b+m)+2\ h^2\ m\ n+(b+m)^2\ n^2+4\ \omega_0^2\ ((b+m)^2+n^2+4\ \omega_0^2))))),\ (16 \\
&\quad \sqrt{2} \\
&\quad m^4 \\
&\quad n \\
&\quad (-2\ h^2\ m+3\ (b+m)\ n^2+(b+m)\ \omega_0^2) \\
&\quad (6\ h^4\ m^2\ n^3\ (-h^2\ m+(b+m)\ n^2)+\omega_0^2 \\
&\qquad (2\ h^6\ m^3\ n+h^4\ m^2\ (41\ (b+m)-34\ n)\ n^3-6\ h^2\ m\ (b+m)\ (9\ (b+m)-8\ n)\ n^5+15\ (b+m)^3 \\
&\qquad n^7+n\ (-5\ h^4\ m^2\ (b+m)-20\ h^4\ m^2\ n-128\ h^2\ m\ (b+m)^2\ n^2+48\ h^2\ m\ (b+m)\ n^3+ \\
&\qquad\quad (85\ b^3+255\ b^2\ m+16\ h^2\ m+255\ b\ m^2+85\ m^3)\ n^4+20\ (b+m)^2\ n^5+60\ (b+m)\ n^6) \\
&\qquad \omega_0^2+(-2\ h^4\ m^2+2\ h^2\ m\ n\ (-13\ (b+m)^2-12\ (b+m)\ n-48\ n^2)+ \\
&\qquad\quad n^3\ (101\ (b+m)^3+116\ (b+m)^2\ n+340\ (b+m)\ n^2+80\ n^3))\ \omega_0^4+ \\
&\qquad (-24\ h^2\ m\ (b+m)+(31\ b^3+93\ b^2\ m+80\ h^2\ m+93\ b\ m^2+31\ m^3)\ n+ \\
&\qquad\quad 124\ (b+m)^2\ n^2+404\ (b+m)\ n^3+464\ n^4)\ \omega_0^6+ \\
&\qquad 4\ (7\ (b+m)^2+31\ (b+m)\ n+124\ n^2)\ \omega_0^8+112\ \omega_0^{10})))\ / \\
&\ (h^3\ (m^2+\omega_0^2)^2\ (n^2+\omega_0^2)^2\ ((h^2\ m-(b+m)\ n^2)^2+2\ (h^2\ m\ (b+m)-4\ h^2\ m\ n+(b+m)^2\ n^2+2\ n^4) \\
&\qquad \omega_0^2+((b+m)^2+8\ n^2)\ \omega_0^4+4\ \omega_0^6) \\
&\quad (h^4\ m^2+4\ \omega_0^2\ (-2\ h^2\ m\ (b+m)+2\ h^2\ m\ n+(b+m)^2\ n^2+4\ \omega_0^2\ ((b+m)^2+n^2+4\ \omega_0^2))))), \\
&-(16\ \sqrt{2}\ m^4\ (n\ (h^2\ (2\ m-3\ n)-3\ (b+m)\ n^2)-(h^2+(b+m)\ n)\ \omega_0^2) \\
&\quad (6\ h^4\ m^2\ n^3\ (-h^2\ m+(b+m)\ n^2)+\omega_0^2\ (2\ h^6\ m^3\ n+h^4\ m^2\ (41\ (b+m)-34\ n)\ n^3- \\
&\qquad 6\ h^2\ m\ (b+m)\ (9\ (b+m)-8\ n)\ n^5+15\ (b+m)^3\ n^7+ \\
&\qquad n\ (-5\ h^4\ m^2\ (b+m)-20\ h^4\ m^2\ n-128\ h^2\ m\ (b+m)^2\ n^2+48\ h^2\ m\ (b+m)\ n^3+ \\
&\qquad\quad (85\ b^3+255\ b^2\ m+16\ h^2\ m+255\ b\ m^2+85\ m^3)\ n^4+20\ (b+m)^2\ n^5+60\ (b+m)\ n^6) \\
&\qquad \omega_0^2+(-2\ h^4\ m^2+2\ h^2\ m\ n\ (-13\ (b+m)^2-12\ (b+m)\ n-48\ n^2)+ \\
&\qquad\quad n^3\ (101\ (b+m)^3+116\ (b+m)^2\ n+340\ (b+m)\ n^2+80\ n^3))\ \omega_0^4+ \\
&\qquad (-24\ h^2\ m\ (b+m)+(31\ b^3+93\ b^2\ m+80\ h^2\ m+93\ b\ m^2+31\ m^3)\ n+ \\
&\qquad\quad 124\ (b+m)^2\ n^2+404\ (b+m)\ n^3+464\ n^4)\ \omega_0^6+ \\
&\qquad 4\ (7\ (b+m)^2+31\ (b+m)\ n+124\ n^2)\ \omega_0^8+112\ \omega_0^{10})))\ / \\
&\ (h^3\ (m^2+\omega_0^2)^2\ (n^2+\omega_0^2)^2\ ((h^2\ m-(b+m)\ n^2)^2+2\ (h^2\ m\ (b+m)-4\ h^2\ m\ n+(b+m)^2\ n^2+2\ n^4) \\
&\qquad \omega_0^2+((b+m)^2+8\ n^2)\ \omega_0^4+4\ \omega_0^6) \\
&\quad (h^4\ m^2+4\ \omega_0^2\ (-2\ h^2\ m\ (b+m)+2\ h^2\ m\ n+(b+m)^2\ n^2+4\ \omega_0^2\ ((b+m)^2+n^2+4\ \omega_0^2)))))\}
\end{aligned}
$$

```
h22 = Simplify[h221 + h222 + h223 + h224 + h225,
        simplifica
  ω0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
       números reais
```

$$\Big\{ \frac{1}{(m^2+\omega_0^2)^2}$$

$$8\,m^4 \left(-\frac{2\,n\,(3\,n^2+\omega_0^2)}{h^2\,(n^2+\omega_0^2)^2} + \Big(-3\,\mathrm{i}\,h^2\,m\,n^2\,(5\,h^4\,m^2+h^2\,m\,(8\,b+8\,m-13\,n)\,n^2+3\,(b+m)^2\,n^4) -\right.$$

$$(12\,h^6\,m^3\,n+h^4\,m^2\,(50\,b+50\,m-161\,n)\,n^3+$$

$$25\,h^2\,m\,(b+m)\,(2\,b+2\,m-3\,n)\,n^5+12\,(b+m)^3\,n^7)\,\omega_0+$$

$$\mathrm{i}\,(h^6\,m^3+2\,h^4\,m^2\,(19\,b+19\,m-103\,n)\,n^2+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^6+$$

$$h^2\,m\,n^4\,(91\,b^2+182\,b\,m+91\,m^2-404\,b\,n-404\,m\,n+123\,n^2))\,\omega_0^2+$$

$$n\,(44\,b^3\,n^4+4\,b^2\,(5\,h^2\,m\,n^2+33\,m\,n^4-60\,n^5)+b\,(-10\,h^4\,m^2+h^2\,m\,(40\,m-719\,n)\,n^2+$$

$$12\,n^4\,(11\,m^2-40\,m\,n+9\,n^2))+m\,(-2\,h^4\,m\,(5\,m+49\,n)+$$

$$4\,n^4\,(11\,m^2-60\,m\,n+27\,n^2)+h^2\,n^2\,(20\,m^2-719\,m\,n+628\,n^2)))\,\omega_0^3+$$

$$\mathrm{i}\,(h^4\,m^2\,(14\,m-37\,n)+40\,b^3\,n^4+3\,h^2\,m\,n^2\,(19\,m^2+104\,m\,n-357\,n^2)+$$

$$4\,n^4\,(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3)+b^2\,(57\,h^2\,m\,n^2+20\,n^4\,(6\,m+11\,n))+$$

$$2\,b\,(7\,h^4\,m^2+3\,h^2\,m\,n^2\,(19\,m+52\,n)+4\,n^4\,(15\,m^2+55\,m\,n-54\,n^2)))\,\omega_0^4+$$

$$(-35\,h^4\,m^2+h^2\,m\,n\,(22\,b^2+44\,b\,m+22\,m^2-223\,b\,n-223\,m\,n-512\,n^2)+$$

$$4\,n^3\,(19\,b^3+19\,m^3+b^2\,(57\,m-50\,n)-50\,m^2\,n-$$

$$99\,m\,n^2+72\,n^3+b\,(57\,m^2-100\,m\,n-99\,n^2)))\,\omega_0^5-$$

$$\mathrm{i}\,(16\,b^3\,n^2+h^2\,m\,(-5\,m^2-140\,m\,n+231\,n^2)+b^2\,(-5\,h^2\,m+4\,(12\,m-95\,n)\,n^2)+$$

$$4\,n^2\,(4\,m^3-95\,m^2\,n+90\,m\,n^2+66\,n^3)-$$

$$2\,b\,(5\,h^2\,m\,(m+14\,n)-4\,n^2\,(6\,m^2-95\,m\,n+45\,n^2)))\,\omega_0^6+$$

$$(20\,b^3\,n+20\,b^2\,n\,(3\,m+4\,n)-5\,h^2\,m\,(m+36\,n)+$$

$$4\,n\,(5\,m^3+20\,m^2\,n-171\,m\,n^2+60\,n^3)+b\,(-5\,h^2\,m+4\,n\,(15\,m^2+40\,m\,n-171\,n^2)))$$

$$\omega_0^7-\mathrm{i}\,(8\,b^3-3\,h^2\,m+8\,m^3+4\,b^2\,(6\,m-25\,n)-100\,m^2\,n-144\,m\,n^2+$$

$$456\,n^3+8\,b\,(3\,m^2-25\,m\,n-18\,n^2))\,\omega_0^8+$$

$$4\,(10\,b^2+10\,m^2+5\,b\,(4\,m-9\,n)-45\,m\,n-24\,n^2)\,\omega_0^9+$$

$$24\,\mathrm{i}\,(3\,b+3\,m-5\,n)\,\omega_0^{10}-48\,\omega_0^{11}\Big)\Big/$$

$$\Big(h^2\,(n^2+\omega_0^2)^2\,(h^2\,m-2\,\mathrm{i}\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2-8\,\mathrm{i}\,\omega_0^3)$$

$$(-\mathrm{i}\,n\,(b^2\,n^2+2\,b\,m\,(-h^2+n^2)+m\,(m\,n^2+h^2\,(-2\,m+n)))-$$

$$3\,(b^2\,n^2-b\,(h^2\,m+n^2\,(-2\,m+n))-m\,(h^2\,(m-2\,n)+n^2\,(-m+n)))\,\omega_0+$$

$$3\,\mathrm{i}\,(2\,h^2\,m+n\,(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2))\,\omega_0^2+$$

$$(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2)\,\omega_0^3+3\,\mathrm{i}\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5)\Big)+$$

$$\Big(3\,h^2\,m\,n^2\,(5\,h^4\,m^2+h^2\,m\,(8\,b+8\,m-13\,n)\,n^2+3\,(b+m)^2\,n^4)+$$

$$\mathrm{i}\,(12\,h^6\,m^3\,n+h^4\,m^2\,(50\,b+50\,m-161\,n)\,n^3+$$

$$25\,h^2\,m\,(b+m)\,(2\,b+2\,m-3\,n)\,n^5+12\,(b+m)^3\,n^7)\,\omega_0-$$

$$(h^6\,m^3+2\,h^4\,m^2\,(19\,b+19\,m-103\,n)\,n^2+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^6+$$

$$h^2\,m\,n^4\,(91\,b^2+182\,b\,m+91\,m^2-404\,b\,n-404\,m\,n+123\,n^2))\,\omega_0^2-$$

$$\mathrm{i}\,n\,(44\,b^3\,n^4+4\,b^2\,(5\,h^2\,m\,n^2+33\,m\,n^4-60\,n^5)+b\,(-10\,h^4\,m^2+$$

$$h^2\,m\,(40\,m-719\,n)\,n^2+12\,n^4\,(11\,m^2-40\,m\,n+9\,n^2))+m\,(-2\,h^4\,m\,(5\,m+49\,n)+$$

$$4\,n^4\,(11\,m^2-60\,m\,n+27\,n^2)+h^2\,n^2\,(20\,m^2-719\,m\,n+628\,n^2)))\,\omega_0^3-$$

$$(h^4\,m^2\,(14\,m-37\,n)+40\,b^3\,n^4+3\,h^2\,m\,n^2\,(19\,m^2+104\,m\,n-357\,n^2)+$$

$$4\,n^4\,(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3)+b^2\,(57\,h^2\,m\,n^2+20\,n^4\,(6\,m+11\,n))+$$

$$2\,b\,(7\,h^4\,m^2+3\,h^2\,m\,n^2\,(19\,m+52\,n)+4\,n^4\,(15\,m^2+55\,m\,n-54\,n^2)))\,\omega_0^4+$$

$$\mathrm{i}\,(35\,h^4\,m^2+h^2\,m\,n\,(-22\,b^2-44\,b\,m-22\,m^2+223\,b\,n+223\,m\,n+512\,n^2)-$$

$$4\,n^3\,(19\,b^3+19\,m^3+b^2\,(57\,m-50\,n)-50\,m^2\,n-$$

$$99\,m\,n^2+72\,n^3+b\,(57\,m^2-100\,m\,n-99\,n^2)))\,\omega_0^5+$$

$$(16\,b^3\,n^2+h^2\,m\,(-5\,m^2-140\,m\,n+231\,n^2)+b^2\,(-5\,h^2\,m+4\,(12\,m-95\,n)\,n^2)+$$

$$4\,n^2\,(4\,m^3-95\,m^2\,n+90\,m\,n^2+66\,n^3)-$$

$$2\,b\,(5\,h^2\,m\,(m+14\,n)-4\,n^2\,(6\,m^2-95\,m\,n+45\,n^2)))\,\omega_0^6-$$

$$\mathrm{i}\,(20\,b^3\,n+20\,b^2\,n\,(3\,m+4\,n)-5\,h^2\,m\,(m+36\,n)+$$

$$4\,n\,(5\,m^3+20\,m^2\,n-171\,m\,n^2+60\,n^3)+b\,(-5\,h^2\,m+4\,n\,(15\,m^2+40\,m\,n-171\,n^2)))$$

$$\omega_0^7+(8\,b^3-3\,h^2\,m+8\,m^3+4\,b^2\,(6\,m-25\,n)-100\,m^2\,n-144\,m\,n^2+$$

$$456\,n^3+8\,b\,(3\,m^2-25\,m\,n-18\,n^2))\,\omega_0^8-$$

$$4\,\mathrm{i}\,(10\,b^2+10\,m^2+5\,b\,(4\,m-9\,n)-45\,m\,n-24\,n^2)\,\omega_0^9-$$

$$24\,(3\,b+3\,m-5\,n)\,\omega_0^{10}+48\,\mathrm{i}\,\omega_0^{11})\Big/$$

$$(h^2\,(n^2+\omega_0^2)^2\,(h^2\,m+2\,\mathrm{i}\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2+8\,\mathrm{i}\,\omega_0^3)$$

$$(n\,(b^2\,n^2+2\,b\,m\,(-h^2+n^2)+m\,(m\,n^2+h^2\,(-2\,m+n)))+$$

$$3\,\mathrm{i}\,(b^2\,n^2-b\,(h^2\,m+n^2\,(-2\,m+n))-m\,(h^2\,(m-2\,n)+n^2\,(-m+n)))\,\omega_0-$$

$$3\,(2\,h^2\,m+n\,(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2))\,\omega_0^2-$$

$$\mathrm{i}\,(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2)\,\omega_0^3-3\,(b+m-3\,n)\,\omega_0^4+3\,\mathrm{i}\,\omega_0^5))+$$

$$(2\,m\,(-3\,h^2\,m\,n+2\,n\,(b+m+3\,n)\,\omega_0^2+10\,\omega_0^4))\Big/$$

$$((n-\mathrm{i}\,\omega_0)\,(n+\mathrm{i}\,\omega_0)\,(h^2\,m-2\,\mathrm{i}\,(n-2\,\mathrm{i}\,\omega_0)\,(b+m+2\,\mathrm{i}\,\omega_0)\,\omega_0)$$

$$(h^2\,m+2\,(n+2\,\mathrm{i}\,\omega_0)\,\omega_0\,(\mathrm{i}\,(b+m)+2\,\omega_0)))+$$

$$(4\,(-b-m)\,(3\,n^2+\omega_0^2)\,(6\,h^4\,m^2\,n^3\,(h^2\,m-(b+m)\,n^2)+\omega_0^2\,(-2\,h^6\,m^3\,n+6\,h^2\,m$$

$$(b+m)\,(9\,(b+m)-8\,n)\,n^5-15\,(b+m)^3\,n^7+h^4\,m^2\,n^3\,(-41\,(b+m)+34\,n)-$$

$$n\,(-5\,h^4\,m^2\,(b+m)-20\,h^4\,m^2\,n-128\,h^2\,m\,(b+m)^2\,n^2+48\,h^2\,m\,(b+m)\,n^3+$$

$$(85\,b^3+255\,b^2\,m+16\,h^2\,m+255\,b\,m^2+85\,m^3)\,n^4+20\,(b+m)^2\,n^5+$$

$$60\,(b+m)\,n^6)\,\omega_0^2+(2\,h^4\,m^2+2\,h^2\,m\,n\,(13\,(b+m)^2+12\,(b+m)\,n+48\,n^2)+$$

$$n^3\,(-101\,(b+m)^3-116\,(b+m)^2\,n-340\,(b+m)\,n^2-80\,n^3))\,\omega_0^4-$$

$$(-24\,h^2\,m\,(b+m)+(31\,b^3+93\,b^2\,m+80\,h^2\,m+93\,b\,m^2+31\,m^3)\,n+$$

$$124\,(b+m)^2\,n^2+404\,(b+m)\,n^3+464\,n^4)\,\omega_0^6-$$

$$4\,(7\,(b+m)^2+31\,(b+m)\,n+124\,n^2)\,\omega_0^8-112\,\omega_0^{10})))\Big/$$

$$(h^2\,(n^2+\omega_0^2)^2\,((h^2\,m-(b+m)\,n^2)^2+2\,(h^2\,m\,(b+m)-4\,h^2\,m\,n+(b+m)^2\,n^2+2\,n^4)\,\omega_0^2+$$

$$((b+m)^2+8\,n^2)\,\omega_0^4+4\,\omega_0^6)$$

$$(h^4\,m^2+4\,\omega_0^2\,(-2\,h^2\,m\,(b+m)+2\,h^2\,m\,n+(b+m)^2\,n^2+4\,\omega_0^2\,((b+m)^2+n^2+4\,\omega_0^2))))\Big),$$

$$\frac{1}{h^3\,(m^2+\omega_0^2)^2}\,2\,\sqrt{2}\,m^4\,\Bigg(-\frac{(3\,n^2+\omega_0^2)\,(7\,n^2+\omega_0^2)}{(n^2+\omega_0^2)^2}+$$

$$(2\,(-3\,h^2\,m\,n^3\,(10\,h^4\,m^2+h^2\,m\,(11\,b+11\,m-21\,n)\,n^2+3\,(b+m)^2\,n^4)+$$

$$\mathrm{i}\,(9\,h^6\,m^3\,n^2+h^4\,m^2\,(79\,b+79\,m-241\,n)\,n^4+$$

$$h^2\,m\,(b+m)\,(62\,b+62\,m-105\,n)\,n^6+12\,(b+m)^3\,n^8)\,\omega_0+$$

$$(-10\,h^6\,m^3\,n+26\,h^4\,m^2\,(2\,b+2\,m-11\,n)\,n^3+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^7+$$

$$h^2\,m\,n^5\,(127\,b^2+254\,b\,m+127\,m^2-582\,b\,n-582\,m\,n+183\,n^2))\,\omega_0^2+$$

$$\mathrm{i}\,(h^6\,m^3+6\,h^4\,m^2\,n^2\,(10\,b+10\,m+11\,n)-4\,(b+m)\,n^6$$

$$(11\,b^2+22\,b\,m+11\,m^2-60\,b\,n-60\,m\,n+27\,n^2)-$$

$$h^2\,m\,n^4\,(28\,b^2+56\,b\,m+28\,m^2-1013\,b\,n-1013\,m\,n+936\,n^2))\,\omega_0^3+$$

$$n \left(h^4 m^2 (29 m - 189 n) + 40 b^3 n^4 + 5 h^2 m n^2 \left(21 m^2 + 58 m n - 303 n^2\right) +\right.$$
$$4 n^4 \left(10 m^3 + 55 m^2 n - 108 m n^2 + 18 n^3\right) + 5 b^2 \left(21 h^2 m n^2 + 4 n^4 (6 m + 11 n)\right) +$$
$$\left.b \left(29 h^4 m^2 + 10 h^2 m n^2 (21 m + 29 n) + 8 n^4 \left(15 m^2 + 55 m n - 54 n^2\right)\right)\right) \omega_0^4 -$$
$$i \left(76 b^3 n^4 - h^4 m^2 (13 m + 67 n) + h^2 m n^2 \left(50 m^2 - 613 m n - 436 n^2\right) +\right.$$
$$4 n^4 \left(19 m^3 - 50 m^2 n - 99 m n^2 + 72 n^3\right) + b^2 \left(50 h^2 m n^2 + 4 (57 m - 50 n) n^4\right) -$$
$$\left.b \left(13 h^4 m^2 + h^2 m n^2 (-100 m + 613 n) + 4 n^4 \left(-57 m^2 + 100 m n + 99 n^2\right)\right)\right) \omega_0^5 +$$
$$\left(-32 h^4 m^2 + h^2 m n \left(17 b^2 + 34 b m + 17 m^2 + 342 b n + 342 m n - 819 n^2\right) -\right.$$
$$4 n^3 \left(4 b^3 + 4 m^3 + b^2 (12 m - 95 n) - 95 m^2 n + 90 m n^2 + 66 n^3 +\right.$$
$$\left.\left.2 b \left(6 m^2 - 95 m n + 45 n^2\right)\right)\right) \omega_0^6 - i \left(20 b^3 n^2 + h^2 m \left(8 m^2 - 71 m n - 472 n^2\right) +\right.$$
$$4 n^2 \left(5 m^3 + 20 m^2 n - 171 m n^2 + 60 n^3\right) + b^2 \left(8 h^2 m + 20 n^2 (3 m + 4 n)\right) +$$
$$\left.b \left(h^2 m (16 m - 71 n) + 4 n^2 \left(15 m^2 + 40 m n - 171 n^2\right)\right)\right) \omega_0^7 +$$
$$\left(h^2 m (46 m - 81 n) - 8 b^3 n - 4 b^2 (6 m - 25 n) n + 4 n \left(-2 m^3 + 25 m^2 n +\right.\right.$$
$$\left.\left.36 m n^2 - 114 n^3\right) + 2 b \left(23 h^2 m + 4 n \left(-3 m^2 + 25 m n + 18 n^2\right)\right)\right) \omega_0^8 +$$
$$4 i \left(15 h^2 m + n \left(-10 b^2 - 20 b m - 10 m^2 + 45 b n + 45 m n + 24 n^2\right)\right) \omega_0^9 +$$
$$\left.\left.24 (3 b + 3 m - 5 n) n \omega_0^{10} + 48 i n \omega_0^{11}\right)\right) \Big/$$
$$\left(\left(n^2 + \omega_0^2\right)^2 \left(h^2 m - 2 i (b + m) n \omega_0 - 4 (b + m - n) \omega_0^2 - 8 i \omega_0^3\right)\right.$$
$$\left(-n \left(b^2 n^2 + 2 b m \left(-h^2 + n^2\right) + m \left(m n^2 + h^2 (-2 m + n)\right)\right) +\right.$$
$$3 i \left(b^2 n^2 - b \left(h^2 m + n^2 (-2 m + n)\right) - m \left(h^2 (m - 2 n) + n^2 (-m + n)\right)\right) \omega_0 +$$
$$3 \left(2 h^2 m + n \left(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2\right)\right) \omega_0^2 -$$
$$\left.\left.i \left(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2\right) \omega_0^3 + 3 (b + m - 3 n) \omega_0^4 + 3 i \omega_0^5\right)\right) -$$
$$\left(2 \left(-3 i h^2 m n^3 \left(10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b + m)^2 n^4\right) +\right.\right.$$
$$\left(9 h^6 m^3 n^2 + h^4 m^2 (79 b + 79 m - 241 n) n^4 +\right.$$
$$\left.h^2 m (b + m) (62 b + 62 m - 105 n) n^6 + 12 (b + m)^3 n^8\right) \omega_0 -$$
$$i \left(10 h^6 m^3 n - 26 h^4 m^2 (2 b + 2 m - 11 n) n^3 - 12 (b + m)^2 (4 b + 4 m - 5 n) n^7 -\right.$$
$$\left.h^2 m n^5 \left(127 b^2 + 254 b m + 127 m^2 - 582 b n - 582 m n + 183 n^2\right)\right) \omega_0^2 +$$
$$\left(h^6 m^3 + 6 h^4 m^2 n^2 (10 b + 10 m + 11 n) - 4 (b + m) n^6\right.$$
$$\left(11 b^2 + 22 b m + 11 m^2 - 60 b n - 60 m n + 27 n^2\right) -$$
$$\left.h^2 m n^4 \left(28 b^2 + 56 b m + 28 m^2 - 1013 b n - 1013 m n + 936 n^2\right)\right) \omega_0^3 +$$
$$i n \left(h^4 m^2 (29 m - 189 n) + 40 b^3 n^4 + 5 h^2 m n^2 \left(21 m^2 + 58 m n - 303 n^2\right) +\right.$$
$$4 n^4 \left(10 m^3 + 55 m^2 n - 108 m n^2 + 18 n^3\right) + 5 b^2 \left(21 h^2 m n^2 + 4 n^4 (6 m + 11 n)\right) +$$
$$\left.b \left(29 h^4 m^2 + 10 h^2 m n^2 (21 m + 29 n) + 8 n^4 \left(15 m^2 + 55 m n - 54 n^2\right)\right)\right) \omega_0^4 +$$
$$\left(-76 b^3 n^4 + h^4 m^2 (13 m + 67 n) + h^2 m n^2 \left(-50 m^2 + 613 m n + 436 n^2\right) +\right.$$
$$4 n^4 \left(-19 m^3 + 50 m^2 n + 99 m n^2 - 72 n^3\right) + b^2 \left(-50 h^2 m n^2 + 4 n^4 (-57 m + 50 n)\right) +$$
$$\left.b \left(13 h^4 m^2 + h^2 m n^2 (-100 m + 613 n) + 4 n^4 \left(-57 m^2 + 100 m n + 99 n^2\right)\right)\right) \omega_0^5 -$$
$$i \left(32 h^4 m^2 - h^2 m n \left(17 b^2 + 34 b m + 17 m^2 + 342 b n + 342 m n - 819 n^2\right) +\right.$$
$$4 n^3 \left(4 b^3 + 4 m^3 + b^2 (12 m - 95 n) - 95 m^2 n + 90 m n^2 + 66 n^3 +\right.$$
$$\left.\left.2 b \left(6 m^2 - 95 m n + 45 n^2\right)\right)\right) \omega_0^6 - \left(20 b^3 n^2 + h^2 m \left(8 m^2 - 71 m n - 472 n^2\right) +\right.$$
$$4 n^2 \left(5 m^3 + 20 m^2 n - 171 m n^2 + 60 n^3\right) + b^2 \left(8 h^2 m + 20 n^2 (3 m + 4 n)\right) +$$
$$\left.b \left(h^2 m (16 m - 71 n) + 4 n^2 \left(15 m^2 + 40 m n - 171 n^2\right)\right)\right) \omega_0^7 -$$
$$i \left(8 b^3 n + 4 b^2 (6 m - 25 n) n + h^2 m (-46 m + 81 n) + 4 n \left(2 m^3 - 25 m^2 n -\right.\right.$$
$$\left.\left.36 m n^2 + 114 n^3\right) - 2 b \left(23 h^2 m + 4 n \left(-3 m^2 + 25 m n + 18 n^2\right)\right)\right) \omega_0^8 +$$
$$4 \left(15 h^2 m + n \left(-10 b^2 - 20 b m - 10 m^2 + 45 b n + 45 m n + 24 n^2\right)\right) \omega_0^9 +$$
$$\left.\left.24 i (3 b + 3 m - 5 n) n \omega_0^{10} + 48 n \omega_0^{11}\right)\right) \Big/$$
$$\left(\left(n^2 + \omega_0^2\right)^2 \left(h^2 m + 2 i (b + m) n \omega_0 - 4 (b + m - n) \omega_0^2 + 8 i \omega_0^3\right)\right.$$
$$\left(i n \left(b^2 n^2 + 2 b m \left(-h^2 + n^2\right) + m \left(m n^2 + h^2 (-2 m + n)\right)\right) -\right.$$

$$3\,(b^2 n^2 - b\,(h^2 m + n^2 (-2 m + n)) - m\,(h^2 (m - 2 n) + n^2 (-m + n)))\,\omega_0 -$$
$$3\,\mathrm{i}\,(2 h^2 m + n\,(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2))\,\omega_0^2 +$$
$$(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2)\,\omega_0^3 - 3\,\mathrm{i}\,(b + m - 3 n)\,\omega_0^4 - 3\,\omega_0^5)) +$$
$$(h^2 m\,(-21 h^2 m n^2 + (-25 h^2 m + 8 n^2 (b + m + 3 n))\,\omega_0^2 + 40 n\,\omega_0^4)) \Big/$$
$$((n - \mathrm{i}\,\omega_0)\,(n + \mathrm{i}\,\omega_0)\,(h^2 m - 2\,\mathrm{i}\,(n - 2\,\mathrm{i}\,\omega_0)\,(b + m + 2\,\mathrm{i}\,\omega_0)\,\omega_0)$$
$$(h^2 m + 2\,(n + 2\,\mathrm{i}\,\omega_0)\,\omega_0\,(\mathrm{i}\,(b + m) + 2\,\omega_0)))) +$$
$$(8 n\,(-2 h^2 m + 3 (b + m) n^2 + (b + m)\,\omega_0^2)\,(6 h^4 m^2 n^3 (-h^2 m + (b + m) n^2) +$$
$$\omega_0^2\,(2 h^6 m^3 n + h^4 m^2 (41 (b + m) - 34 n) n^3 - 6 h^2 m (b + m) (9 (b + m) - 8 n) n^5 +$$
$$15 (b + m)^3 n^7 + n\,(-5 h^4 m^2 (b + m) - 20 h^4 m^2 n - 128 h^2 m (b + m)^2 n^2 + 48 h^2 m$$
$$(b + m) n^3 + (85 b^3 + 255 b^2 m + 16 h^2 m + 255 b m^2 + 85 m^3) n^4 + 20 (b + m)^2$$
$$n^5 + 60 (b + m) n^6)\,\omega_0^2 + (-2 h^4 m^2 + 2 h^2 m n\,(-13 (b + m)^2 - 12 (b + m) n -$$
$$48 n^2) + n^3\,(101 (b + m)^3 + 116 (b + m)^2 n + 340 (b + m) n^2 + 80 n^3))\,\omega_0^4 +$$
$$(-24 h^2 m (b + m) + (31 b^3 + 93 b^2 m + 80 h^2 m + 93 b m^2 + 31 m^3) n +$$
$$124 (b + m)^2 n^2 + 404 (b + m) n^3 + 464 n^4)\,\omega_0^6 +$$
$$4\,(7 (b + m)^2 + 31 (b + m) n + 124 n^2)\,\omega_0^8 + 112\,\omega_0^{10}))) \Big/$$
$$((n^2 + \omega_0^2)^2\,((h^2 m - (b + m) n^2)^2 + 2\,(h^2 m (b + m) - 4 h^2 m n + (b + m)^2 n^2 + 2 n^4)\,\omega_0^2 +$$
$$((b + m)^2 + 8 n^2)\,\omega_0^4 + 4\,\omega_0^6)$$
$$(h^4 m^2 + 4\,\omega_0^2\,(-2 h^2 m (b + m) + 2 h^2 m n + (b + m)^2 n^2 + 4\,\omega_0^2\,((b + m)^2 + n^2 + 4\,\omega_0^2))))),$$

$$\frac{1}{h^3\,(m^2 + \omega_0^2)^2}\,2\sqrt{2}\,m^4\,\Big(-\frac{(3 n^2 + \omega_0^2)\,(7 n^2 + \omega_0^2)}{(n^2 + \omega_0^2)^2} +$$
$$(2\,(-3 h^2 m n^3\,(10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b + m)^2 n^4) +$$
$$\mathrm{i}\,(9 h^6 m^3 n^2 + h^4 m^2 (79 b + 79 m - 241 n) n^4 +$$
$$h^2 m (b + m) (62 b + 62 m - 105 n) n^6 + 12 (b + m)^3 n^8)\,\omega_0 +$$
$$(-10 h^6 m^3 n + 26 h^4 m^2 (2 b + 2 m - 11 n) n^3 + 12 (b + m)^2 (4 b + 4 m - 5 n) n^7 +$$
$$h^2 m n^5\,(127 b^2 + 254 b m + 127 m^2 - 582 b n - 582 m n + 183 n^2))\,\omega_0^2 +$$
$$\mathrm{i}\,(h^6 m^3 + 6 h^4 m^2 n^2 (10 b + 10 m + 11 n) - 4 (b + m) n^6$$
$$(11 b^2 + 22 b m + 11 m^2 - 60 b n - 60 m n + 27 n^2) -$$
$$h^2 m n^4\,(28 b^2 + 56 b m + 28 m^2 - 1013 b n - 1013 m n + 936 n^2))\,\omega_0^3 +$$
$$n\,(h^4 m^2 (29 m - 189 n) + 40 b^3 n^4 + 5 h^2 m n^2\,(21 m^2 + 58 m n - 303 n^2) +$$
$$4 n^4\,(10 m^3 + 55 m^2 n - 108 m n^2 + 18 n^3) + 5 b^2\,(21 h^2 m n^2 + 4 n^4 (6 m + 11 n)) +$$
$$b\,(29 h^4 m^2 + 10 h^2 m n^2 (21 m + 29 n) + 8 n^4\,(15 m^2 + 55 m n - 54 n^2)))\,\omega_0^4 -$$
$$\mathrm{i}\,(76 b^3 n^4 - h^4 m^2 (13 m + 67 n) + h^2 m n^2\,(50 m^2 - 613 m n - 436 n^2) +$$
$$4 n^4\,(19 m^3 - 50 m^2 n - 99 m n^2 + 72 n^3) + b^2\,(50 h^2 m n^2 + 4 (57 m - 50 n) n^4) -$$
$$b\,(13 h^4 m^2 + h^2 m n^2 (-100 m + 613 n) + 4 n^4\,(-57 m^2 + 100 m n + 99 n^2)))\,\omega_0^5 +$$
$$(-32 h^4 m^2 + h^2 m n\,(17 b^2 + 34 b m + 17 m^2 + 342 b n + 342 m n - 819 n^2) -$$
$$4 n^3\,(4 b^3 + 4 m^3 + b^2 (12 m - 95 n) - 95 m^2 n + 90 m n^2 + 66 n^3 +$$
$$2 b\,(6 m^2 - 95 m n + 45 n^2)))\,\omega_0^6 - \mathrm{i}\,(20 b^3 n^2 + h^2 m\,(8 m^2 - 71 m n - 472 n^2) +$$
$$4 n^2\,(5 m^3 + 20 m^2 n - 171 m n^2 + 60 n^3) + b^2\,(8 h^2 m + 20 n^2 (3 m + 4 n)) +$$
$$b\,(h^2 m (16 m - 71 n) + 4 n^2\,(15 m^2 + 40 m n - 171 n^2)))\,\omega_0^7 +$$
$$(h^2 m (46 m - 81 n) - 8 b^3 n - 4 b^2 (6 m - 25 n) n + 4 n\,(-2 m^3 + 25 m^2 n +$$
$$36 m n^2 - 114 n^3) + 2 b\,(23 h^2 m + 4 n\,(-3 m^2 + 25 m n + 18 n^2)))\,\omega_0^8 +$$
$$4\,\mathrm{i}\,(15 h^2 m + n\,(-10 b^2 - 20 b m - 10 m^2 + 45 b n + 45 m n + 24 n^2))\,\omega_0^9 +$$
$$24 (3 b + 3 m - 5 n)\,n\,\omega_0^{10} + 48\,\mathrm{i}\,n\,\omega_0^{11})) \Big/$$

$$\left(\left(n^2+\omega_0^2\right)^2 \left(h^2 m-2 i (b+m) n \omega_0-4 (b+m-n) \omega_0^2-8 i \omega_0^3\right)\right.$$
$$\left(-n \left(b^2 n^2+2 b m \left(-h^2+n^2\right)+m \left(m n^2+h^2 (-2 m+n)\right)\right)+\right.$$
$$3 i \left(b^2 n^2-b \left(h^2 m+n^2 (-2 m+n)\right)-m \left(h^2 (m-2 n)+n^2 (-m+n)\right)\right) \omega_0+$$
$$3 \left(2 h^2 m+n \left(b^2+2 b m+m^2-3 b n-3 m n+n^2\right)\right) \omega_0^2-$$
$$\left.\left. i \left(b^2+2 b m+m^2-9 b n-9 m n+9 n^2\right) \omega_0^3+3 (b+m-3 n) \omega_0^4+3 i \omega_0^5\right)\right)-$$
$$\left(2 \left(-3 i h^2 m n^3 \left(10 h^4 m^2+h^2 m (11 b+11 m-21 n) n^2+3 (b+m)^2 n^4\right)+\right.\right.$$
$$\left(9 h^6 m^3 n^2+h^4 m^2 (79 b+79 m-241 n) n^4+\right.$$
$$\left. h^2 m (b+m) (62 b+62 m-105 n) n^6+12 (b+m)^3 n^8\right) \omega_0-$$
$$i \left(10 h^6 m^3 n-26 h^4 m^2 (2 b+2 m-11 n) n^3-12 (b+m)^2 (4 b+4 m-5 n) n^7-\right.$$
$$\left. h^2 m n^5 \left(127 b^2+254 b m+127 m^2-582 b n-582 m n+183 n^2\right)\right) \omega_0^2+$$
$$\left(h^6 m^3+6 h^4 m^2 n^2 (10 b+10 m+11 n)-4 (b+m) n^6\right.$$
$$\left(11 b^2+22 b m+11 m^2-60 b n-60 m n+27 n^2\right)-$$
$$\left. h^2 m n^4 \left(28 b^2+56 b m+28 m^2-1013 b n-1013 m n+936 n^2\right)\right) \omega_0^3+$$
$$i n \left(h^4 m^2 (29 m-189 n)+40 b^3 n^4+5 h^2 m n^2 \left(21 m^2+58 m n-303 n^2\right)+\right.$$
$$4 n^4 \left(10 m^3+55 m^2 n-108 m n^2+18 n^3\right)+5 b^2 \left(21 h^2 m n^2+4 n^4 (6 m+11 n)\right)+$$
$$\left. b \left(29 h^4 m^2+10 h^2 m n^2 (21 m+29 n)+8 n^4 \left(15 m^2+55 m n-54 n^2\right)\right)\right) \omega_0^4+$$
$$\left(-76 b^3 n^4+h^4 m^2 (13 m+67 n)+h^2 m n^2 \left(-50 m^2+613 m n+436 n^2\right)+\right.$$
$$4 n^4 \left(-19 m^3+50 m^2 n+99 m n^2-72 n^3\right)+b^2 \left(-50 h^2 m n^2+4 n^4 (-57 m+50 n)\right)+$$
$$\left. b \left(13 h^4 m^2+h^2 m n^2 (-100 m+613 n)+4 n^4 \left(-57 m^2+100 m n+99 n^2\right)\right)\right) \omega_0^5-$$
$$i \left(32 h^4 m^2-h^2 m n \left(17 b^2+34 b m+17 m^2+342 b n+342 m n-819 n^2\right)+\right.$$
$$4 n^3 \left(4 b^3+4 m^3+b^2 (12 m-95 n)-95 m^2 n+90 m n^2+66 n^3+\right.$$
$$\left.\left. 2 b \left(6 m^2-95 m n+45 n^2\right)\right)\right) \omega_0^6-\left(20 b^3 n^2+h^2 m \left(8 m^2-71 m n-472 n^2\right)+\right.$$
$$4 n^2 \left(5 m^3+20 m^2 n-171 m n^2+60 n^3\right)+b^2 \left(8 h^2 m+20 n^2 (3 m+4 n)\right)+$$
$$\left. b \left(h^2 m (16 m-71 n)+4 n^2 \left(15 m^2+40 m n-171 n^2\right)\right)\right) \omega_0^7-$$
$$i \left(8 b^3 n+4 b^2 (6 m-25 n) n+h^2 m (-46 m+81 n)+4 n \left(2 m^3-25 m^2 n-\right.\right.$$
$$\left.\left. 36 m n^2+114 n^3\right)-2 b \left(23 h^2 m+4 n \left(-3 m^2+25 m n+18 n^2\right)\right)\right) \omega_0^8+$$
$$4 \left(15 h^2 m+n \left(-10 b^2-20 b m-10 m^2+45 b n+45 m n+24 n^2\right)\right) \omega_0^9+$$
$$\left.\left. 24 i (3 b+3 m-5 n) n \omega_0^{10}+48 n \omega_0^{11}\right)\right) \Big/$$
$$\left(\left(n^2+\omega_0^2\right)^2 \left(h^2 m+2 i (b+m) n \omega_0-4 (b+m-n) \omega_0^2+8 i \omega_0^3\right)\right.$$
$$\left(i n \left(b^2 n^2+2 b m \left(-h^2+n^2\right)+m \left(m n^2+h^2 (-2 m+n)\right)\right)-\right.$$
$$3 \left(b^2 n^2-b \left(h^2 m+n^2 (-2 m+n)\right)-m \left(h^2 (m-2 n)+n^2 (-m+n)\right)\right) \omega_0-$$
$$3 i \left(2 h^2 m+n \left(b^2+2 b m+m^2-3 b n-3 m n+n^2\right)\right) \omega_0^2+$$
$$\left.\left.\left(b^2+2 b m+m^2-9 b n-9 m n+9 n^2\right) \omega_0^3-3 i (b+m-3 n) \omega_0^4-3 \omega_0^5\right)\right)+$$
$$\left(h^2 m \left(-21 h^2 m n^2+\left(-25 h^2 m+8 n^2 (b+m+3 n)\right) \omega_0^2+40 n \omega_0^4\right)\right) \Big/$$
$$\left(\left(n-i \omega_0\right) \left(n+i \omega_0\right)\right.$$
$$\left(h^2 m-2 i \left(n-2 i \omega_0\right) \left(b+m+2 i \omega_0\right) \omega_0\right)$$
$$\left.\left.\left(h^2 m+2 \left(n+2 i \omega_0\right) \omega_0 \left(i (b+m)+2 \omega_0\right)\right)\right)\right)-$$
$$\left(8 \left(n \left(h^2 (2 m-3 n)-3 (b+m) n^2\right)-\left(h^2+(b+m) n\right) \omega_0^2\right)\right.$$
$$\left(6 h^4 m^2 n^3 \left(-h^2 m+(b+m) n^2\right)+\omega_0^2 \left(2 h^6 m^3 n+h^4 m^2 (41 (b+m)-34 n) n^3-\right.\right.$$
$$6 h^2 m (b+m) (9 (b+m)-8 n) n^5+15 (b+m)^3 n^7+$$
$$n \left(-5 h^4 m^2 (b+m)-20 h^4 m^2 n-128 h^2 m (b+m)^2 n^2+48 h^2 m (b+m) n^3+\right.$$
$$\left(85 b^3+255 b^2 m+16 h^2 m+255 b m^2+85 m^3\right) n^4+20 (b+m)^2 n^5+$$
$$\left. 60 (b+m) n^6\right) \omega_0^2+\left(-2 h^4 m^2+2 h^2 m n \left(-13 (b+m)^2-12 (b+m) n-48 n^2\right)+\right.$$
$$\left. n^3 \left(101 (b+m)^3+116 (b+m)^2 n+340 (b+m) n^2+80 n^3\right)\right) \omega_0^4+$$
$$\left(-24 h^2 m (b+m)+\left(31 b^3+93 b^2 m+80 h^2 m+93 b m^2+31 m^3\right) n+\right.$$

$$124\,(b+m)^2 n^2 + 404\,(b+m)\,n^3 + 464\,n^4\Big)\,\omega_0^6 +$$
$$4\,\left(7\,(b+m)^2 + 31\,(b+m)\,n + 124\,n^2\right)\omega_0^8 + 112\,\omega_0^{10}\Big)\Big)\Big)\Big/$$
$$\Big(\left(n^2+\omega_0^2\right)^2 \Big(\left(h^2\,m - (b+m)\,n^2\right)^2 + 2\,\left(h^2\,m\,(b+m) - 4\,h^2\,m\,n + (b+m)^2\,n^2 + 2\,n^4\right)\omega_0^2 +$$
$$\left((b+m)^2 + 8\,n^2\right)\omega_0^4 + 4\,\omega_0^6\Big)$$
$$\left(h^4\,m^2 + 4\,\omega_0^2\,\left(-2\,h^2\,m\,(b+m) + 2\,h^2\,m\,n + (b+m)^2\,n^2 + 4\,\omega_0^2\,\left((b+m)^2 + n^2 + 4\,\omega_0^2\right)\right)\right)\Big)\Big\}$$

(* Componentes do número complexo G32 *)

```
(* G32=Simply[
  pb.(6 bb[h11,h21]+bb[h20b,h30]+3 bb[h20,h21b]+3 bb[q,h22]+2 bb[qb,h31]),
  ω0∈Reals && m>0 && n<0 && h>0 &&b>0] *)
```

números reais

```
G321 = Simplify[pb.(6 bb[h11, h21]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```

simplifica   números reais

$$
\begin{aligned}
&\Big(24 m^5 \Big(3 h^2 m n^4 \big(25 h^4 m^2 + 5 h^2 m (6 b + 6 m - 11 n) n^2 + 9 (b + m)^2 n^4\big) - \\
&\quad \mathrm{i} \big(90 h^6 m^3 n^3 + h^4 m^2 (334 b + 334 m - 829 n) n^5 + \\
&\qquad 57 h^2 m (b + m) (4 b + 4 m - 5 n) n^7 + 36 (b + m)^3 n^9\big) \omega_0 - \\
&\quad \big(14 h^6 m^3 n^2 + 3 h^4 m^2 (144 b + 144 m - 529 n) n^4 + 36 (b + m)^2 (6 b + 6 m - 5 n) n^8 + \\
&\qquad h^2 m n^6 \big(666 b^2 + 666 m^2 + 2 b (666 m - 979 n) - 1958 m n + 489 n^2\big)\big) \omega_0^2 - \\
&\quad \mathrm{i} \big(10 h^6 m^3 n + h^4 m^2 n^3 (-76 b - 76 m + 1339 n) - \\
&\qquad 12 (b + m) n^7 \big(40 b^2 + 80 b m + 40 m^2 - 90 b n - 90 m n + 27 n^2\big) - \\
&\qquad 2 h^2 m n^5 \big(370 b^2 + 740 b m + 370 m^2 - 2474 b n - 2474 m n + 1559 n^2\big)\big) \omega_0^3 + \\
&\quad \big(-h^6 m^3 - h^4 m^2 n^2 (150 b + 150 m + 83 n) + 2 h^2 m n^4 \\
&\qquad\quad \big(48 b^2 + 96 b m + 48 m^2 - 2577 b n - 2577 m n + 3694 n^2\big) + 24 n^6 \big(18 b^3 + 18 m^3 + \\
&\qquad\qquad 2 b^2 (27 m - 50 n) - 100 m^2 n + 81 m n^2 - 9 n^3 + b \big(54 m^2 - 200 m n + 81 n^2\big)\big)\big) \omega_0^4 + \\
&\quad \mathrm{i}\, n \big(h^4 m^2 (42 m - 437 n) + 8 b^3 n^4 + 2 h^2 m n^2 \big(178 m^2 + 505 m n - 3685 n^2\big) + \\
&\qquad 8 n^4 \big(m^3 + 270 m^2 n - 540 m n^2 + 162 n^3\big) + 4 b^2 \big(89 h^2 m n^2 + 6 n^4 (m + 90 n)\big) + \\
&\qquad 2 b \big(21 h^4 m^2 + h^2 m n^2 (356 m + 505 n) + 12 n^4 \big(m^2 + 180 m n - 180 n^2\big)\big)\big) \omega_0^5 + \\
&\quad \big(288 b^3 n^4 - h^4 m^2 (12 m + 107 n) + 2 h^2 m n^2 \big(85 m^2 - 915 m n - 753 n^2\big) + \\
&\qquad 8 n^4 \big(36 m^3 - 5 m^2 n - 486 m n^2 + 360 n^3\big) + 2 b^2 \big(85 h^2 m n^2 + 4 (108 m - 5 n) n^4\big) - \\
&\qquad 2 b \big(6 h^4 m^2 + 5 h^2 m n^2 (-34 m + 183 n) + 8 n^4 \big(-54 m^2 + 5 m n + 243 n^2\big)\big)\big) \omega_0^6 - \\
&\quad \mathrm{i} \big(29 h^4 m^2 - 2 h^2 m n \big(14 b^2 + 28 b m + 14 m^2 + 482 b n + 482 m n - 1167 n^2\big) + 8 n^3 \big(20 b^3 + \\
&\qquad\qquad 20 m^3 + 60 b^2 (m - 3 n) - 180 m^2 n + 9 m n^2 + 324 n^3 + b \big(60 m^2 - 360 m n + 9 n^2\big)\big)\big) \omega_0^7 + \\
&\quad \big(16 b^3 n^2 + h^2 m \big(21 m^2 - 94 m n - 1244 n^2\big) + 16 n^2 \big(m^3 + 50 m^2 n - 162 m n^2 + 3 n^3\big) + \\
&\qquad b^2 \big(21 h^2 m + 16 n^2 (3 m + 50 n)\big) + \\
&\qquad 2 b \big(h^2 m (21 m - 47 n) + 8 n^2 \big(3 m^2 + 100 m n - 162 n^2\big)\big)\big) \omega_0^8 - \\
&\quad \mathrm{i} \big(36 b^3 n + 4 b^2 (27 m - 20 n) n + h^2 m (-97 m + 102 n) + 4 n \\
&\qquad\quad \big(9 m^3 - 20 m^2 n - 360 m n^2 + 432 n^3\big) - b \big(97 h^2 m + 4 n \big(-27 m^2 + 40 m n + 360 n^2\big)\big)\big) \omega_0^9 - \\
&\quad \big(8 b^3 + 117 h^2 m + 8 m^3 + 12 b^2 (2 m - 15 n) - 180 m^2 n + 144 m n^2 + \\
&\qquad 960 n^3 + 24 b \big(m^2 - 15 m n + 6 n^2\big)\big) \omega_0^{10} - \\
&\quad 4 \mathrm{i} \big(10 b^2 + 20 b m + 10 m^2 - 81 b n - 81 m n + 24 n^2\big) \omega_0^{11} + \\
&\quad 72 (b + m - 3 n) \omega_0^{12} + 48 \mathrm{i}\, \omega_0^{13}\Big)\Big) \Big/ \\
&\Big(h^2 \big(m^2 + \omega_0^2\big)^2 \big(n^2 + \omega_0^2\big)^2 \big(h^2 m - (b + m) n^2 + 2 \mathrm{i} (b + m - n) n \omega_0 + \\
&\qquad (b + m - 4 n) \omega_0^2 + 2 \mathrm{i}\, \omega_0^3\big) \\
&\quad \big(h^2 m - 2 \mathrm{i} (b + m) n \omega_0 - 4 (b + m - n) \omega_0^2 - 8 \mathrm{i}\, \omega_0^3\big) \\
&\quad \big(-n \big(b^2 n^2 + 2 b m \big(-h^2 + n^2\big) + m \big(m n^2 + h^2 (-2 m + n)\big)\big) + \\
&\qquad 3 \mathrm{i} \big(b^2 n^2 - b \big(h^2 m + n^2 (-2 m + n)\big) - m \big(h^2 (m - 2 n) + n^2 (-m + n)\big)\big) \omega_0 + \\
&\qquad 3 \big(2 h^2 m + n \big(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2\big)\big) \omega_0^2 - \\
&\qquad \mathrm{i} \big(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2\big) \omega_0^3 + 3 (b + m - 3 n) \omega_0^4 + 3 \mathrm{i}\, \omega_0^5\big)\Big)
\end{aligned}
$$

```
G322 = Simplify[pb.(bb[h20b, h30]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```

simplifica   números reais

$$\Big(12\,m^6\,\Big(55\,h^4\,m^2\,n^3 - i\,h^2\,m\,n^2\,\big(137\,h^2\,m - 27\,(b+m)\,n^2\big)\,\omega_0 + \big(-h^4\,m^2\,n + h^2\,m\,(46\,b + 46\,m - 145\,n)\,n^3 + 4\,(b+m)^2\,n^5\big)\,\omega_0^2 - i\,\big(113\,h^4\,m^2 + 20\,(b+m)^2\,n^4 - 2\,h^2\,m\,n^2\,(28\,b + 28\,m + 193\,n)\big)\,\omega_0^3 + 2\,n\,\big(h^2\,m\,(63\,m - 8\,n) - 12\,b^2\,n^2 - 12\,m^2\,n^2 + 8\,n^4 + 3\,b\,m\,(21\,h^2 - 8\,n^2)\big)\,\omega_0^4 - i\,\big(3\,h^2\,m\,(17\,m - 214\,n) + 8\,b^2\,n^2 + 8\,n^2\,(m^2 + 10\,n^2) + b\,m\,(51\,h^2 + 16\,n^2)\big)\,\omega_0^5 + \big(385\,h^2\,m - 4\,n\,(7\,b^2 + 14\,b\,m + 7\,m^2 + 24\,n^2)\big)\,\omega_0^6 + 4\,i\,\big(3\,b^2 + 6\,b\,m + 3\,m^2 - 8\,n^2\big)\,\omega_0^7 - 112\,n\,\omega_0^8 + 48\,i\,\omega_0^9\Big)\Big) \Big/ \Big((-i\,n + \omega_0)\,(i\,n + \omega_0)\,(m^2 + \omega_0^2)^2\,\big(h^2\,m - (b+m)\,n^2 + 2\,i\,(b + m - n)\,n\,\omega_0 + (b + m - 4\,n)\,\omega_0^2 + 2\,i\,\omega_0^3\big)\,\big(h^2\,m + 2\,i\,(b+m)\,n\,\omega_0 - 4\,(b + m - n)\,\omega_0^2 + 8\,i\,\omega_0^3\big)\,\big(-h^2\,m + 2\,i\,(b+m)\,n\,\omega_0 + 4\,(b + m - n)\,\omega_0^2 + 8\,i\,\omega_0^3\big)\,\big(-h^2\,m + 3\,i\,(b+m)\,n\,\omega_0 + 9\,(b + m - n)\,\omega_0^2 + 27\,i\,\omega_0^3\big)\Big)$$

```
G323 = Simplify[pb.(3 bb[h20, h21b]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```

simplifica   números reais

$$\Big(12\,\mathrm{i}\,m^6\Big(-3\,\mathrm{i}\,h^2 m n^3\left(25 h^4 m^2+5 h^2 m\,(6 b+6 m-11 n)\,n^2+9\,(b+m)^2 n^4\right)+\left(-45 h^6 m^3 n^2+2 h^4 m^2(59 b+59 m-224 n)\,n^4+3 h^2 m\,(b+m)\,(49 b+49 m-111 n)\,n^6+36\,(b+m)^3 n^8\right)\omega_0-$$
$$\mathrm{i}\left(79 h^6 m^3 n+3 h^4 m^2(48 b+48 m-29 n)\,n^3-12\,(b+m)^2(9 b+9 m-20 n)\,n^7-3 h^2 m n^5\left(45 b^2+90 b m+45 m^2-423 b n-423 m n+131 n^2\right)\right)\omega_0^2+$$
$$\left(7 h^6 m^3+56 h^4 m^2(6 b+6 m-13 n)\,n^2+h^2 m n^4\left(409 b^2+818 b m+409 m^2+645 b n+645 m n-1661 n^2\right)+4\,(b+m)\,n^6\left(15 b^2+30 b m+15 m^2+167 b n+167 m n-81 n^2\right)\right)\omega_0^3+$$
$$\mathrm{i}\,n\left(h^4 m^2(2 m-873 n)+396 b^3 n^4+h^2 m n^2\left(495 m^2-2867 m n-961 n^2\right)-4 n^4\left(-99 m^3+101 m^2 n+165 m n^2+6 n^3\right)+b^2\left(495 h^2 m n^2+4\,(297 m-101 n)\,n^4\right)+b\left(2 h^4 m^2+h^2 m\,(990 m-2867 n)\,n^2-4 n^4\left(-297 m^2+202 m n+165 n^2\right)\right)\right)\omega_0^4+$$
$$\left(-140 b^3 n^4+2 h^4 m^2(49 m+12 n)+h^2 m n^2\left(233 m^2+2993 m n-3523 n^2\right)+4 n^4\left(-35 m^3+555 m^2 n-291 m n^2+122 n^3\right)+b^2\left(233 h^2 m n^2+60 n^4(-7 m+37 n)\right)+b\left(98 h^4 m^2+h^2 m n^2(466 m+2993 n)-12 n^4\left(35 m^2-370 m n+97 n^2\right)\right)\right)\omega_0^5-$$
$$\mathrm{i}\left(235 h^4 m^2+h^2 m n\left(-205 b^2-410 b m-205 m^2+391 b n+391 m n+3949 n^2\right)-4 n^3\left(93 b^3+93 m^3+163 m^2 n-709 m n^2+298 n^3+b^2(279 m+163 n)+b\left(279 m^2+326 m n-709 n^2\right)\right)\right)\omega_0^6+$$
$$\left(-220 b^3 n^2+h^2 m\left(35 m^2+863 m n+449 n^2\right)+4 n^2\left(-55 m^3+489 m^2 n-49 m n^2+10 n^3\right)+b^2\left(35 h^2 m+12 n^2(-55 m+163 n)\right)+b\left(h^2 m\,(70 m+863 n)-4 n^2\left(165 m^2-978 m n+49 n^2\right)\right)\right)\omega_0^7+$$
$$\mathrm{i}\left(11 h^2 m\,(5 m-73 n)+84 b^3 n+4 b^2 n\,(63 m+269 n)+4 n\left(21 m^3+269 m^2 n-707 m n^2+470 n^3\right)+b\left(55 h^2 m+4 n\left(63 m^2+538 m n-707 n^2\right)\right)\right)\omega_0^8+\left(-56 b^3+263 h^2 m-56 m^3+404 m^2 n+940 m n^2-808 n^3+b^2(-168 m+404 n)+b\left(-168 m^2+808 m n+940 n^2\right)\right)\omega_0^9+$$
$$4\,\mathrm{i}\left(65 b^2+130 b m+65 m^2-163 b n-163 m n+166 n^2\right)\omega_0^{10}+8\,(37 b+37 m-45 n)\,\omega_0^{11}\Big)\Big)\Big/$$
$$\Big((m-\mathrm{i}\,\omega_0)^2(m+\mathrm{i}\,\omega_0)^2(-\mathrm{i}\,n+\omega_0)^2(\mathrm{i}\,n+\omega_0)\left(h^2 m-(b+m)\,n^2+2\,\mathrm{i}\,(b+m-n)\,n\,\omega_0+(b+m-4 n)\,\omega_0^2+2\,\mathrm{i}\,\omega_0^3\right)$$
$$\left(h^2 m+2\,\mathrm{i}\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2+8\,\mathrm{i}\,\omega_0^3\right)\left(-h^2 m+2\,\mathrm{i}\,(b+m)\,n\,\omega_0+4\,(b+m-n)\,\omega_0^2+8\,\mathrm{i}\,\omega_0^3\right)$$
$$\Big(\mathrm{i}\,n\left(b^2 n^2+2 b m\left(-h^2+n^2\right)+m\left(m n^2+h^2(-2 m+n)\right)\right)-3\left(b^2 n^2-b\left(h^2 m+n^2(-2 m+n)\right)-m\left(h^2(m-2 n)+n^2(-m+n)\right)\right)\omega_0-$$
$$3\,\mathrm{i}\left(2 h^2 m+n\left(b^2+2 b m+m^2-3 b n-3 m n+n^2\right)\right)\omega_0^2+\left(b^2+2 b m+m^2-9 b n-9 m n+9 n^2\right)\omega_0^3-3\,\mathrm{i}\,(b+m-3 n)\,\omega_0^4-3\,\omega_0^5\Big)\Big)$$

```
G324 = Simplify[pb.(3 bb[q, h22]), ω0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
```

simplifica   números reais

$$\frac{1}{(m+\mathrm{i}\,\omega_0)\left(m^2+\omega_0^2\right)^2}\,12\,m^5\Bigg(-1\Big/\left(h^2\left(h^2 m-(n-\mathrm{i}\,\omega_0)^2(b+m+2\,\mathrm{i}\,\omega_0)\right)\right)$$
$$(-\mathrm{i}\,m+\omega_0)\,(\mathrm{i}\,n+\omega_0)\left(-\frac{\left(3 n^2+\omega_0^2\right)\left(7 n^2+\omega_0^2\right)}{\left(n^2+\omega_0^2\right)^2}+\right.$$
$$\Big(2\Big(-3 h^2 m n^3\left(10 h^4 m^2+h^2 m\,(11 b+11 m-21 n)\,n^2+3\,(b+m)^2 n^4\right)+$$
$$\mathrm{i}\,\Big(9 h^6 m^3 n^2+h^4 m^2(79 b+79 m-241 n)\,n^4+h^2 m\,(b+m)\,(62 b+62 m-105 n)$$

$$n^6 + 12\,(b+m)^3\,n^8)\,\omega_0 + (-10\,h^6\,m^3\,n + 26\,h^4\,m^2\,(2\,b + 2\,m - 11\,n)\,n^3 +$$

$$12\,(b+m)^2\,(4\,b + 4\,m - 5\,n)\,n^7 + h^2\,m\,n^5\,(127\,b^2 + 254\,b\,m + 127\,m^2 - 582$$

$$b\,n - 582\,m\,n + 183\,n^2))\,\omega_0^2 + i\,(h^6\,m^3 + 6\,h^4\,m^2\,n^2\,(10\,b + 10\,m + 11\,n) -$$

$$4\,(b+m)\,n^6\,(11\,b^2 + 22\,b\,m + 11\,m^2 - 60\,b\,n - 60\,m\,n + 27\,n^2) -$$

$$h^2\,m\,n^4\,(28\,b^2 + 56\,b\,m + 28\,m^2 - 1013\,b\,n - 1013\,m\,n + 936\,n^2))\,\omega_0^3 +$$

$$n\,(h^4\,m^2\,(29\,m - 189\,n) + 40\,b^3\,n^4 + 5\,h^2\,m\,n^2\,(21\,m^2 + 58\,m\,n - 303\,n^2) + 4\,n^4$$

$$(10\,m^3 + 55\,m^2\,n - 108\,m\,n^2 + 18\,n^3) + 5\,b^2\,(21\,h^2\,m\,n^2 + 4\,n^4\,(6\,m + 11\,n)) +$$

$$b\,(29\,h^4\,m^2 + 10\,h^2\,m\,n^2\,(21\,m + 29\,n) + 8\,n^4\,(15\,m^2 + 55\,m\,n - 54\,n^2)))\,\omega_0^4 -$$

$$i\,(76\,b^3\,n^4 - h^4\,m^2\,(13\,m + 67\,n) + h^2\,m\,n^2\,(50\,m^2 - 613\,m\,n - 436\,n^2) + 4\,n^4$$

$$(19\,m^3 - 50\,m^2\,n - 99\,m\,n^2 + 72\,n^3) + b^2\,(50\,h^2\,m\,n^2 + 4\,(57\,m - 50\,n)\,n^4) -$$

$$b\,(13\,h^4\,m^2 + h^2\,m\,n^2\,(-100\,m + 613\,n) + 4\,n^4\,(-57\,m^2 + 100\,m\,n + 99\,n^2)))$$

$$\omega_0^5 + (-32\,h^4\,m^2 + h^2\,m\,n\,(17\,b^2 + 34\,b\,m + 17\,m^2 + 342\,b\,n + 342\,m\,n - 819\,n^2) -$$

$$4\,n^3\,(4\,b^3 + 4\,m^3 + b^2\,(12\,m - 95\,n) - 95\,m^2\,n + 90\,m\,n^2 + 66\,n^3 + 2\,b\,(6\,m^2 -$$

$$95\,m\,n + 45\,n^2)))\,\omega_0^6 - i\,(20\,b^3\,n^2 + h^2\,m\,(8\,m^2 - 71\,m\,n - 472\,n^2) +$$

$$4\,n^2\,(5\,m^3 + 20\,m^2\,n - 171\,m\,n^2 + 60\,n^3) + b^2\,(8\,h^2\,m + 20\,n^2\,(3\,m + 4\,n)) +$$

$$b\,(h^2\,m\,(16\,m - 71\,n) + 4\,n^2\,(15\,m^2 + 40\,m\,n - 171\,n^2)))\,\omega_0^7 +$$

$$(h^2\,m\,(46\,m - 81\,n) - 8\,b^3\,n - 4\,b^2\,(6\,m - 25\,n)\,n + 4\,n\,(-2\,m^3 + 25\,m^2\,n +$$

$$36\,m\,n^2 - 114\,n^3) + 2\,b\,(23\,h^2\,m + 4\,n\,(-3\,m^2 + 25\,m\,n + 18\,n^2)))\,\omega_0^8 +$$

$$4\,i\,(15\,h^2\,m + n\,(-10\,b^2 - 20\,b\,m - 10\,m^2 + 45\,b\,n + 45\,m\,n + 24\,n^2))\,\omega_0^9 +$$

$$24\,(3\,b + 3\,m - 5\,n)\,n\,\omega_0^{10} + 48\,i\,n\,\omega_0^{11}))\,/$$

$$((n^2 + \omega_0^2)^2\,(h^2\,m - 2\,i\,(b+m)\,n\,\omega_0 - 4\,(b + m - n)\,\omega_0^2 - 8\,i\,\omega_0^3)$$

$$(-n\,(b^2\,n^2 + 2\,b\,m\,(-h^2 + n^2) + m\,(m\,n^2 + h^2\,(-2\,m + n))) +$$

$$3\,i\,(b^2\,n^2 - b\,(h^2\,m + n^2\,(-2\,m + n)) - m\,(h^2\,(m - 2\,n) + n^2\,(-m + n)))\,\omega_0 +$$

$$3\,(2\,h^2\,m + n\,(b^2 + 2\,b\,m + m^2 - 3\,b\,n - 3\,m\,n + n^2))\,\omega_0^2 -$$

$$i\,(b^2 + 2\,b\,m + m^2 - 9\,b\,n - 9\,m\,n + 9\,n^2)\,\omega_0^3 + 3\,(b + m - 3\,n)\,\omega_0^4 + 3\,i\,\omega_0^5)) -$$

$$(2\,(-3\,i\,h^2\,m\,n^3\,(10\,h^4\,m^2 + h^2\,m\,(11\,b + 11\,m - 21\,n)\,n^2 + 3\,(b+m)^2\,n^4) +$$

$$(9\,h^6\,m^3\,n^2 + h^4\,m^2\,(79\,b + 79\,m - 241\,n)\,n^4 +$$

$$h^2\,m\,(b+m)\,(62\,b + 62\,m - 105\,n)\,n^6 + 12\,(b+m)^3\,n^8)\,\omega_0 -$$

$$i\,(10\,h^6\,m^3\,n - 26\,h^4\,m^2\,(2\,b + 2\,m - 11\,n)\,n^3 - 12\,(b+m)^2\,(4\,b + 4\,m - 5\,n)\,n^7 -$$

$$h^2\,m\,n^5\,(127\,b^2 + 254\,b\,m + 127\,m^2 - 582\,b\,n - 582\,m\,n + 183\,n^2))\,\omega_0^2 +$$

$$(h^6\,m^3 + 6\,h^4\,m^2\,n^2\,(10\,b + 10\,m + 11\,n) - 4\,(b+m)\,n^6$$

$$(11\,b^2 + 22\,b\,m + 11\,m^2 - 60\,b\,n - 60\,m\,n + 27\,n^2) -$$

$$h^2\,m\,n^4\,(28\,b^2 + 56\,b\,m + 28\,m^2 - 1013\,b\,n - 1013\,m\,n + 936\,n^2))\,\omega_0^3 +$$

$$i\,n\,(h^4\,m^2\,(29\,m - 189\,n) + 40\,b^3\,n^4 + 5\,h^2\,m\,n^2\,(21\,m^2 + 58\,m\,n - 303\,n^2) + 4\,n^4$$

$$(10\,m^3 + 55\,m^2\,n - 108\,m\,n^2 + 18\,n^3) + 5\,b^2\,(21\,h^2\,m\,n^2 + 4\,n^4\,(6\,m + 11\,n)) +$$

$$b\,(29\,h^4\,m^2 + 10\,h^2\,m\,n^2\,(21\,m + 29\,n) + 8\,n^4\,(15\,m^2 + 55\,m\,n - 54\,n^2)))\,\omega_0^4 +$$

$$(-76\,b^3\,n^4 + h^4\,m^2\,(13\,m + 67\,n) + h^2\,m\,n^2\,(-50\,m^2 + 613\,m\,n + 436\,n^2) +$$

$$4\,n^4\,(-19\,m^3 + 50\,m^2\,n + 99\,m\,n^2 - 72\,n^3) +$$

$$b^2\,(-50\,h^2\,m\,n^2 + 4\,n^4\,(-57\,m + 50\,n)) + b\,(13\,h^4\,m^2 +$$

$$h^2\,m\,n^2\,(-100\,m + 613\,n) + 4\,n^4\,(-57\,m^2 + 100\,m\,n + 99\,n^2)))\,\omega_0^5 -$$

$$i\,(32\,h^4\,m^2 - h^2\,m\,n\,(17\,b^2 + 34\,b\,m + 17\,m^2 + 342\,b\,n + 342\,m\,n - 819\,n^2) +$$

$$4\,n^3\,(4\,b^3 + 4\,m^3 + b^2\,(12\,m - 95\,n) - 95\,m^2\,n + 90\,m\,n^2 + 66\,n^3 + 2\,b\,(6\,m^2 -$$

$$95\,m\,n + 45\,n^2)))\,\omega_0^6 - (20\,b^3\,n^2 + h^2\,m\,(8\,m^2 - 71\,m\,n - 472\,n^2) +$$

$$4\,n^2\,(5\,m^3 + 20\,m^2\,n - 171\,m\,n^2 + 60\,n^3) + b^2\,(8\,h^2\,m + 20\,n^2\,(3\,m + 4\,n)) +$$

$$b\,(h^2\,m\,(16\,m - 71\,n) + 4\,n^2\,(15\,m^2 + 40\,m\,n - 171\,n^2)))\,\omega_0^7 -$$

$$\mathrm{i}\,\left(8\,b^3\,n+4\,b^2\,(6\,m-25\,n)\,n+h^2\,m\,(-46\,m+81\,n)+4\,n\,\left(2\,m^3-25\,m^2\,n-\right.\right.$$

$$\left.\left.36\,m\,n^2+114\,n^3\right)-2\,b\,\left(23\,h^2\,m+4\,n\,\left(-3\,m^2+25\,m\,n+18\,n^2\right)\right)\right)\,\omega_0^8+$$

$$4\,\left(15\,h^2\,m+n\,\left(-10\,b^2-20\,b\,m-10\,m^2+45\,b\,n+45\,m\,n+24\,n^2\right)\right)\,\omega_0^9+$$

$$24\,\mathrm{i}\,(3\,b+3\,m-5\,n)\,n\,\omega_0^{10}+48\,n\,\omega_0^{11}))\Big/$$

$$\left(\left(n^2+\omega_0^2\right)^2\,\left(h^2\,m+2\,\mathrm{i}\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2+8\,\mathrm{i}\,\omega_0^3\right)\right.$$

$$\left(\mathrm{i}\,n\,\left(b^2\,n^2+2\,b\,m\,\left(-h^2+n^2\right)+m\,\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)-\right.$$

$$3\,\left(b^2\,n^2-b\,\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\,\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)\,\omega_0-$$

$$3\,\mathrm{i}\,\left(2\,h^2\,m+n\,\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\,\omega_0^2+$$

$$\left.\left.\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\,\omega_0^3-3\,\mathrm{i}\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5\right)\right)+$$

$$\left(h^2\,m\,\left(-21\,h^2\,m\,n^2+\left(-25\,h^2\,m+8\,n^2\,(b+m+3\,n)\right)\,\omega_0^2+40\,n\,\omega_0^4\right)\right)\Big/$$

$$\left(\left(n-\mathrm{i}\,\omega_0\right)\,\left(n+\mathrm{i}\,\omega_0\right)\,\left(h^2\,m-2\,\mathrm{i}\,\left(n-2\,\mathrm{i}\,\omega_0\right)\,\left(b+m+2\,\mathrm{i}\,\omega_0\right)\,\omega_0\right)\right.$$

$$\left.\left(h^2\,m+2\,\left(n+2\,\mathrm{i}\,\omega_0\right)\,\omega_0\,\left(\mathrm{i}\,(b+m)+2\,\omega_0\right)\right)\right)+$$

$$\left(8\,n\,\left(-2\,h^2\,m+3\,(b+m)\,n^2+(b+m)\,\omega_0^2\right)\,\left(6\,h^4\,m^2\,n^3\,\left(-h^2\,m+(b+m)\,n^2\right)+\right.\right.$$

$$\omega_0^2\,\left(2\,h^6\,m^3\,n+h^4\,m^2\,(41\,(b+m)-34\,n)\,n^3-6\,h^2\,m\,(b+m)\,(9\,(b+m)-8\,n)\,n^5+\right.$$

$$15\,(b+m)^3\,n^7+n\,\left(-5\,h^4\,m^2\,(b+m)-20\,h^4\,m^2\,n-128\,h^2\,m\,(b+m)^2\,n^2+\right.$$

$$48\,h^2\,m\,(b+m)\,n^3+\left(85\,b^3+255\,b^2\,m+16\,h^2\,m+255\,b\,m^2+85\,m^3\right)\,n^4+$$

$$\left.20\,(b+m)^2\,n^5+60\,(b+m)\,n^6\right)\,\omega_0^2+$$

$$\left(-2\,h^4\,m^2+2\,h^2\,m\,n\,\left(-13\,(b+m)^2-12\,(b+m)\,n-48\,n^2\right)+\right.$$

$$\left.n^3\,\left(101\,(b+m)^3+116\,(b+m)^2\,n+340\,(b+m)\,n^2+80\,n^3\right)\right)\,\omega_0^4+$$

$$\left(-24\,h^2\,m\,(b+m)+\left(31\,b^3+93\,b^2\,m+80\,h^2\,m+93\,b\,m^2+31\,m^3\right)\,n+\right.$$

$$\left.124\,(b+m)^2\,n^2+404\,(b+m)\,n^3+464\,n^4\right)\,\omega_0^6+$$

$$\left.\left.\left.4\,\left(7\,(b+m)^2+31\,(b+m)\,n+124\,n^2\right)\,\omega_0^8+112\,\omega_0^{10}\right)\right)\right)\Big/$$

$$\left(\left(n^2+\omega_0^2\right)^2\,\left(\left(h^2\,m-(b+m)\,n^2\right)^2+2\,\left(h^2\,m\,(b+m)-4\,h^2\,m\,n+(b+m)^2\,n^2+\right.\right.\right.$$

$$\left.\left.2\,n^4\right)\,\omega_0^2+\left((b+m)^2+8\,n^2\right)\,\omega_0^4+4\,\omega_0^6\right)\,\left(h^4\,m^2+\right.$$

$$\left.\left.4\,\omega_0^2\,\left(-2\,h^2\,m\,(b+m)+2\,h^2\,m\,n+(b+m)^2\,n^2+4\,\omega_0^2\,\left((b+m)^2+n^2+4\,\omega_0^2\right)\right)\right)\right)\right)+$$

$$1\Big/\left(1+\frac{\mathrm{i}\,\omega_0}{m+\mathrm{i}\,\omega_0}+\frac{b+\frac{h^2\,m}{(\mathrm{i}\,n+\omega_0)^2}}{m+\mathrm{i}\,\omega_0}\right)\,\left(2\,\left(-\frac{2\,n\,\left(3\,n^2+\omega_0^2\right)}{h^2\,\left(n^2+\omega_0^2\right)^2}+\right.\right.$$

$$\left(-3\,\mathrm{i}\,h^2\,m\,n^2\,\left(5\,h^4\,m^2+h^2\,m\,(8\,b+8\,m-13\,n)\,n^2+3\,(b+m)^2\,n^4\right)-\right.$$

$$\left(12\,h^6\,m^3\,n+h^4\,m^2\,(50\,b+50\,m-161\,n)\,n^3+\right.$$

$$\left.25\,h^2\,m\,(b+m)\,(2\,b+2\,m-3\,n)\,n^5+12\,(b+m)^3\,n^7\right)\,\omega_0+$$

$$\mathrm{i}\,\left(h^6\,m^3+2\,h^4\,m^2\,(19\,b+19\,m-103\,n)\,n^2+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^6+\right.$$

$$\left.h^2\,m\,n^4\,\left(91\,b^2+182\,b\,m+91\,m^2-404\,b\,n-404\,m\,n+123\,n^2\right)\right)\,\omega_0^2+$$

$$n\,\left(44\,b^3\,n^4+4\,b^2\,\left(5\,h^2\,m\,n^2+33\,m\,n^4-60\,n^5\right)+\right.$$

$$b\,\left(-10\,h^4\,m^2+h^2\,m\,(40\,m-719\,n)\,n^2+12\,n^4\,\left(11\,m^2-40\,m\,n+9\,n^2\right)\right)+$$

$$m\,\left(-2\,h^4\,m\,(5\,m+49\,n)+4\,n^4\,\left(11\,m^2-60\,m\,n+27\,n^2\right)+\right.$$

$$\left.\left.h^2\,n^2\,\left(20\,m^2-719\,m\,n+628\,n^2\right)\right)\right)\,\omega_0^3+$$

$$\mathrm{i}\,\left(h^4\,m^2\,(14\,m-37\,n)+40\,b^3\,n^4+3\,h^2\,m\,n^2\,\left(19\,m^2+104\,m\,n-357\,n^2\right)+4\,n^4\right.$$

$$\left(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3\right)+b^2\,\left(57\,h^2\,m\,n^2+20\,n^4\,(6\,m+11\,n)\right)+$$

$$\left.2\,b\,\left(7\,h^4\,m^2+3\,h^2\,m\,n^2\,(19\,m+52\,n)+4\,n^4\,\left(15\,m^2+55\,m\,n-54\,n^2\right)\right)\right)\,\omega_0^4+$$

$$\left(-35\,h^4\,m^2+h^2\,m\,n\,\left(22\,b^2+44\,b\,m+22\,m^2-223\,b\,n-223\,m\,n-512\,n^2\right)+\right.$$

$$4\,n^3\,\left(19\,b^3+19\,m^3+b^2\,(57\,m-50\,n)-50\,m^2\,n-\right.$$

$$\left.\left.99\,m\,n^2+72\,n^3+b\,\left(57\,m^2-100\,m\,n-99\,n^2\right)\right)\right)\,\omega_0^5-$$

$$\mathrm{i}\,\left(16\,b^3\,n^2+h^2\,m\,\left(-5\,m^2-140\,m\,n+231\,n^2\right)+b^2\,\left(-5\,h^2\,m+4\,(12\,m-95\,n)\,n^2\right)+\right.$$

$$4\,n^2\left(4\,m^3-95\,m^2\,n+90\,m\,n^2+66\,n^3\right)-$$
$$2\,b\left(5\,h^2\,m\,(m+14\,n)-4\,n^2\left(6\,m^2-95\,m\,n+45\,n^2\right)\right)\Big)\,\omega_0^6+$$
$$\Big(20\,b^3\,n+20\,b^2\,n\,(3\,m+4\,n)-5\,h^2\,m\,(m+36\,n)+4\,n\left(5\,m^3+20\,m^2\,n-\right.$$
$$\left.171\,m\,n^2+60\,n^3\right)+b\left(-5\,h^2\,m+4\,n\left(15\,m^2+40\,m\,n-171\,n^2\right)\right)\Big)\,\omega_0^7-$$
$$i\,\Big(8\,b^3-3\,h^2\,m+8\,m^3+4\,b^2\,(6\,m-25\,n)-100\,m^2\,n-144\,m\,n^2+$$
$$456\,n^3+8\,b\left(3\,m^2-25\,m\,n-18\,n^2\right)\Big)\,\omega_0^8+$$
$$4\left(10\,b^2+10\,m^2+5\,b\,(4\,m-9\,n)-45\,m\,n-24\,n^2\right)\omega_0^9+$$
$$24\,i\,(3\,b+3\,m-5\,n)\,\omega_0^{10}-48\,\omega_0^{11}\Big)\Big/$$
$$\Big(h^2\left(n^2+\omega_0^2\right)^2\left(h^2\,m-2\,i\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2-8\,i\,\omega_0^3\right)$$
$$\Big(-i\,n\left(b^2\,n^2+2\,b\,m\left(-h^2+n^2\right)+m\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)-3$$
$$\left(b^2\,n^2-b\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)$$
$$\omega_0+3\,i\left(2\,h^2\,m+n\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\omega_0^2+$$
$$\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\omega_0^3+3\,i\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5\Big)\Big)+$$
$$\Big(3\,h^2\,m\,n^2\left(5\,h^4\,m^2+h^2\,m\,(8\,b+8\,m-13\,n)\,n^2+3\,(b+m)^2\,n^4\right)+$$
$$i\,\Big(12\,h^6\,m^3\,n+h^4\,m^2\,(50\,b+50\,m-161\,n)\,n^3+$$
$$25\,h^2\,m\,(b+m)\,(2\,b+2\,m-3\,n)\,n^5+12\,(b+m)^3\,n^7\Big)\,\omega_0-$$
$$\Big(h^6\,m^3+2\,h^4\,m^2\,(19\,b+19\,m-103\,n)\,n^2+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^6+$$
$$h^2\,m\,n^4\left(91\,b^2+182\,b\,m+91\,m^2-404\,b\,n-404\,m\,n+123\,n^2\right)\Big)\,\omega_0^2-$$
$$i\,n\,\Big(44\,b^3\,n^4+4\,b^2\left(5\,h^2\,m\,n^2+33\,m\,n^4-60\,n^5\right)+$$
$$b\left(-10\,h^4\,m^2+h^2\,m\,(40\,m-719\,n)\,n^2+12\,n^4\left(11\,m^2-40\,m\,n+9\,n^2\right)\right)+$$
$$m\,\Big(-2\,h^4\,m\,(5\,m+49\,n)+4\,n^4\left(11\,m^2-60\,m\,n+27\,n^2\right)+$$
$$h^2\,n^2\left(20\,m^2-719\,m\,n+628\,n^2\right)\Big)\Big)\,\omega_0^3-$$
$$\Big(h^4\,m^2\,(14\,m-37\,n)+40\,b^3\,n^4+3\,h^2\,m\,n^2\left(19\,m^2+104\,m\,n-357\,n^2\right)+4\,n^4$$
$$\left(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3\right)+b^2\left(57\,h^2\,m\,n^2+20\,n^4\,(6\,m+11\,n)\right)+$$
$$2\,b\left(7\,h^4\,m^2+3\,h^2\,m\,n^2\,(19\,m+52\,n)+4\,n^4\left(15\,m^2+55\,m\,n-54\,n^2\right)\right)\Big)\,\omega_0^4+$$
$$i\,\Big(35\,h^4\,m^2+h^2\,m\,n\left(-22\,b^2-44\,b\,m-22\,m^2+223\,b\,n+223\,m\,n+512\,n^2\right)-$$
$$4\,n^3\,\Big(19\,b^3+19\,m^3+b^2\,(57\,m-50\,n)-50\,m^2\,n-$$
$$99\,m\,n^2+72\,n^3+b\left(57\,m^2-100\,m\,n-99\,n^2\right)\Big)\Big)\,\omega_0^5+$$
$$\Big(16\,b^3\,n^2+h^2\,m\left(-5\,m^2-140\,m\,n+231\,n^2\right)+b^2\left(-5\,h^2\,m+4\,(12\,m-95\,n)\,n^2\right)+$$
$$4\,n^2\left(4\,m^3-95\,m^2\,n+90\,m\,n^2+66\,n^3\right)-$$
$$2\,b\left(5\,h^2\,m\,(m+14\,n)-4\,n^2\left(6\,m^2-95\,m\,n+45\,n^2\right)\right)\Big)\,\omega_0^6-$$
$$i\,\Big(20\,b^3\,n+20\,b^2\,n\,(3\,m+4\,n)-5\,h^2\,m\,(m+36\,n)+4\,n\left(5\,m^3+20\,m^2\,n-\right.$$
$$\left.171\,m\,n^2+60\,n^3\right)+b\left(-5\,h^2\,m+4\,n\left(15\,m^2+40\,m\,n-171\,n^2\right)\right)\Big)\,\omega_0^7+$$
$$\Big(8\,b^3-3\,h^2\,m+8\,m^3+4\,b^2\,(6\,m-25\,n)-100\,m^2\,n-144\,m\,n^2+$$
$$456\,n^3+8\,b\left(3\,m^2-25\,m\,n-18\,n^2\right)\Big)\,\omega_0^8-$$
$$4\,i\left(10\,b^2+10\,m^2+5\,b\,(4\,m-9\,n)-45\,m\,n-24\,n^2\right)\omega_0^9-$$
$$24\,(3\,b+3\,m-5\,n)\,\omega_0^{10}+48\,i\,\omega_0^{11}\Big)\Big/$$
$$\Big(h^2\left(n^2+\omega_0^2\right)^2\left(h^2\,m+2\,i\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2+8\,i\,\omega_0^3\right)$$
$$\Big(n\left(b^2\,n^2+2\,b\,m\left(-h^2+n^2\right)+m\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)+3\,i$$
$$\left(b^2\,n^2-b\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)$$
$$\omega_0-3\left(2\,h^2\,m+n\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\omega_0^2-i$$
$$\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\omega_0^3-3\,(b+m-3\,n)\,\omega_0^4+3\,i\,\omega_0^5\Big)\Big)+$$
$$\left(2\,m\left(-3\,h^2\,m\,n+2\,n\,(b+m+3\,n)\,\omega_0^2+10\,\omega_0^4\right)\right)\Big/\Big(\left(n-i\,\omega_0\right)\left(n+i\,\omega_0\right)$$
$$\left(h^2\,m-2\,i\left(n-2\,i\,\omega_0\right)\left(b+m+2\,i\,\omega_0\right)\omega_0\right)$$
$$\left(h^2\,m+2\left(n+2\,i\,\omega_0\right)\omega_0\left(i\,(b+m)+2\,\omega_0\right)\right)\Big)+$$

$$\left(4\,(-b-m)\,\left(3\,n^2+\omega_0^2\right)\,\left(6\,h^4\,m^2\,n^3\,\left(h^2\,m-(b+m)\,n^2\right)+\omega_0^2\right.\right.$$

$$\left(-2\,h^6\,m^3\,n+6\,h^2\,m\,(b+m)\,(9\,(b+m)-8\,n)\,n^5-15\,(b+m)^3\,n^7+\right.$$

$$h^4\,m^2\,n^3\,(-41\,(b+m)+34\,n)-n\,\left(-5\,h^4\,m^2\,(b+m)-20\,h^4\,m^2\,n-\right.$$

$$128\,h^2\,m\,(b+m)^2\,n^2+48\,h^2\,m\,(b+m)\,n^3+\left(85\,b^3+255\,b^2\,m+16\,h^2\,m+\right.$$

$$\left.\left.255\,b\,m^2+85\,m^3\right)\,n^4+20\,(b+m)^2\,n^5+60\,(b+m)\,n^6\right)\,\omega_0^2+$$

$$\left(2\,h^4\,m^2+2\,h^2\,m\,n\,\left(13\,(b+m)^2+12\,(b+m)\,n+48\,n^2\right)+\right.$$

$$\left.n^3\,\left(-101\,(b+m)^3-116\,(b+m)^2\,n-340\,(b+m)\,n^2-80\,n^3\right)\right)\,\omega_0^4-$$

$$\left(-24\,h^2\,m\,(b+m)+\left(31\,b^3+93\,b^2\,m+80\,h^2\,m+93\,b\,m^2+31\,m^3\right)\,n+\right.$$

$$\left.124\,(b+m)^2\,n^2+404\,(b+m)\,n^3+464\,n^4\right)\,\omega_0^6-$$

$$\left.\left.\left.4\,\left(7\,(b+m)^2+31\,(b+m)\,n+124\,n^2\right)\,\omega_0^8-112\,\omega_0^{10}\right)\right)\right)\Big/$$

$$\left(h^2\,\left(n^2+\omega_0^2\right)^2\,\left(\left(h^2\,m-(b+m)\,n^2\right)^2+2\,\left(h^2\,m\,(b+m)-4\,h^2\,m\,n+\right.\right.\right.$$

$$\left.(b+m)^2\,n^2+2\,n^4\right)\,\omega_0^2+\left((b+m)^2+8\,n^2\right)\,\omega_0^4+4\,\omega_0^6\Big)\,\left(h^4\,m^2+4\,\omega_0^2\right.$$

$$\left.\left.\left.\left(-2\,h^2\,m\,(b+m)+2\,h^2\,m\,n+(b+m)^2\,n^2+4\,\omega_0^2\,\left((b+m)^2+n^2+4\,\omega_0^2\right)\right)\right)\right)\right)-$$

$$1\Big/\left(h^2\,\left(-n+i\,\omega_0\right)\right)\,\left(-\frac{\left(3\,n^2+\omega_0^2\right)\,\left(7\,n^2+\omega_0^2\right)}{\left(n^2+\omega_0^2\right)^2}+\right.$$

$$\left(2\,\left(-3\,h^2\,m\,n^3\,\left(10\,h^4\,m^2+h^2\,m\,(11\,b+11\,m-21\,n)\,n^2+3\,(b+m)^2\,n^4\right)+i\right.\right.$$

$$\left(9\,h^6\,m^3\,n^2+h^4\,m^2\,(79\,b+79\,m-241\,n)\,n^4+\right.$$

$$\left.h^2\,m\,(b+m)\,(62\,b+62\,m-105\,n)\,n^6+12\,(b+m)^3\,n^8\right)\,\omega_0+$$

$$\left(-10\,h^6\,m^3\,n+26\,h^4\,m^2\,(2\,b+2\,m-11\,n)\,n^3+12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^7+\right.$$

$$\left.h^2\,m\,n^5\,\left(127\,b^2+254\,b\,m+127\,m^2-582\,b\,n-582\,m\,n+183\,n^2\right)\right)$$

$$\omega_0^2+i\,\left(h^6\,m^3+6\,h^4\,m^2\,n^2\,(10\,b+10\,m+11\,n)-4\,(b+m)\,n^6\right.$$

$$\left(11\,b^2+22\,b\,m+11\,m^2-60\,b\,n-60\,m\,n+27\,n^2\right)-$$

$$\left.h^2\,m\,n^4\,\left(28\,b^2+56\,b\,m+28\,m^2-1013\,b\,n-1013\,m\,n+936\,n^2\right)\right)\,\omega_0^3+n$$

$$\left(h^4\,m^2\,(29\,m-189\,n)+40\,b^3\,n^4+5\,h^2\,m\,n^2\,\left(21\,m^2+58\,m\,n-303\,n^2\right)+\right.$$

$$4\,n^4\,\left(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3\right)+$$

$$5\,b^2\,\left(21\,h^2\,m\,n^2+4\,n^4\,(6\,m+11\,n)\right)+$$

$$\left.b\,\left(29\,h^4\,m^2+10\,h^2\,m\,n^2\,(21\,m+29\,n)+8\,n^4\,\left(15\,m^2+55\,m\,n-54\,n^2\right)\right)\right)\,\omega_0^4-$$

$$i\,\left(76\,b^3\,n^4-h^4\,m^2\,(13\,m+67\,n)+h^2\,m\,n^2\,\left(50\,m^2-613\,m\,n-436\,n^2\right)+4\,n^4\right.$$

$$\left(19\,m^3-50\,m^2\,n-99\,m\,n^2+72\,n^3\right)+b^2\,\left(50\,h^2\,m\,n^2+4\,(57\,m-50\,n)\,n^4\right)-$$

$$\left.b\,\left(13\,h^4\,m^2+h^2\,m\,n^2\,(-100\,m+613\,n)+4\,n^4\,\left(-57\,m^2+100\,m\,n+99\,n^2\right)\right)\right)$$

$$\omega_0^5+\left(-32\,h^4\,m^2+h^2\,m\,n\,\left(17\,b^2+34\,b\,m+17\,m^2+342\,b\,n+342\,m\,n-819\,n^2\right)-\right.$$

$$4\,n^3\,\left(4\,b^3+4\,m^3+b^2\,(12\,m-95\,n)-95\,m^2\,n+90\,m\,n^2+66\,n^3+2\,b\,\left(6\,m^2-\right.\right.$$

$$\left.\left.\left.95\,m\,n+45\,n^2\right)\right)\right)\,\omega_0^6-i\,\left(20\,b^3\,n^2+h^2\,m\,\left(8\,m^2-71\,m\,n-472\,n^2\right)+\right.$$

$$4\,n^2\,\left(5\,m^3+20\,m^2\,n-171\,m\,n^2+60\,n^3\right)+b^2\,\left(8\,h^2\,m+20\,n^2\,(3\,m+4\,n)\right)+$$

$$\left.b\,\left(h^2\,m\,(16\,m-71\,n)+4\,n^2\,\left(15\,m^2+40\,m\,n-171\,n^2\right)\right)\right)\,\omega_0^7+$$

$$\left(h^2\,m\,(46\,m-81\,n)-8\,b^3\,n-4\,b^2\,(6\,m-25\,n)\,n+4\,n\,\left(-2\,m^3+25\,m^2\,n+\right.\right.$$

$$\left.\left.36\,m\,n^2-114\,n^3\right)+2\,b\,\left(23\,h^2\,m+4\,n\,\left(-3\,m^2+25\,m\,n+18\,n^2\right)\right)\right)\,\omega_0^8+4$$

$$i\,\left(15\,h^2\,m+n\,\left(-10\,b^2-20\,b\,m-10\,m^2+45\,b\,n+45\,m\,n+24\,n^2\right)\right)$$

$$\left.\left.\omega_0^9+24\,(3\,b+3\,m-5\,n)\,n\,\omega_0^{10}+48\,i\,n\,\omega_0^{11}\right)\right)\Big/$$

$$\left(\left(n^2+\omega_0^2\right)^2\,\left(h^2\,m-2\,i\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2-8\,i\,\omega_0^3\right)\right.$$

$$\left(-n\,\left(b^2\,n^2+2\,b\,m\,\left(-h^2+n^2\right)+m\,\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)+3\,i\right.$$

$$\left(b^2\,n^2-b\,\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\,\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)$$

$$\omega_0+3\,\left(2\,h^2\,m+n\,\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\,\omega_0^2-i$$

$$\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\omega_0^3+3\,(b+m-3\,n)\,\omega_0^4+3\,\mathrm{i}\,\omega_0^5\Big)\Big)-$$
$$\Big(2\,\Big(-3\,\mathrm{i}\,h^2\,m\,n^3\left(10\,h^4\,m^2+h^2\,m\,(11\,b+11\,m-21\,n)\,n^2+3\,(b+m)^2\,n^4\right)+$$
$$\Big(9\,h^6\,m^3\,n^2+h^4\,m^2\,(79\,b+79\,m-241\,n)\,n^4+$$
$$h^2\,m\,(b+m)\,(62\,b+62\,m-105\,n)\,n^6+12\,(b+m)^3\,n^8\Big)\,\omega_0-\mathrm{i}$$
$$\Big(10\,h^6\,m^3\,n-26\,h^4\,m^2\,(2\,b+2\,m-11\,n)\,n^3-12\,(b+m)^2\,(4\,b+4\,m-5\,n)\,n^7-$$
$$h^2\,m\,n^5\left(127\,b^2+254\,b\,m+127\,m^2-582\,b\,n-582\,m\,n+183\,n^2\right)\Big)$$
$$\omega_0^2+\Big(h^6\,m^3+6\,h^4\,m^2\,n^2\,(10\,b+10\,m+11\,n)-4\,(b+m)\,n^6$$
$$\left(11\,b^2+22\,b\,m+11\,m^2-60\,b\,n-60\,m\,n+27\,n^2\right)-$$
$$h^2\,m\,n^4\left(28\,b^2+56\,b\,m+28\,m^2-1013\,b\,n-1013\,m\,n+936\,n^2\right)\Big)\,\omega_0^3+\mathrm{i}\,n$$
$$\Big(h^4\,m^2\,(29\,m-189\,n)+40\,b^3\,n^4+5\,h^2\,m\,n^2\left(21\,m^2+58\,m\,n-303\,n^2\right)+$$
$$4\,n^4\left(10\,m^3+55\,m^2\,n-108\,m\,n^2+18\,n^3\right)+$$
$$5\,b^2\left(21\,h^2\,m\,n^2+4\,n^4\,(6\,m+11\,n)\right)+$$
$$b\left(29\,h^4\,m^2+10\,h^2\,m\,n^2\,(21\,m+29\,n)+8\,n^4\left(15\,m^2+55\,m\,n-54\,n^2\right)\right)\Big)\,\omega_0^4+$$
$$\Big(-76\,b^3\,n^4+h^4\,m^2\,(13\,m+67\,n)+h^2\,m\,n^2\left(-50\,m^2+613\,m\,n+436\,n^2\right)+$$
$$4\,n^4\left(-19\,m^3+50\,m^2\,n+99\,m\,n^2-72\,n^3\right)+$$
$$b^2\left(-50\,h^2\,m\,n^2+4\,n^4\,(-57\,m+50\,n)\right)+$$
$$b\left(13\,h^4\,m^2+h^2\,m\,n^2\,(-100\,m+613\,n)+4\,n^4\left(-57\,m^2+100\,m\,n+99\,n^2\right)\right)\Big)$$
$$\omega_0^5-\mathrm{i}\,\Big(32\,h^4\,m^2-h^2\,m\,n\left(17\,b^2+34\,b\,m+17\,m^2+342\,b\,n+342\,m\,n-819\,n^2\right)+$$
$$4\,n^3\Big(4\,b^3+4\,m^3+b^2\,(12\,m-95\,n)-95\,m^2\,n+90\,m\,n^2+66\,n^3+2\,b\,\big(6\,m^2-$$
$$95\,m\,n+45\,n^2\big)\Big)\Big)\,\omega_0^6-\Big(20\,b^3\,n^2+h^2\,m\left(8\,m^2-71\,m\,n-472\,n^2\right)+$$
$$4\,n^2\left(5\,m^3+20\,m^2\,n-171\,m\,n^2+60\,n^3\right)+b^2\left(8\,h^2\,m+20\,n^2\,(3\,m+4\,n)\right)+$$
$$b\left(h^2\,m\,(16\,m-71\,n)+4\,n^2\left(15\,m^2+40\,m\,n-171\,n^2\right)\right)\Big)\,\omega_0^7-\mathrm{i}$$
$$\Big(8\,b^3\,n+4\,b^2\,(6\,m-25\,n)\,n+h^2\,m\,(-46\,m+81\,n)+4\,n\,\big(2\,m^3-25\,m^2\,n-$$
$$36\,m\,n^2+114\,n^3\big)-2\,b\left(23\,h^2\,m+4\,n\left(-3\,m^2+25\,m\,n+18\,n^2\right)\right)\Big)\,\omega_0^8+4$$
$$\left(15\,h^2\,m+n\left(-10\,b^2-20\,b\,m-10\,m^2+45\,b\,n+45\,m\,n+24\,n^2\right)\right)$$
$$\omega_0^9+24\,\mathrm{i}\,(3\,b+3\,m-5\,n)\,n\,\omega_0^{10}+48\,n\,\omega_0^{11}\Big)\Big)\Big/$$
$$\Big(\left(n^2+\omega_0^2\right)^2\left(h^2\,m+2\,\mathrm{i}\,(b+m)\,n\,\omega_0-4\,(b+m-n)\,\omega_0^2+8\,\mathrm{i}\,\omega_0^3\right)$$
$$\Big(\mathrm{i}\,n\left(b^2\,n^2+2\,b\,m\left(-h^2+n^2\right)+m\left(m\,n^2+h^2\,(-2\,m+n)\right)\right)-3$$
$$\left(b^2\,n^2-b\left(h^2\,m+n^2\,(-2\,m+n)\right)-m\left(h^2\,(m-2\,n)+n^2\,(-m+n)\right)\right)$$
$$\omega_0-3\,\mathrm{i}\left(2\,h^2\,m+n\left(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\right)\right)\omega_0^2+$$
$$\left(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\right)\omega_0^3-3\,\mathrm{i}\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5\Big)\Big)+$$
$$\left(h^2\,m\left(-21\,h^2\,m\,n^2+\left(-25\,h^2\,m+8\,n^2\,(b+m+3\,n)\right)\omega_0^2+40\,n\,\omega_0^4\right)\right)\Big/$$
$$\Big(\left(n-\mathrm{i}\,\omega_0\right)\left(n+\mathrm{i}\,\omega_0\right)\left(h^2\,m-2\,\mathrm{i}\left(n-2\,\mathrm{i}\,\omega_0\right)\left(b+m+2\,\mathrm{i}\,\omega_0\right)\omega_0\right)$$
$$\left(h^2\,m+2\left(n+2\,\mathrm{i}\,\omega_0\right)\omega_0\left(\mathrm{i}\,(b+m)+2\,\omega_0\right)\right)\Big)+$$
$$\Big(8\,n\left(-2\,h^2\,m+3\,(b+m)\,n^2+(b+m)\,\omega_0^2\right)\Big(6\,h^4\,m^2\,n^3\left(-h^2\,m+(b+m)\,n^2\right)+\omega_0^2$$
$$\Big(2\,h^6\,m^3\,n+h^4\,m^2\,(41\,(b+m)-34\,n)\,n^3-6\,h^2\,m\,(b+m)\,(9\,(b+m)-8\,n)\,n^5+$$
$$15\,(b+m)^3\,n^7+n\,\Big(-5\,h^4\,m^2\,(b+m)-20\,h^4\,m^2\,n-128\,h^2\,m\,(b+m)^2\,n^2+$$
$$48\,h^2\,m\,(b+m)\,n^3+\left(85\,b^3+255\,b^2\,m+16\,h^2\,m+255\,b\,m^2+85\,m^3\right)n^4+$$
$$20\,(b+m)^2\,n^5+60\,(b+m)\,n^6\Big)\,\omega_0^2+$$
$$\Big(-2\,h^4\,m^2+2\,h^2\,m\,n\left(-13\,(b+m)^2-12\,(b+m)\,n-48\,n^2\right)+$$
$$n^3\left(101\,(b+m)^3+116\,(b+m)^2\,n+340\,(b+m)\,n^2+80\,n^3\right)\Big)\,\omega_0^4+$$
$$\Big(-24\,h^2\,m\,(b+m)+\left(31\,b^3+93\,b^2\,m+80\,h^2\,m+93\,b\,m^2+31\,m^3\right)n+$$
$$124\,(b+m)^2\,n^2+404\,(b+m)\,n^3+464\,n^4\Big)\,\omega_0^6+$$
$$4\left(7\,(b+m)^2+31\,(b+m)\,n+124\,n^2\right)\omega_0^8+112\,\omega_0^{10}\Big)\Big)\Big)\Big/$$
$$\Big(\left(n^2+\omega_0^2\right)^2\Big(\left(h^2\,m-(b+m)\,n^2\right)^2+2\left(h^2\,m\,(b+m)-4\,h^2\,m\,n+(b+m)^2\,n^2+2\,n^4\right)$$

$$\omega_0^2 + \left((b+m)^2 + 8 n^2\right) \omega_0^4 + 4 \omega_0^6\right) \left(h^4 m^2 + 4 \omega_0^2\right.$$
$$\left.\left(-2 h^2 m (b+m) + 2 h^2 m n + (b+m)^2 n^2 + 4 \omega_0^2 \left((b+m)^2 + n^2 + 4 \omega_0^2\right)\right)\right)\left.\right)\left.\right)\left.\right)\left.\right)$$

G325 = Simplify[pb.(2 bb[qb, h31]), ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 &&b > 0]
simplifica　　números reais

$$\frac{1}{\left(m - i \omega_0\right)^2 \left(m + i \omega_0\right)^3 \left(-i n + \omega_0\right)^2 \left(i n + \omega_0\right)^3}$$
$$24 m^6 \left(-1 \Big/ \left(h^2 m - \left(n - i \omega_0\right)^2 \left(b + m + 2 i \omega_0\right)\right) \left(-i m + \omega_0\right) \left(-i n + \omega_0\right) \left(i n + \omega_0\right)\right.$$
$$\left(\left(3 h^2 m n^3 \left(10 h^4 m^2 + h^2 m (11 b + 11 m - 21 n) n^2 + 3 (b+m)^2 n^4\right) -\right.\right.$$
$$i \left(69 h^6 m^3 n^2 + h^4 m^2 (175 b + 175 m - 397 n) n^4 + 7 h^2 m\right.$$
$$\left.(b+m) (14 b + 14 m - 15 n) n^6 + 12 (b+m)^3 n^8\right) \omega_0 -$$
$$\left(38 h^6 m^3 n + 6 h^4 m^2 (57 b + 57 m - 170 n) n^3 + 12 (b+m)^2 (8 b + 8 m - 5 n)\right.$$
$$\left.n^7 + 3 h^2 m n^5 \left(127 b^2 + 254 b m + 127 m^2 - 294 b n - 294 m n + 61 n^2\right)\right) \omega_0^2 +$$
$$i \left(3 h^6 m^3 + 2 h^4 m^2 (127 b + 127 m - 661 n) n^2 + 4 (b+m) n^6\right.$$
$$\left(77 b^2 + 77 m^2 + 2 b (77 m - 60 n) - 120 m n + 27 n^2\right) + h^2 m n^4$$
$$\left.\left(674 b^2 + 1348 b m + 674 m^2 - 2899 b n - 2899 m n + 1428 n^2\right)\right) \omega_0^3 +$$
$$n \left(h^4 m^2 (9 m - 767 n) + 496 b^3 n^4 + h^2 m n^2 \left(515 m^2 - 4612 m n + 4405 n^2\right) +\right.$$
$$4 n^4 \left(124 m^3 - 385 m^2 n + 216 m n^2 - 18 n^3\right) + b^2$$
$$\left(515 h^2 m n^2 + 4 (372 m - 385 n) n^4\right) + b$$
$$\left.\left(9 h^4 m^2 + 2 h^2 m (515 m - 2306 n) n^2 + 8 n^4 \left(186 m^2 - 385 m n + 108 n^2\right)\right)\right) \omega_0^4 -$$
$$i \left(388 b^3 n^4 - h^4 m^2 (37 m + 25 n) + h^2 m n^2 \left(46 m^2 - 3383 m n + 6658 n^2\right) +\right.$$
$$4 n^4 \left(97 m^3 - 620 m^2 n + 693 m n^2 - 144 n^3\right) + 2 b^2$$
$$\left(23 h^2 m n^2 + 2 (291 m - 620 n) n^4\right) + b \left(-37 h^4 m^2 +\right.$$
$$\left.\left.h^2 m (92 m - 3383 n) n^2 + 4 n^4 \left(291 m^2 - 1240 m n + 693 n^2\right)\right)\right) \omega_0^5 +$$
$$\left(-90 h^4 m^2 + h^2 m n \left(137 b^2 + 274 b m + 137 m^2 + 506 b n + 506 m n - 4721 n^2\right) -\right.$$
$$4 n^3 \left(16 b^3 + 16 m^3 + b^2 (48 m - 485 n) - 485 m^2 n +\right.$$
$$\left.\left.1116 m n^2 - 462 n^3 + 2 b \left(24 m^2 - 485 m n + 558 n^2\right)\right)\right) \omega_0^6 -$$
$$i \left(116 b^3 n^2 + h^2 m \left(50 m^2 - 627 m n - 752 n^2\right) + 4 n^2 \left(29 m^3 + 80 m^2 n -\right.\right.$$
$$\left.873 m n^2 + 744 n^3\right) + b^2 \left(50 h^2 m + 4 n^2 (87 m + 80 n)\right) + b$$
$$\left.\left(h^2 m (100 m - 627 n) + 4 n^2 \left(87 m^2 + 160 m n - 873 n^2\right)\right)\right) \omega_0^7 +$$
$$\left(15 h^2 m (16 m - 51 n) - 80 b^3 n + 20 b^2 n (-12 m + 29 n) + 4 n \left(-20 m^3 + 145 m^2 n +\right.\right.$$
$$\left.\left.144 m n^2 - 582 n^3\right) + 8 b \left(30 h^2 m + n \left(-30 m^2 + 145 m n + 72 n^2\right)\right)\right) \omega_0^8 +$$
$$2 i \left(8 b^3 + 147 h^2 m + 8 m^3 + 8 b^2 (3 m - 25 n) - 200 m^2 n + 522 m\right.$$
$$\left.n^2 + 192 n^3 + b \left(24 m^2 - 400 m n + 522 n^2\right)\right) \omega_0^9 -$$
$$8 \left(10 b^2 + 20 b m + 10 m^2 - 90 b n - 90 m n + 87 n^2\right) \omega_0^{10} -$$
$$\left.48 i (3 b + 3 m - 10 n) \omega_0^{11} + 96 \omega_0^{12}\right) \Big/$$
$$\left(\left(-h^2 m + 2 i (b+m) n \omega_0 + 4 (b + m - n) \omega_0^2 + 8 i \omega_0^3\right)^2\right.$$
$$\left(-n \left(b^2 n^2 + 2 b m \left(-h^2 + n^2\right) + m \left(m n^2 + h^2 (-2 m + n)\right)\right) +\right.$$
$$3 i \left(b^2 n^2 - b \left(h^2 m + n^2 (-2 m + n)\right) - m \left(h^2 (m - 2 n) + n^2 (-m + n)\right)\right) \omega_0 +$$
$$3 \left(2 h^2 m + n \left(b^2 + 2 b m + m^2 - 3 b n - 3 m n + n^2\right)\right) \omega_0^2 -$$
$$\left.\left.i \left(b^2 + 2 b m + m^2 - 9 b n - 9 m n + 9 n^2\right) \omega_0^3 + 3 (b + m - 3 n) \omega_0^4 + 3 i \omega_0^5\right)\right) -$$
$$\left(2 h^2 m \left(i n + \omega_0\right)^2 \left(2 h^2 m - 3 (b+m) n^2 + i (13 (b+m) - 12 n) n \omega_0 +\right.\right.$$

$$12\ (b+m-4\ n)\ \omega_0^2+44\ i\ \omega_0^3\Big)\ \Big(12\ h^2\ m\ n^2+$$

$$\omega_0\ \Big(-i\ n\ \Big(7\ h^2\ m+15\ (b+m)\ n^2\Big)+\omega_0\ \Big(3\ h^2\ m+5\ n^2\ (-7\ (b+m)+6\ n)+$$

$$\omega_0\ \Big(5\ i\ (b+m-14\ n)\ n-\Big(7\ (b+m)+10\ n+14\ i\ \omega_0\Big)\ \omega_0\Big)\Big)\Big)\Big)\Big)\Big/$$

$$\Big(\Big(-h^2\ m+2\ \Big(b+m+2\ i\ \omega_0\Big)\ \omega_0\ \Big(i\ n+2\ \omega_0\Big)\Big)^3\ \Big(h^2\ m-(b+m)\ n^2+$$

$$\omega_0\ \Big(2\ i\ (b+m-n)\ n+\Big(b+m-4\ n+2\ i\ \omega_0\Big)\ \omega_0\Big)\Big)\Big)+$$

$$1\Big/\Big(h^2\ m-2\ i\ \Big(n-2\ i\ \omega_0\Big)\ \Big(b+m+2\ i\ \omega_0\Big)\ \omega_0\Big)^2\ \Big(i\ n+\omega_0\Big)$$

$$\Big(-21\ i\ h^2\ m\ n^3+\omega_0\ \Big(-49\ h^2\ m\ n^2+6\ (b+m)\ n^4+\omega_0\ \Big(3\ i$$

$$\Big(5\ h^2\ m\ n-6\ (b+m)\ n^3+4\ n^4\Big)+\omega_0\ \Big(-9\ h^2\ m+2\ n^2\ (-5\ (b+m)+18\ n)-$$

$$2\ i\ n\ (3\ (b+m)+10\ n)\ \omega_0-4\ (b+m-3\ n)\ \omega_0^2-8\ i\ \omega_0^3\Big)\Big)\Big)\Big)+$$

$$\Big(h^2\ m\ \Big(i\ n+\omega_0\Big)\ \Big(21\ i\ h^2\ m\ n^3+\omega_0\ \Big(80\ h^2\ m\ n^2-15\ (b+m)\ n^4+\omega_0\ \Big(-i\ n\ \Big(79\ h^2\ m+$$

$$n^2\ (-61\ (b+m)+39\ n)\Big)+\omega_0\ \Big(-12\ h^2\ m+(61\ (b+m)-155\ n)\ n^2+$$

$$\omega_0\ \Big(13\ i\ n\ (b+m+11\ n)+\Big(28\ (b+m)-59\ n+86\ i\ \omega_0\Big)\ \omega_0\Big)\Big)\Big)\Big)\Big)\Big)\Big/$$

$$\Big(\Big(h^2\ m-2\ \Big(b+m+2\ i\ \omega_0\Big)\ \omega_0\ \Big(i\ n+2\ \omega_0\Big)\Big)^2\ \Big(-h^2\ m+3\ \Big(b+m+3\ i\ \omega_0\Big)$$

$$\omega_0\ \Big(i\ n+3\ \omega_0\Big)\Big)\Big)\Big)+$$

$$1\Bigg/\left(1+\frac{i\ \omega_0}{m+i\ \omega_0}+\frac{b+\frac{h^2\ m}{(i\ n+\omega_0)^2}}{m+i\ \omega_0}\right)\ \Big(1\Big/\Big(i\ m+\omega_0\Big)\ \Big(m-i\ \omega_0\Big)$$

$$\Big(\Big(3\ h^2\ m\ n^3\ \Big(10\ h^4\ m^2+h^2\ m\ (11\ b+11\ m-21\ n)\ n^2+3\ (b+m)^2\ n^4\Big)-$$

$$i\ \Big(69\ h^6\ m^3\ n^2+h^4\ m^2\ (175\ b+175\ m-397\ n)\ n^4+$$

$$7\ h^2\ m\ (b+m)\ (14\ b+14\ m-15\ n)\ n^6+12\ (b+m)^3\ n^8\Big)\ \omega_0-$$

$$\Big(38\ h^6\ m^3\ n+6\ h^4\ m^2\ (57\ b+57\ m-170\ n)\ n^3+12\ (b+m)^2\ (8\ b+8\ m-5\ n)\ n^7+$$

$$3\ h^2\ m\ n^5\ \Big(127\ b^2+254\ b\ m+127\ m^2-294\ b\ n-294\ m\ n+61\ n^2\Big)\Big)\ \omega_0^2+$$

$$i\ \Big(3\ h^6\ m^3+2\ h^4\ m^2\ (127\ b+127\ m-661\ n)\ n^2+4\ (b+m)\ n^6$$

$$\Big(77\ b^2+77\ m^2+2\ b\ (77\ m-60\ n)-120\ m\ n+27\ n^2\Big)+$$

$$h^2\ m\ n^4\ \Big(674\ b^2+1348\ b\ m+674\ m^2-2899\ b\ n-2899\ m\ n+1428\ n^2\Big)\Big)\ \omega_0^3+$$

$$n\ \Big(h^4\ m^2\ (9\ m-767\ n)+496\ b^3\ n^4+h^2\ m\ n^2\ \Big(515\ m^2-4612\ m\ n+4405\ n^2\Big)+$$

$$4\ n^4\ \Big(124\ m^3-385\ m^2\ n+216\ m\ n^2-18\ n^3\Big)+$$

$$b^2\ \Big(515\ h^2\ m\ n^2+4\ (372\ m-385\ n)\ n^4\Big)+b\ \Big(9\ h^4\ m^2+$$

$$2\ h^2\ m\ (515\ m-2306\ n)\ n^2+8\ n^4\ \Big(186\ m^2-385\ m\ n+108\ n^2\Big)\Big)\Big)\ \omega_0^4-$$

$$i\ \Big(388\ b^3\ n^4-h^4\ m^2\ (37\ m+25\ n)+h^2\ m\ n^2\ \Big(46\ m^2-3383\ m\ n+6658\ n^2\Big)+$$

$$4\ n^4\ \Big(97\ m^3-620\ m^2\ n+693\ m\ n^2-144\ n^3\Big)+$$

$$2\ b^2\ \Big(23\ h^2\ m\ n^2+2\ (291\ m-620\ n)\ n^4\Big)+b\ \Big(-37\ h^4\ m^2+$$

$$h^2\ m\ (92\ m-3383\ n)\ n^2+4\ n^4\ \Big(291\ m^2-1240\ m\ n+693\ n^2\Big)\Big)\Big)\ \omega_0^5+$$

$$\Big(-90\ h^4\ m^2+h^2\ m\ n\ \Big(137\ b^2+274\ b\ m+137\ m^2+506\ b\ n+506\ m\ n-4721\ n^2\Big)-$$

$$4\ n^3\ \Big(16\ b^3+16\ m^3+b^2\ (48\ m-485\ n)-485\ m^2\ n+$$

$$1116\ m\ n^2-462\ n^3+2\ b\ \Big(24\ m^2-485\ m\ n+558\ n^2\Big)\Big)\Big)\ \omega_0^6-$$

$$i\ \Big(116\ b^3\ n^2+h^2\ m\ \Big(50\ m^2-627\ m\ n-752\ n^2\Big)+4\ n^2\ \Big(29\ m^3+80\ m^2\ n-$$

$$873\ m\ n^2+744\ n^3\Big)+b^2\ \Big(50\ h^2\ m+4\ n^2\ (87\ m+80\ n)\Big)+$$

$$b\ \Big(h^2\ m\ (100\ m-627\ n)+4\ n^2\ \Big(87\ m^2+160\ m\ n-873\ n^2\Big)\Big)\Big)\ \omega_0^7+$$

$$\Big(15\ h^2\ m\ (16\ m-51\ n)-80\ b^3\ n+20\ b^2\ n\ (-12\ m+29\ n)+$$

$$4\ n\ \Big(-20\ m^3+145\ m^2\ n+144\ m\ n^2-582\ n^3\Big)+$$

$$8\ b\ \Big(30\ h^2\ m+n\ \Big(-30\ m^2+145\ m\ n+72\ n^2\Big)\Big)\Big)\ \omega_0^8+$$

$$2\ i\ \Big(8\ b^3+147\ h^2\ m+8\ m^3+8\ b^2\ (3\ m-25\ n)-200\ m^2\ n+522\ m\ n^2+$$

$$192\ n^3+b\ \Big(24\ m^2-400\ m\ n+522\ n^2\Big)\Big)\ \omega_0^9-8\ \Big(10\ b^2+20\ b\ m+10\ m^2-$$

$$90\ b\ n-90\ m\ n+87\ n^2\Big)\ \omega_0^{10}-48\ i\ (3\ b+3\ m-10\ n)\ \omega_0^{11}+96\ \omega_0^{12}\Big)\Big/$$

$$\Big(\big(-h^2\,m+2\,\mathrm{i}\,(b+m)\,n\,\omega_0+4\,(b+m-n)\,\omega_0^2+8\,\mathrm{i}\,\omega_0^3\big)^2$$
$$\Big(-n\,\big(b^2\,n^2+2\,b\,m\,\big(-h^2+n^2\big)+m\,\big(m\,n^2+h^2\,(-2\,m+n)\big)\big)+$$
$$3\,\mathrm{i}\,\big(b^2\,n^2-b\,\big(h^2\,m+n^2\,(-2\,m+n)\big)-m\,\big(h^2\,(m-2\,n)+n^2\,(-m+n)\big)\big)\,\omega_0+$$
$$3\,\big(2\,h^2\,m+n\,\big(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2\big)\big)\,\omega_0^2-$$
$$\mathrm{i}\,\big(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\big)\,\omega_0^3+3\,(b+m-3\,n)\,\omega_0^4+3\,\mathrm{i}\,\omega_0^5\Big)\Big)-$$
$$\Big(2\,h^2\,m\,\big(\mathrm{i}\,n+\omega_0\big)^2\,\big(2\,h^2\,m-3\,(b+m)\,n^2+\mathrm{i}\,(13\,(b+m)-12\,n)\,n\,\omega_0+$$
$$12\,(b+m-4\,n)\,\omega_0^2+44\,\mathrm{i}\,\omega_0^3\big)\,\big(12\,h^2\,m\,n^2+$$
$$\omega_0\,\big(-\mathrm{i}\,n\,\big(7\,h^2\,m+15\,(b+m)\,n^2\big)+\omega_0\,\big(3\,h^2\,m+5\,n^2\,(-7\,(b+m)+6\,n)+$$
$$\omega_0\,\big(5\,\mathrm{i}\,(b+m-14\,n)\,n-\big(7\,(b+m)+10\,n+14\,\mathrm{i}\,\omega_0\big)\,\omega_0\big)\big)\big)\big)\Big)\Big/$$
$$\Big(\big(-h^2\,m+2\,\big(b+m+2\,\mathrm{i}\,\omega_0\big)\,\omega_0\,\big(\mathrm{i}\,n+2\,\omega_0\big)\big)^3\,\big(h^2\,m-(b+m)\,n^2+$$
$$\omega_0\,\big(2\,\mathrm{i}\,(b+m-n)\,n+\big(b+m-4\,n+2\,\mathrm{i}\,\omega_0\big)\,\omega_0\big)\big)\Big)+$$
$$1\Big/\big(h^2\,m-2\,\mathrm{i}\,\big(n-2\,\mathrm{i}\,\omega_0\big)\,\big(b+m+2\,\mathrm{i}\,\omega_0\big)\,\omega_0\big)^2\,\big(\mathrm{i}\,n+\omega_0\big)$$
$$\Big(-21\,\mathrm{i}\,h^2\,m\,n^3+\omega_0\,\big(-49\,h^2\,m\,n^2+6\,(b+m)\,n^4+\omega_0\,\big(3\,\mathrm{i}\,\big(5\,h^2\,m\,n-$$
$$6\,(b+m)\,n^3+4\,n^4\big)+\omega_0\,\big(-9\,h^2\,m+2\,n^2\,(-5\,(b+m)+18\,n)-$$
$$2\,\mathrm{i}\,n\,(3\,(b+m)+10\,n)\,\omega_0-4\,(b+m-3\,n)\,\omega_0^2-8\,\mathrm{i}\,\omega_0^3\big)\big)\big)\Big)+$$
$$\Big(h^2\,m\,\big(\mathrm{i}\,n+\omega_0\big)\,\big(21\,\mathrm{i}\,h^2\,m\,n^3+\omega_0\,\big(80\,h^2\,m\,n^2-15\,(b+m)\,n^4+$$
$$\omega_0\,\big(-\mathrm{i}\,n\,\big(79\,h^2\,m+n^2\,(-61\,(b+m)+39\,n)\big)+$$
$$\omega_0\,\big(-12\,h^2\,m+(61\,(b+m)-155\,n)\,n^2+\omega_0\,\big(13\,\mathrm{i}\,n\,(b+m+11\,n)+$$
$$\big(28\,(b+m)-59\,n+86\,\mathrm{i}\,\omega_0\big)\,\omega_0\big)\big)\big)\big)\big)\Big)\Big/$$
$$\Big(\big(h^2\,m-2\,\big(b+m+2\,\mathrm{i}\,\omega_0\big)\,\omega_0\,\big(\mathrm{i}\,n+2\,\omega_0\big)\big)^2\,\big(-h^2\,m+3\,\big(b+m+3\,\mathrm{i}\,\omega_0\big)$$
$$\omega_0\,\big(\mathrm{i}\,n+3\,\omega_0\big)\big)\big)\Big)+$$
$$\big(-\mathrm{i}\,n+\omega_0\big)\,\Big(\Big(3\,\mathrm{i}\,h^2\,m\,n^2\,\big(5\,h^4\,m^2+h^2\,m\,(8\,b+8\,m-13\,n)\,n^2+3\,(b+m)^2\,n^4\big)+$$
$$\big(12\,h^6\,m^3\,n+h^4\,m^2\,(50\,b+50\,m-191\,n)\,n^3+$$
$$h^2\,m\,(b+m)\,(50\,b+50\,m-27\,n)\,n^5+12\,(b+m)^3\,n^7\big)\,\omega_0-$$
$$\mathrm{i}\,\big(h^6\,m^3+2\,h^4\,m^2\,(19\,b+19\,m-145\,n)\,n^2+48\,(b+m)^3\,n^6+$$
$$h^2\,m\,n^4\,\big(91\,b^2+182\,b\,m+91\,m^2-202\,b\,n-202\,m\,n+27\,n^2\big)\big)\,\omega_0^2-$$
$$n\,\big(44\,b^3\,n^4+4\,b^2\,\big(5\,h^2\,m\,n^2+n^4\,(33\,m+41\,n)\big)+$$
$$b\,\big(-10\,h^4\,m^2+h^2\,m\,(40\,m-437\,n)\,n^2+4\,n^4\,\big(33\,m^2+82\,m\,n-33\,n^2\big)\big)+$$
$$m\,\big(-2\,h^4\,m\,(5\,m+89\,n)+4\,n^4\,\big(11\,m^2+41\,m\,n-33\,n^2\big)+$$
$$h^2\,n^2\,\big(20\,m^2-437\,m\,n+242\,n^2\big)\big)\big)\,\omega_0^3-$$
$$\mathrm{i}\,\big(h^4\,m^2\,(14\,m-9\,n)+40\,b^3\,n^4+h^2\,m\,n^2\,\big(57\,m^2+196\,m\,n-565\,n^2\big)+$$
$$8\,n^4\,\big(5\,m^3-105\,m^2\,n+145\,m\,n^2-21\,n^3\big)+3\,b^2\,\big(19\,h^2\,m\,n^2+40\,(m-7\,n)\,n^4\big)+$$
$$2\,b\,\big(7\,h^4\,m^2+h^2\,m\,n^2\,(57\,m+98\,n)+20\,n^4\,\big(3\,m^2-42\,m\,n+29\,n^2\big)\big)\big)\,\omega_0^4+$$
$$\big(33\,h^4\,m^2+h^2\,m\,n\,\big(-22\,b^2-44\,b\,m-22\,m^2+163\,b\,n+163\,m\,n+332\,n^2\big)-$$
$$4\,n^3\,\big(19\,b^3+19\,m^3+b^2\,(57\,m-385\,n)-385\,m^2\,n+$$
$$931\,m\,n^2-320\,n^3+b\,\big(57\,m^2-770\,m\,n+931\,n^2\big)\big)\big)\,\omega_0^5+$$
$$\mathrm{i}\,\big(16\,b^3\,n^2+h^2\,m\,\big(-5\,m^2-78\,m\,n+123\,n^2\big)+b^2\,\big(-5\,h^2\,m+16\,(3\,m-71\,n)\,n^2\big)+$$
$$8\,n^2\,\big(2\,m^3-142\,m^2\,n+689\,m\,n^2-467\,n^3\big)-$$
$$2\,b\,\big(h^2\,m\,(5\,m+39\,n)-4\,n^2\,\big(6\,m^2-284\,m\,n+689\,n^2\big)\big)\big)\,\omega_0^6-$$
$$\big(20\,b^3\,n+12\,b^2\,n\,(5\,m+9\,n)-h^2\,m\,(19\,m+94\,n)+4\,n\,\big(5\,m^3+27\,m^2\,n-$$
$$895\,m\,n^2+1296\,n^3\big)+b\,\big(-19\,h^2\,m+4\,n\,\big(15\,m^2+54\,m\,n-895\,n^2\big)\big)\big)\,\omega_0^7+$$
$$\mathrm{i}\,\big(8\,b^3+11\,h^2\,m+8\,b^2\,(3\,m-31\,n)+8\,b\,\big(3\,m^2-62\,m\,n-35\,n^2\big)+$$
$$8\,\big(m^3-31\,m^2\,n-35\,m\,n^2+403\,n^3\big)\big)\,\omega_0^8-$$
$$4\,\big(21\,b^2+42\,b\,m+21\,m^2-179\,b\,n-179\,m\,n-64\,n^2\big)\,\omega_0^9-$$

$$8\,i\,(29\,b+29\,m-75\,n)\,\omega_0^{10}+192\,\omega_0^{11}\Big)\Big/$$
$$\Big(\big(-h^2 m+2\,i\,(b+m)\,n\,\omega_0+4\,(b+m-n)\,\omega_0^2+8\,i\,\omega_0^3\big)^2$$
$$\Big(-i\,n\,\big(b^2 n^2+2\,b\,m\,(-h^2+n^2)+m\,(m\,n^2+h^2\,(-2\,m+n))\big)-3\,\big(b^2 n^2-b\,(h^2 m+n^2\,(-2\,m+n))-m\,(h^2\,(m-2\,n)+n^2\,(-m+n))\big)\,\omega_0+3\,i\,\big(2\,h^2 m+n\,(b^2+2\,b\,m+m^2-3\,b\,n-3\,m\,n+n^2)\big)\,\omega_0^2+\big(b^2+2\,b\,m+m^2-9\,b\,n-9\,m\,n+9\,n^2\big)\,\omega_0^3+3\,i\,(b+m-3\,n)\,\omega_0^4-3\,\omega_0^5\Big)\Big)+$$
$$\Big(2\,(i\,n+\omega_0)^2\,\Big(3\,h^2 m\,(b+m)\,n+\omega_0\,\Big(-3\,i\,h^2 m\,(3\,(b+m)-4\,n)+4\,\omega_0\,\big(7\,h^2 m+(b+m)^2\,n+\omega_0\,(-i\,(b+m)\,(b+m-4\,n)+4\,(b+m-n+i\,\omega_0)\,\omega_0)\big)\Big)\Big)$$
$$\Big(12\,h^2 m\,n^2+\omega_0\,\Big(-i\,n\,\big(7\,h^2 m+15\,(b+m)\,n^2\big)+\omega_0\,\Big(3\,h^2 m+5\,n^2\,(-7\,(b+m)+6\,n)+\omega_0\,\big(5\,i\,(b+m-14\,n)\,n-(7\,(b+m)+10\,n+14\,i\,\omega_0)\,\omega_0\big)\Big)\Big)\Big)\Big)\Big/$$
$$\Big(\big(-h^2 m+2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)\big)^3\,\big(h^2 m-(b+m)\,n^2+\omega_0\,(2\,i\,(b+m-n)\,n+(b+m-4\,n+2\,i\,\omega_0)\,\omega_0)\big)\Big)+$$
$$\Big(2\,(i\,n+\omega_0)\,\Big(-6\,i\,h^2 m\,n^2+\omega_0\,\Big(-5\,h^2 m\,n+12\,(b+m)\,n^3+\omega_0\,\Big(-i\,\big(h^2 m+6\,(3\,(b+m)-4\,n)\,n^2\big)+2\,\omega_0\,\big(2\,n\,(b+m+9\,n)+\omega_0\,(-3\,i\,(b+m)+4\,i\,n+6\,\omega_0)\big)\Big)\Big)\Big)\Big)\Big/$$
$$\big(h^2 m-2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)\big)^2+\Big((i\,n+\omega_0)$$
$$\Big(13\,i\,h^4 m^2 n^2+\omega_0\,\Big(h^2 m\,n\,\big(32\,h^2 m-29\,(b+m)\,n^2\big)+\omega_0\,\Big(-i\,\big(19\,h^4 m^2+4\,(b+m)^2 n^4+h^2 m\,n^2\,(-71\,(b+m)+67\,n)\big)+\omega_0\,\Big(n\,\big(-5\,h^2 m\,(b+m)-157\,h^2 m\,n-16\,(b+m)^2 n^2+16\,(b+m)\,n^3\big)+\omega_0\,\Big(i\,\big(63\,h^2 m\,(b+m)-19\,h^2 m\,n+8\,(b+m)^2 n^2-64\,(b+m)\,n^3+16\,n^4\big)+\omega_0\,\Big(-141\,h^2 m+16\,n\,\big(-(b+m)^2-2\,(b+m)\,n+4\,n^2\big)+4\,\omega_0\,\big(i\,(3\,(b+m)^2-16\,(b+m)\,n-8\,n^2)-4\,(3\,(b+m)-4\,n+3\,i\,\omega_0)\,\omega_0\big)\Big)\Big)\Big)\Big)\Big)\Big)\Big/$$
$$\Big(\big(h^2 m-2\,(b+m+2\,i\,\omega_0)\,\omega_0\,(i\,n+2\,\omega_0)\big)^2\,\big(-h^2 m+3\,(b+m+3\,i\,\omega_0)\,\omega_0\,(i\,n+3\,\omega_0)\big)\Big)\Big)$$

G32 = Simplify[G321 + G322 + G323 + G324 + G325,
simplifica
ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0];
números reais

ReG32 =
Simplify[ComplexExpand[Re[G32]], ω_0 ∈ Reals && m > 0 && n < 0 && h > 0 && b > 0]
simplifica expande funções ⋯ parte real números reais

(* Substituindo os valorr de ω_0 e h *)

$\omega_0 = \sqrt{(-n) * (b+m)}$

$\sqrt{-(b+m)\,n}$

```
h = Sqrt[((b + m) n (-b - m + n))/m]
```

$$\sqrt{\frac{(b+m)\,n\,(-b-m+n)}{m}}$$

```
(* Primeiro coeficiente de Lyapunov em função dos parâmteros * )

l1 = FullSimplify[l1, m > 0 && n < 0 &&b > 0]
```
simplifica completamente

$$\left(2\,m^3\,n\left((b+m)^2+12\,(b+m)\,n-n^2\right)\right)\Big/\left((b+m-n)\left(-m^2+(b+m)\,n\right)\left((b+m)^2-6\,(b+m)\,n+n^2\right)\left((b+m)^2-3\,(b+m)\,n+n^2\right)\right)$$

```
(* Segundo coeficiente de Lyapunov em função dos parâmteros * )

l2 = FullSimplify[1/12 ReG32, m > 0 && n < 0 &&b > 0]
```
simplifica completamente

$$\Big(m^6\,\big(9\,(b+m)^{14}-277\,(b+m)^{13}\,n-608\,(b+m)^{12}\,n^2+54\,271\,(b+m)^{11}\,n^3-522\,318\,(b+m)^{10}\,n^4+2\,396\,645\,(b+m)^9\,n^5-5\,068\,501\,(b+m)^8\,n^6+2\,025\,066\,(b+m)^7\,n^7+4\,809\,361\,(b+m)^6\,n^8-4\,193\,735\,(b+m)^5\,n^9+601\,386\,(b+m)^4\,n^{10}+172\,127\,(b+m)^3\,n^{11}-49\,228\,(b+m)^2\,n^{12}+4015\,(b+m)\,n^{13}-117\,n^{14}\big)\Big)\Big/\Big(9\,(b+m)\,n\left(m^2-(b+m)\,n\right)^2\left((b+m)^2-11\,(b+m)\,n+n^2\right)\left((b+m)^5-10\,(b+m)^4\,n+29\,(b+m)^3\,n^2-29\,(b+m)^2\,n^3+10\,(b+m)\,n^4-n^5\right)^3\Big)$$

Modelo Tridimensional

Cáculo de L1 para n= -1, b=m.

```
(* Substituindo os valores *)

n := -1

m := b

FullSimplify[a]
```
simplifica completamente

$1 + 3\,b$

$\omega_0 = \sqrt{(-n) * (b + m)}$

$\sqrt{2}\ \sqrt{b}$

$h := \sqrt{\frac{(b + m)\ n\ (-b - m + n)}{m}}$

(* Primeiro coeficiente de Lyapunov em função de b *)

l1 = FullSimplify[l1]
simplifica completamente

$$\frac{2\,b^2\,(-1 + 4\,(-6 + b)\,b)}{(2 + b)\,(1 + 2\,b)\,\left(1 + 6\,b + 4\,b^2\right)\,(1 + 4\,b\,(3 + b))}$$

Solve[(-1 + 4 (-6 + b) b) == 0, {b}]
resolve

$\left\{\left\{b \to \frac{1}{2}\left(6 - \sqrt{37}\right)\right\}, \left\{b \to \frac{1}{2}\left(6 + \sqrt{37}\right)\right\}\right\}$

Modelo Tridimensional Cálculo de L2 para $n = -1,\ b = \left(6 + \sqrt{37}\right)/2,\ m = b$

(* Componentes do sistema (2.1))

$f1[x\_, y\_, z\_] := n * x - y^2$

f2[x_, y_, z_] := m * (z - y)

f3[x_, y_, z_] := a * y - b * z + x * y

(*Substituindo os valores*)

n := -1

m := b

$b := \left(6 + \sqrt{37}\right)/2$

```
N[(6 + √37)/2, 50]
valor numérico

6.0413812651491098444998421226010335310424850473932
```

```
(* Pontos de equilíbrio *)
```

```
Solve[n * x - y^2 == 0 && m * (z - y) == 0 && a * y - b * z + x * y == 0, {x, y, z}]
resolve
```

$$\left\{\{x \to 0,\ y \to 0,\ z \to 0\},\ \left\{x \to \frac{1}{2}\left(6+\sqrt{37}-2a\right),\ y \to -\frac{\sqrt{-6-\sqrt{37}+2a}}{\sqrt{2}},\ z \to -\frac{\sqrt{-6-\sqrt{37}+2a}}{\sqrt{2}}\right\},\ \left\{x \to \frac{1}{2}\left(6+\sqrt{37}-2a\right),\ y \to \frac{\sqrt{-6-\sqrt{37}+2a}}{\sqrt{2}},\ z \to \frac{\sqrt{-6-\sqrt{37}+2a}}{\sqrt{2}}\right\}\right\}$$

```
P0 := {b - a, √(n * (b - a)), √(n * (b - a))}
```

```
(* Parte linear do campo de vetores *)
```

```
Df[{x_, y_, z_}] := {{Derivative[1, 0, 0][f1][x, y, z],
                      derivação
    Derivative[0, 1, 0][f1][x, y, z], Derivative[0, 0, 1][f1][x, y, z]},
    derivação                          derivação
   {Derivative[1, 0, 0][f2][x, y, z], Derivative[0, 1, 0][f2][x, y, z],
    derivação                          derivação
    Derivative[0, 0, 1][f2][x, y, z]}, {Derivative[1, 0, 0][f3][x, y, z],
    derivação                            derivação
    Derivative[0, 1, 0][f3][x, y, z], Derivative[0, 0, 1][f3][x, y, z]}}
    derivação                          derivação
```

```
A := Df[P0]
```

```
A
```

$$\left\{\left\{-1,\ -2\sqrt{\frac{1}{2}\left(-6-\sqrt{37}\right)+a},\ 0\right\},\ \left\{0,\ \frac{1}{2}\left(-6-\sqrt{37}\right),\ \frac{1}{2}\left(6+\sqrt{37}\right)\right\},\ \left\{\sqrt{\frac{1}{2}\left(-6-\sqrt{37}\right)+a},\ \frac{1}{2}\left(6+\sqrt{37}\right),\ \frac{1}{2}\left(-6-\sqrt{37}\right)\right\}\right\}$$

```
(* Polinômio Característico *)
```

```
Det[A - λ * IdentityMatrix[3]]
```
determinante matriz identidade

$$\frac{1}{4}\left(-12\sqrt{2}\sqrt{\frac{1}{2}\left(-6-\sqrt{37}\right)+a}\sqrt{-6-\sqrt{37}+2a}-2\sqrt{74}\sqrt{\frac{1}{2}\left(-6-\sqrt{37}\right)+a}\sqrt{-6-\sqrt{37}+2a}-24\lambda-4\sqrt{37}\lambda-28\lambda^2-4\sqrt{37}\lambda^2-4\lambda^3\right)$$

```
pc := λ^3 + (b + m - n) * λ^2 - n * (b + m) * λ + 2 * (b - a) * m * n

FullSimplify[Solve[(b + m - n) * (-n) * (b + m) == 2 * (b - a) * m * n, {a}]]
```
simplifica comple··· resolve

$$\left\{\left\{a\to 10+\frac{3\sqrt{37}}{2}\right\}\right\}$$

```
(* Valor da Bifurcação *)

a := FullSimplify[(b^2 + 4 b m + m^2 - (b + m) n)/(2 m)]
```
simplifica completamente

```
Solve[pc == 0, {λ}]
```
resolve

$$\left\{\left\{\lambda\to -7-\sqrt{37}\right\},\left\{\lambda\to -i\sqrt{6+\sqrt{37}}\right\},\left\{\lambda\to i\sqrt{6+\sqrt{37}}\right\}\right\}$$

```
(* Autovalores *)

Eigenvalues[A] /. -i Sqrt[6 + Sqrt[37]] → -I * ω0 /. i Sqrt[6 + Sqrt[37]] → I * ω0
```
autovalores unidade imaginária unidade

$$\left\{-7-\sqrt{37},\ i\,\omega_0,\ -i\,\omega_0\right\}$$

```
Eigensystem[A] /. Sqrt[6 + Sqrt[37]] → ω0 /. 6 + Sqrt[37] → ω0^2 /. Sqrt[37 (6 + Sqrt[37])] → h1 /.
  Sqrt[7 + Sqrt[37]] → h2 /. -6 - Sqrt[37] → -(ω0^2)
```
autovalores e autovetores

$$\left\{\left\{-7-\sqrt{37},\ i\,\omega_0,\ -i\,\omega_0\right\},\left\{\left\{-\frac{2\,h2}{8+\sqrt{37}},-\frac{\omega_0^2}{8+\sqrt{37}},1\right\},\left\{-\frac{2\left(-i\,h1-6\,i\,\omega_0+\omega_0^2\right)}{\sqrt{7+\sqrt{37}}\left(2\,i\,\omega_0+\omega_0^2\right)},\frac{\omega_0^2}{2\,i\,\omega_0+\omega_0^2},1\right\},\left\{-\frac{2\left(i\,h1+6\,i\,\omega_0+\omega_0^2\right)}{\sqrt{7+\sqrt{37}}\left(-2\,i\,\omega_0+\omega_0^2\right)},\frac{\omega_0^2}{-2\,i\,\omega_0+\omega_0^2},1\right\}\right\}\right\}$$

```
lb1 := I * ω0
```
unidade

```
lb2 := -I * ω0
```
unidade

```
lb3 := -b - m + n
```

```
(* Autovetor q e seu conjudado qb *)
```

$$q := \left\{-\frac{2\left(-\mathrm{i}\,h1 - 6\,\mathrm{i}\,\omega_0 + \omega_0^2\right)}{h2 * \left(2\,\mathrm{i}\,\omega_0 + \omega_0^2\right)}, \frac{\omega_0^2}{2\,\mathrm{i}\,\omega_0 + \omega_0^2}, 1\right\}$$

```
qb = FullSimplify[ComplexExpand[Conjugate[q]]]
```

simplifica comple··· expande funções ··· conjugado

$$\left\{-\frac{2\left(\mathrm{i}\,h1 + \omega_0\left(6\,\mathrm{i} + \omega_0\right)\right)}{h2\,\omega_0\left(-2\,\mathrm{i} + \omega_0\right)}, \frac{\omega_0}{-2\,\mathrm{i} + \omega_0}, 1\right\}$$

$$\text{Eigensystem[Transpose[A]]} \;/.\; \sqrt{6+\sqrt{37}} \to \omega_0 \;/.\; 6+\sqrt{37} \to \omega_0\text{^}2 \;/.$$

autovalores e a··· transposição

$$\sqrt{37\left(6+\sqrt{37}\right)} \to h1 \;/.\; \sqrt{7+\sqrt{37}} \to h2 \;/.\; -6-\sqrt{37} \to -\left(\omega_0\text{^}2\right)$$

$$\left\{\left\{-7-\sqrt{37}, \mathrm{i}\,\omega_0, -\mathrm{i}\,\omega_0\right\},\right.$$

$$\left.\left\{\left\{-\frac{h2}{\omega_0^2}, -\frac{8+\sqrt{37}}{\omega_0^2}, 1\right\}, \left\{-\frac{\mathrm{i}\,h2}{-\mathrm{i}+\omega_0}, -\frac{-2\,\mathrm{i}-\omega_0}{\sqrt{\omega_0^2}}, 1\right\}, \left\{\frac{\mathrm{i}\,h2}{\mathrm{i}+\omega_0}, -\frac{2\,\mathrm{i}-\omega_0}{\sqrt{\omega_0^2}}, 1\right\}\right\}\right\}$$

```
(* Autovertor p e seu conjugado pb*)
```

$$p := \left(1\Big/\left(\frac{2\left(h1 + \omega_0\left(8 + \omega_0\left(2\,\mathrm{i} + \omega_0\right)\right)\right)}{\omega_0\left(2 + \omega_0\left(\mathrm{i} + \omega_0\right)\right)}\right)\right) * \left\{\frac{\mathrm{i}\,h2}{\mathrm{i} + \omega_0}, -\frac{2\,\mathrm{i} - \omega_0}{\omega_0}, 1\right\}$$

$$pb := \left(1\Big/\left(\frac{2\left(h1 + \omega_0\left(8 + \omega_0\left(2\,\mathrm{i} + \omega_0\right)\right)\right)}{\omega_0\left(2 + \omega_0\left(\mathrm{i} + \omega_0\right)\right)}\right)\right) * \left\{-\frac{\mathrm{i}\,h2}{-\mathrm{i} + \omega_0}, 1 + \frac{2\,\mathrm{i}}{\omega_0}, 1\right\}$$

```
(* Verificação da Normalização *)
```

```
FullSimplify[pb.q]
```

simplifica completamente

1

```
(* Funções multilineares simétricas B, C, D e E *)
```

c··· der··· númer

```
(* Função B *)

bb[{x1_, x2_, x3_}, {y1_, y2_, y3_}] := {-2 x2 y2, 0 , x1 * y2 + x2 * y1}

(* Função C *)
      constа

cc[{x1_, x2_, x3_}, {y1_, y2_, y3_}, {u1_, u2_, u3_}] := {0, 0, 0}

(* Função D *)
      derivа

dd[{x1_, x2_, x3_}, {y1_, y2_, y3_}, {u1_, u2_, u3_}, {v1_, v2_, v3_}] := {0, 0, 0}

(* Função E *)
      númer

ee[{x1_, x2_, x3_}, {y1_, y2_, y3_},
  {u1_, u2_, u3_}, {v1_, v2_, v3_}, {w1_, w2_, w3_}] := {0, 0, 0}

(* Parte linear do campo de vetores *)
```

A = Simplify[Df[P0]] /. $\sqrt{6+\sqrt{37}} \to \omega_0$ /. $6+\sqrt{37} \to \omega_0$^2 /.
simplifica
$\sqrt{37\left(6+\sqrt{37}\right)} \to$ h1 /. $\sqrt{7+\sqrt{37}} \to$ h2 /. $-6-\sqrt{37} \to -(\omega_0$^2) /. $\sqrt{37} \to$ r1

$$\left\{\{-1, -2\,h2, 0\}, \left\{0, -3-\frac{r1}{2}, \frac{\omega_0^2}{2}\right\}, \left\{h2, \frac{\omega_0^2}{2}, -3-\frac{r1}{2}\right\}\right\}$$

```
(* Inversa da matriz A *)

AI = FullSimplify[Inverse[A]]
   simplifica comple··· matriz inversa
```

$$\left\{\left\{\frac{-(6+r1)^2+\omega_0^4}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}, \frac{4\,h2\,(6+r1)}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}, \frac{4\,h2\,\omega_0^2}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}\right\},\right.$$
$$\left\{-\frac{2\,h2\,\omega_0^2}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}, -\frac{2\,(6+r1)}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}, -\frac{2\,\omega_0^2}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}\right\},$$
$$\left.\left\{-\frac{2\,h2\,(6+r1)}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}, \frac{8\,h2^2-2\,\omega_0^2}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}, -\frac{2\,(6+r1)}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}\right\}\right\}$$

(* Matriz D2 = 2 $i\,\omega_0$ I *)
unidad

D2 = 2 $i\,\omega_0$ IdentityMatrix[3]
matriz identidade

$$\{\{2\,i\,\omega_0, 0, 0\}, \{0, 2\,i\,\omega_0, 0\}, \{0, 0, 2\,i\,\omega_0\}\}$$

```
(* Matriz DA = 2iω0I-A *)
```

unidade ir

```
DA = D2 - A
```

$$\left\{\{1+2\,\mathrm{i}\,\omega_0,\ 2\,\mathrm{h2},\ 0\},\ \left\{0,\ 3+\frac{\mathrm{r1}}{2}+2\,\mathrm{i}\,\omega_0,\ -\frac{\omega_0^2}{2}\right\},\ \left\{-\mathrm{h2},\ -\frac{\omega_0^2}{2},\ 3+\frac{\mathrm{r1}}{2}+2\,\mathrm{i}\,\omega_0\right\}\right\}$$

```
(* Inversa da matriz DA *)
```

```
DAI = FullSimplify[Inverse[DA]]
```

simplifica comple··· matriz inversa

$$\Big\{\Big\{\big((6+\mathrm{r1})^2+\omega_0\,(8\,\mathrm{i}\,(6+\mathrm{r1})-\omega_0\,(16+\omega_0^2))\big)\Big/\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big),$$

$$-\Big((4\,\mathrm{h2}\,(6+\mathrm{r1}+4\,\mathrm{i}\,\omega_0))\Big/\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big),$$

$$-\Big((4\,\mathrm{h2}\,\omega_0^2)\Big/\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big)\Big\},\ \Big\{(2\,\mathrm{h2}\,\omega_0^2)\Big/\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big),$$

$$-\Big((2\,(6+\mathrm{r1}+4\,\mathrm{i}\,\omega_0)\,(-\mathrm{i}+2\,\omega_0))\Big/\big(\mathrm{i}\,(6+\mathrm{r1})^2+\omega_0\,(-2\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,\mathrm{i}\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))+\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big),$$

$$(2\,(\mathrm{i}-2\,\omega_0)\,\omega_0^2)\Big/\big(\mathrm{i}\,(6+\mathrm{r1})^2+\omega_0\,(-2\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,\mathrm{i}\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))+\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big\},$$

$$\Big\{(2\,\mathrm{h2}\,(6+\mathrm{r1}+4\,\mathrm{i}\,\omega_0))\Big/\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big),$$

$$(-8\,\mathrm{h2}^2+2\,\omega_0^2+4\,\mathrm{i}\,\omega_0^3)\Big/\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big),$$

$$-\Big((2\,(6+\mathrm{r1}+4\,\mathrm{i}\,\omega_0)\,(-\mathrm{i}+2\,\omega_0))\Big/\big(\mathrm{i}\,(6+\mathrm{r1})^2+\omega_0\,(-2\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,\mathrm{i}\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))+\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big)\Big\}\Big\}$$

```
(* Calculo do vetor complexo h20 *)
```

```
h20 = FullSimplify[DAI.bb[q, q]]
```

simplifica completamente

$$\Big\{-\Big((2\,\mathrm{i}\,\omega_0^2\,(8\,\mathrm{h1}-\mathrm{i}\,(6+\mathrm{r1})^2+\omega_0\,(8\,(12+\mathrm{r1})+\mathrm{i}\,\omega_0\,(24+\omega_0^2))))\Big/\Big((2\,\mathrm{i}+\omega_0)^2\,\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big)\Big),$$

$$\big(4\,\omega_0^2\,(2\,\mathrm{i}\,\mathrm{h1}+\omega_0\,(-4\,(-3\,\mathrm{i}+\mathrm{h1})-(26+\mathrm{h2}^2+4\,\mathrm{i}\,\omega_0)\,\omega_0))\big)\Big/\Big(\mathrm{h2}\,(2\,\mathrm{i}+\omega_0)^2\,\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big),$$

$$-\Big(\big(4\,(6+\mathrm{r1}+4\,\mathrm{i}\,\omega_0)\,(-2\,\mathrm{i}\,\mathrm{h1}+\omega_0\,(4\,(-3\,\mathrm{i}+\mathrm{h1})+(26+\mathrm{h2}^2+4\,\mathrm{i}\,\omega_0)\,\omega_0))\big)\Big/\Big(\mathrm{h2}\,(2\,\mathrm{i}+\omega_0)^2\,\big((6+\mathrm{r1})^2+\omega_0\,(2\,\mathrm{i}\,(6+\mathrm{r1})\,(10+\mathrm{r1})+\omega_0\,(4\,(\mathrm{h2}^2-4\,(7+\mathrm{r1}))-\mathrm{i}\,\omega_0\,(32+\omega_0\,(-\mathrm{i}+2\,\omega_0))))\big)\Big)\Big)\Big\}$$

```
(* Vetor complexo h20b *)
```

```
h20b =
 Simplify[ComplexExpand[Conjugate[h20]], ω0 ∈ Reals && h1 > 0 && h2 > 0 && r1 > 0]
```

simplifica | expande funções ⋯ | conjugado | números reais

$$\left\{\left(2\, i\, \omega_0^2 \left(8\, h1 + i\, (6 + r1)^2 + 8\, (12 + r1)\, \omega_0 - 24\, i\, \omega_0^2 - i\, \omega_0^4\right)\right) \Big/ \left(\left(-2\, i + \omega_0\right)^2 \left((6 + r1)^2 - 2\, i \left(60 + 16\, r1 + r1^2\right) \omega_0 + 4 \left(h2^2 - 4\, (7 + r1)\right) \omega_0^2 + 32\, i\, \omega_0^3 - \omega_0^4 + 2\, i\, \omega_0^5\right)\right),\right.$$

$$-\left(\left(4\, \omega_0^2 \left(2\, i\, h1 + 4 \left(3\, i + h1\right) \omega_0 + \left(26 + h2^2\right) \omega_0^2 - 4\, i\, \omega_0^3\right)\right) \Big/ \left(h2 \left(-2\, i + \omega_0\right)^2 \left((6 + r1)^2 - 2\, i \left(60 + 16\, r1 + r1^2\right) \omega_0 + 4 \left(h2^2 - 4\, (7 + r1)\right) \omega_0^2 + 32\, i\, \omega_0^3 - \omega_0^4 + 2\, i\, \omega_0^5\right)\right)\right),$$

$$\left.-\left(\left(4 \left(6 + r1 - 4\, i\, \omega_0\right) \left(2\, i\, h1 + 4 \left(3\, i + h1\right) \omega_0 + \left(26 + h2^2\right) \omega_0^2 - 4\, i\, \omega_0^3\right)\right) \Big/ \left(h2 \left(-2\, i + \omega_0\right)^2 \left((6 + r1)^2 - 2\, i \left(60 + 16\, r1 + r1^2\right) \omega_0 + 4 \left(h2^2 - 4\, (7 + r1)\right) \omega_0^2 + 32\, i\, \omega_0^3 - \omega_0^4 + 2\, i\, \omega_0^5\right)\right)\right)\right\}$$

```
(* Calculo do vetor complexo h11 *)
```

```
h11 = Simplify[-AI.bb[q, qb]]
```

simplifica

$$\left\{-\frac{2\, \omega_0^2 \left(-(6 + r1)^2 + 8\, \omega_0^2 + \omega_0^4\right)}{\left(-2\, i + \omega_0\right) \left(2\, i + \omega_0\right) \left(-(6 + r1)^2 - 4\, h2^2\, \omega_0^2 + \omega_0^4\right)},\right.$$

$$-\frac{4 \left(2 + h2^2\right) \omega_0^4}{h2 \left(-2\, i + \omega_0\right) \left(2\, i + \omega_0\right) \left((6 + r1)^2 + 4\, h2^2\, \omega_0^2 - \omega_0^4\right)},$$

$$\left.-\frac{4 \left(2 + h2^2\right) (6 + r1)\, \omega_0^2}{h2 \left(-2\, i + \omega_0\right) \left(2\, i + \omega_0\right) \left((6 + r1)^2 + 4\, h2^2\, \omega_0^2 - \omega_0^4\right)}\right\}$$

```
(* Cálculo do número complexo G21 *)
```

```
G21 = FullSimplify[pb.(2 bb[q, h11] + bb[qb, h20]),
   ω0 ∈ Reals && h1 > 0 && h2 > 0 && r1 > 0]
```

simplifica completamente

números reais

$$\frac{1}{(2\,i+\omega_0)^2\,(h1+\omega_0\,(8+\omega_0\,(2\,i+\omega_0)))}\,\omega_0^2\,(2+\omega_0\,(i+\omega_0))$$

$$\Bigg(\Big(2\,\omega_0^2\,\big(-4\,i\,h1\,(2+h2^2)-h2^2\,(6+r1)^2-24\,i\,(2+h2^2)\,\omega_0+4\,(2+3\,h2^2)\,\omega_0^2+h2^2\,\omega_0^4\big)\Big)\Big/$$

$$\big(h2^2\,(-2\,i+\omega_0)\,((6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4)\big)+$$

$$\Big(4\,(i\,h1+\omega_0\,(6\,i+\omega_0))\,\big(-2\,i\,h1+\omega_0\,(4\,(-3\,i+h1)+(26+h2^2+4\,i\,\omega_0)\,\omega_0)\big)\Big)\Big/$$

$$\Big(h2^2\,(-2\,i+\omega_0)\,\big((6+r1)^2+\omega_0\,(2\,i\,(6+r1)\,(10+r1)+\omega_0\,(4\,(h2^2-4\,(7+r1))-i\,\omega_0\,(32+\omega_0\,(-i+2\,\omega_0))))\big)\Big)-$$

$$\Big(i\,\omega_0^2\,\big(8\,h1-i\,(6+r1)^2+\omega_0\,(8\,(12+r1)+i\,\omega_0\,(24+\omega_0^2))\big)\Big)\Big/\Big((-2\,i+\omega_0)\,\big((6+r1)^2+\omega_0\,(2\,i\,(6+r1)\,(10+r1)+\omega_0\,(4\,(h2^2-4\,(7+r1))-i\,\omega_0\,(32+\omega_0\,(-i+2\,\omega_0))))\big)\Big)-$$

$$\Bigg(4\,i\,\omega_0^2\,\Bigg(\frac{2\,(2+h2^2)\,\omega_0^2}{(6+r1)^2+4\,h2^2\,\omega_0^2-\omega_0^4}+\big(2\,h1+\omega_0\,(12+4\,i\,h1+i\,(26+h2^2+4\,i\,\omega_0)\,\omega_0)\big)\Big/$$

$$\big(i\,(6+r1)^2+\omega_0\,(-2\,(6+r1)\,(10+r1)+\omega_0\,(4\,i\,(h2^2-4\,(7+r1))+\omega_0\,(32+\omega_0\,(-i+2\,\omega_0))))\big)\Bigg)\Bigg)\Bigg/\,(-2+\omega_0\,(-3\,i+\omega_0))\Bigg)$$

$$h1 := \sqrt{37\,(6+\sqrt{37})}$$

$$h2 := \sqrt{7+\sqrt{37}}$$

$$r1 := \sqrt{37}$$

G21

$$\Bigg(\omega_0^2\,(2+\omega_0\,(i+\omega_0))\Bigg(\Bigg(2\,\omega_0^2\Big(-\left(6+\sqrt{37}\right)^2\left(7+\sqrt{37}\right)-4\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\,\left(9+\sqrt{37}\right)-24\,i\,\left(9+\sqrt{37}\right)\omega_0+4\left(2+3\left(7+\sqrt{37}\right)\right)\omega_0^2+\left(7+\sqrt{37}\right)\omega_0^4\Big)\Bigg)\Big/\left(\left(7+\sqrt{37}\right)(-2\,i+\omega_0)\left(\left(6+\sqrt{37}\right)^2+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)\right)+\Bigg(4\left(i\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\,(6\,i+\omega_0)\right)\Big(-2\,i\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\Big(4\Big(-3\,i+\sqrt{37\left(6+\sqrt{37}\right)}\Big)+\left(33+\sqrt{37}+4\,i\,\omega_0\right)\omega_0\Big)\Big)\Bigg)\Big/\Big(\left(7+\sqrt{37}\right)(-2\,i+\omega_0)\Big(\left(6+\sqrt{37}\right)^2+\omega_0\Big(2\,i\left(6+\sqrt{37}\right)\left(10+\sqrt{37}\right)+\omega_0\left(4\left(7+\sqrt{37}-4\left(7+\sqrt{37}\right)\right)-i\,\omega_0\,(32+\omega_0\,(-i+2\,\omega_0))\right)\Big)\Big)\Big)-\Bigg(i\,\omega_0^2\Big(-i\left(6+\sqrt{37}\right)^2+8\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8\left(12+\sqrt{37}\right)+i\,\omega_0\left(24+\omega_0^2\right)\right)\Big)\Bigg)\Big/\Big((-2\,i+\omega_0)\Big(\left(6+\sqrt{37}\right)^2+\omega_0\Big(2\,i\left(6+\sqrt{37}\right)\left(10+\sqrt{37}\right)+\omega_0\left(4\left(7+\sqrt{37}-4\left(7+\sqrt{37}\right)\right)-i\,\omega_0\,(32+\omega_0\,(-i+2\,\omega_0))\right)\Big)\Big)\Big)-\Bigg(4\,i\,\omega_0^2\Bigg(\frac{2\left(9+\sqrt{37}\right)\omega_0^2}{\left(6+\sqrt{37}\right)^2+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4}+\Big(2\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\Big(12+4\,i\sqrt{37\left(6+\sqrt{37}\right)}+i\left(33+\sqrt{37}+4\,i\,\omega_0\right)\omega_0\Big)\Big)\Big/\Big(i\left(6+\sqrt{37}\right)^2+\omega_0\Big(-2\left(6+\sqrt{37}\right)\left(10+\sqrt{37}\right)+\omega_0\left(4\,i\left(7+\sqrt{37}-4\left(7+\sqrt{37}\right)\right)+\omega_0\,(32+\omega_0\,(-i+2\,\omega_0))\right)\Big)\Big)\Bigg)\Bigg)\Big/(-2+\omega_0\,(-3\,i+\omega_0))\Bigg)\Bigg)\Big/\left((2\,i+\omega_0)^2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\,(8+\omega_0\,(2\,i+\omega_0))\right)\right)$$

(* Cálculo do número complexo G21b *)

```
G21b = Simplify[ComplexExpand[Conjugate[G21]], ω0 ∈ Reals]
```

simplifica | expande funções ⋯ | conjugado | números r

$$
\begin{aligned}
&\Bigg(\omega_0^2 \Bigg(296\,\mathrm{i}\,\sqrt{6+\sqrt{37}}\,\left(783\,253+128\,766\,\sqrt{37}\right) + \\
&\quad 296\left(2\,575\,320\,\mathrm{i}+423\,380\,\mathrm{i}\,\sqrt{37}+264\,661\,\sqrt{6+\sqrt{37}}+43\,510\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0 + \\
&\quad \Bigg(918\,687\,541\,\mathrm{i}\,\sqrt{6+\sqrt{37}}+151\,031\,303\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)} - \\
&\qquad 148\left(7\,712\,803+1\,267\,977\,\sqrt{37}\right)\Bigg)\omega_0^2 + \Bigg(200\,231\,087\,\sqrt{6+\sqrt{37}} + \\
&\qquad 32\,917\,733\,\sqrt{37\left(6+\sqrt{37}\right)}+4\,\mathrm{i}\left(861\,767\,877+141\,673\,765\,\sqrt{37}\right)\Bigg)\omega_0^3 + \\
&\quad \Bigg(-4\,330\,586\,350-711\,943\,818\,\sqrt{37}-346\,318\,742\,\mathrm{i}\,\sqrt{6+\sqrt{37}} - \\
&\qquad 56\,933\,338\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^4 + \Bigg(-2\,234\,198\,824\,\sqrt{6+\sqrt{37}} + \\
&\qquad 7\left(341\,137\,879\,\mathrm{i}+56\,082\,177\,\mathrm{i}\,\sqrt{37}-52\,471\,040\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\Bigg)\omega_0^5 + \\
&\quad \Bigg(-2\,379\,433\,023-391\,179\,305\,\sqrt{37}+1\,011\,761\,633\,\mathrm{i}\,\sqrt{6+\sqrt{37}} + \\
&\qquad 166\,289\,163\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^6 + \Bigg(-235\,686\,929\,\sqrt{6+\sqrt{37}} - \\
&\qquad 38\,787\,819\,\sqrt{37\left(6+\sqrt{37}\right)}+2\,\mathrm{i}\left(92\,903\,155+15\,272\,441\,\sqrt{37}\right)\Bigg)\omega_0^7 + \\
&\quad \Bigg(-552\,132\,722-90\,773\,302\,\sqrt{37}-27\,758\,362\,\mathrm{i}\,\sqrt{6+\sqrt{37}} - \\
&\qquad 4\,786\,422\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^8 + \\
&\quad \left(23\,088\,000\,\sqrt{6+\sqrt{37}}+3\,277\,320\,\sqrt{37\left(6+\sqrt{37}\right)}+23\,\mathrm{i}\left(2\,787\,311+461\,657\,\sqrt{37}\right)\right) \\
&\quad \omega_0^9 + \Bigg(21\,069\,437+3\,709\,547\,\sqrt{37} - \\
&\qquad 8\,392\,081\,\mathrm{i}\,\sqrt{6+\sqrt{37}}-1\,030\,331\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{10} - \\
&\quad \left(2\,379\,090\,\mathrm{i}+606\,598\,\mathrm{i}\,\sqrt{37}+661\,819\,\sqrt{6+\sqrt{37}}+171\,977\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{11} +
\end{aligned}
$$

$$\left(2\,164\,006 + 418\,898\sqrt{37} + 694\,934\,i\sqrt{6+\sqrt{37}} + 204\,410\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{12} -$$

$$\left(-43\,387\,i + 78\,883\,i\sqrt{37} + 115\,144\sqrt{6+\sqrt{37}} + 14\,768\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{13} +$$

$$\left(-402\,493 - 71\,323\sqrt{37} + 81\,659\,i\sqrt{6+\sqrt{37}} + 19\,241\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{14} +$$

$$\left(218\,018\,i + 26\,854\,i\sqrt{37} + 8029\sqrt{6+\sqrt{37}} + 1807\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{15} +$$

$$2\left(777\,i\sqrt{6+\sqrt{37}} + 179\,i\sqrt{37\left(6+\sqrt{37}\right)} + 85\left(93+7\sqrt{37}\right)\right)\omega_0^{16} +$$

$$\left(13\,355\,i + 1853\,i\sqrt{37} + 296\sqrt{6+\sqrt{37}} + 56\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{17} +$$

$$\left.\left.\left(2875 + 397\sqrt{37}\right)\omega_0^{18} + 26\,i\left(7+\sqrt{37}\right)\omega_0^{19} + 8\left(7+\sqrt{37}\right)\omega_0^{20}\right)\right) \Big/$$

$$\left(\left(7+\sqrt{37}\right)\left(4+\omega_0^2\right)\left(73 + 12\sqrt{37} + 4\left(7+\sqrt{37}\right)\omega_0^2 - \omega_0^4\right)\right.$$

$$\left(37\left(6+\sqrt{37}\right) + 16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0 + 64\,\omega_0^2 + 2\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^3 + 20\,\omega_0^4 + \omega_0^6\right)$$

$$\left(10\,657 + 1752\sqrt{37} + \left(52\,604 + 8648\sqrt{37}\right)\omega_0^2 -\right.$$

$$\left.\left.2\left(89 + 28\sqrt{37}\right)\omega_0^4 - 104\left(-4+\sqrt{37}\right)\omega_0^6 + 129\,\omega_0^8 + 4\,\omega_0^{10}\right)\right)$$

(* Cálculo da parte real do número complexo G21 *)

```
ReG21 = Simplify[ComplexExpand[Re[G21]], ω0 ∈ Reals]
```

simplifica · expande funções ⋯ · parte real · números r

$$\Big(\omega_0^3 \Big(296\sqrt{6+\sqrt{37}}\,\big(264\,661+43\,510\sqrt{37}\big) - 148\big(7\,712\,803+1\,267\,977\sqrt{37}\big)\omega_0 + \sqrt{6+\sqrt{37}}\,\big(200\,231\,087+32\,917\,733\sqrt{37}\big)\omega_0^2 - 2\big(2\,165\,293\,175+355\,971\,909\sqrt{37}\big)\omega_0^3 - 8\sqrt{6+\sqrt{37}}\,\big(279\,274\,853+45\,912\,160\sqrt{37}\big)\omega_0^4 - 11\big(216\,312\,093+35\,561\,755\sqrt{37}\big)\omega_0^5 - \sqrt{6+\sqrt{37}}\,\big(235\,686\,929+38\,787\,819\sqrt{37}\big)\omega_0^6 - 2\big(276\,066\,361+45\,386\,651\sqrt{37}\big)\omega_0^7 + 120\sqrt{6+\sqrt{37}}\,\big(192\,400+27\,311\sqrt{37}\big)\omega_0^8 + \big(21\,069\,437+3\,709\,547\sqrt{37}\big)\omega_0^9 - \sqrt{6+\sqrt{37}}\,\big(661\,819+171\,977\sqrt{37}\big)\omega_0^{10} + \big(2\,164\,006+418\,898\sqrt{37}\big)\omega_0^{11} - 8\sqrt{6+\sqrt{37}}\,\big(14\,393+1846\sqrt{37}\big)\omega_0^{12} - 7\big(57\,499+10\,189\sqrt{37}\big)\omega_0^{13} + \sqrt{6+\sqrt{37}}\,\big(8029+1807\sqrt{37}\big)\omega_0^{14} + 170\big(93+7\sqrt{37}\big)\omega_0^{15} + 8\sqrt{6+\sqrt{37}}\,\big(37+7\sqrt{37}\big)\omega_0^{16} + \big(2875+397\sqrt{37}\big)\omega_0^{17} + 8\big(7+\sqrt{37}\big)\omega_0^{19}\Big)\Big) \Big/$$

$$\Big(\big(7+\sqrt{37}\big)\big(4+\omega_0^2\big)\big(73+12\sqrt{37}+4\big(7+\sqrt{37}\big)\omega_0^2-\omega_0^4\big)\Big(37\big(6+\sqrt{37}\big)+16\sqrt{37\big(6+\sqrt{37}\big)}\,\omega_0+64\,\omega_0^2+2\sqrt{37\big(6+\sqrt{37}\big)}\,\omega_0^3+20\,\omega_0^4+\omega_0^6\Big)\Big(10\,657+1752\sqrt{37}+\big(52\,604+8648\sqrt{37}\big)\omega_0^2 - 2\big(89+28\sqrt{37}\big)\omega_0^4 - 104\big(-4+\sqrt{37}\big)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)\Big)$$

```
(* Cálculo de l1 *)
```

$$\mathbf{l1} = \frac{1}{2}\,\mathbf{Simplify[ReG21]}$$
simplifica

$$\begin{aligned}
&\Big(\omega_0^3\Big(296\sqrt{6+\sqrt{37}}\,\left(264\,661+43\,510\sqrt{37}\right)-\\
&\quad 148\left(7\,712\,803+1\,267\,977\sqrt{37}\right)\omega_0+\sqrt{6+\sqrt{37}}\,\left(200\,231\,087+32\,917\,733\sqrt{37}\right)\omega_0^2-\\
&\quad 2\left(2\,165\,293\,175+355\,971\,909\sqrt{37}\right)\omega_0^3-8\sqrt{6+\sqrt{37}}\\
&\quad \left(279\,274\,853+45\,912\,160\sqrt{37}\right)\omega_0^4-11\left(216\,312\,093+35\,561\,755\sqrt{37}\right)\omega_0^5-\\
&\quad \sqrt{6+\sqrt{37}}\,\left(235\,686\,929+38\,787\,819\sqrt{37}\right)\omega_0^6-2\left(276\,066\,361+45\,386\,651\sqrt{37}\right)\omega_0^7+\\
&\quad 120\sqrt{6+\sqrt{37}}\,\left(192\,400+27\,311\sqrt{37}\right)\omega_0^8+\left(21\,069\,437+3\,709\,547\sqrt{37}\right)\omega_0^9-\\
&\quad \sqrt{6+\sqrt{37}}\,\left(661\,819+171\,977\sqrt{37}\right)\omega_0^{10}+\left(2\,164\,006+418\,898\sqrt{37}\right)\omega_0^{11}-\\
&\quad 8\sqrt{6+\sqrt{37}}\,\left(14\,393+1846\sqrt{37}\right)\omega_0^{12}-7\left(57\,499+10\,189\sqrt{37}\right)\omega_0^{13}+\\
&\quad \sqrt{6+\sqrt{37}}\,\left(8029+1807\sqrt{37}\right)\omega_0^{14}+170\left(93+7\sqrt{37}\right)\omega_0^{15}+\\
&\quad 8\sqrt{6+\sqrt{37}}\,\left(37+7\sqrt{37}\right)\omega_0^{16}+\left(2875+397\sqrt{37}\right)\omega_0^{17}+8\left(7+\sqrt{37}\right)\omega_0^{19}\Big)\Big)\Big/\\
&\Big(2\left(7+\sqrt{37}\right)\left(4+\omega_0^2\right)\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)\\
&\quad\left(37\left(6+\sqrt{37}\right)+16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0+64\,\omega_0^2+2\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^3+20\,\omega_0^4+\omega_0^6\right)\\
&\quad\Big(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-\\
&\qquad 2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)\Big)
\end{aligned}$$

$$\boldsymbol{\omega_0} := \sqrt{6+\sqrt{37}}$$

l1 = Simplify[l1]
simplifica

0

Unset[ω_0]
elimina alocação

(* Matriz D3 = 3iω_0I *)
unidad

D3 = 3 i ω_0 IdentityMatrix[3]
matriz identidade

{{3 i ω_0, 0, 0}, {0, 3 i ω_0, 0}, {0, 0, 3 i ω_0}}

(* Matriz TA = 3iω_0I-A *)
unidade ir

TA = D3 - A

$$\left\{\left\{1+3\,\mathrm{i}\,\omega_0,\ 2\sqrt{7+\sqrt{37}},\ 0\right\},\ \left\{0,\ 3+\frac{\sqrt{37}}{2}+3\,\mathrm{i}\,\omega_0,\ -\frac{\omega_0^2}{2}\right\},\ \left\{-\sqrt{7+\sqrt{37}},\ -\frac{\omega_0^2}{2},\ 3+\frac{\sqrt{37}}{2}+3\,\mathrm{i}\,\omega_0\right\}\right\}$$

(* Matriz inversa da matriz TA *)

TAI = FullSimplify[Inverse[TA]]

simplifica comple··· matriz inversa

$$\left\{\left\{\frac{1}{1+\omega_0\left(3\,\mathrm{i}-\frac{4\left(7+\sqrt{37}\right)\omega_0}{-73-12\sqrt{37}-12\,\mathrm{i}\left(6+\sqrt{37}\right)\omega_0+36\,\omega_0^2+\omega_0^4}\right)},\ \left(4\sqrt{7+\sqrt{37}}\left(6+\sqrt{37}+6\,\mathrm{i}\,\omega_0\right)\right)\Big/\right.\right.$$
$$\left(-73-12\sqrt{37}+\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right),$$
$$\left(4\sqrt{7+\sqrt{37}}\ \omega_0^2\right)\Big/$$
$$\left.\left(-73-12\sqrt{37}+\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right)\right\},$$
$$\left\{-\left(\left(2\sqrt{7+\sqrt{37}}\ \omega_0^2\right)\Big/\left(-73-12\sqrt{37}+\right.\right.\right.$$
$$\left.\left.\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right)\right),$$
$$\left(2\left(-\mathrm{i}+3\,\omega_0\right)\left(-\mathrm{i}\left(6+\sqrt{37}\right)+6\,\omega_0\right)\right)\Big/\left(-73-12\sqrt{37}+\right.$$
$$\left.\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right),$$
$$\left.\frac{1}{-18-\frac{2\,\mathrm{i}\left(7+\sqrt{37}\right)}{-\mathrm{i}+3\,\omega_0}+\frac{73+12\sqrt{37}+12\,\mathrm{i}\left(6+\sqrt{37}\right)\omega_0-\omega_0^4}{2\,\omega_0^2}}\right\},$$
$$\left\{-\left(\left(2\sqrt{7+\sqrt{37}}\left(6+\sqrt{37}+6\,\mathrm{i}\,\omega_0\right)\right)\Big/\left(-73-12\sqrt{37}+\right.\right.\right.$$
$$\left.\left.\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right)\right),$$
$$\left(8\left(7+\sqrt{37}\right)-2\,\omega_0^2-6\,\mathrm{i}\,\omega_0^3\right)\Big/\left(-73-12\sqrt{37}+\right.$$
$$\left.\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right),$$
$$\left(2\left(-\mathrm{i}+3\,\omega_0\right)\left(-\mathrm{i}\left(6+\sqrt{37}\right)+6\,\omega_0\right)\right)\Big/$$
$$\left.\left.\left(-73-12\sqrt{37}+\omega_0\left(-3\,\mathrm{i}\left(97+16\sqrt{37}\right)+\omega_0\left(32\left(7+\sqrt{37}\right)+\omega_0\left(108\,\mathrm{i}+\omega_0+3\,\mathrm{i}\,\omega_0^2\right)\right)\right)\right)\right\}\right\}$$

(* Cálculo do vetor complexo h30 *)

h30 = Simplify[TAI.(3 bb[q, h20])]

simplifica

$$\left\{\left(48\,\omega_0^3\left(11\,692+1924\sqrt{37}-5809\,\mathrm{i}\sqrt{6+\sqrt{37}}-955\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}+\right.\right.\right.$$
$$2\left(8827\,\mathrm{i}+1453\,\mathrm{i}\sqrt{37}+9583\sqrt{6+\sqrt{37}}+1597\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$
$$\left(31\,605+5175\sqrt{37}+19\,832\,\mathrm{i}\sqrt{6+\sqrt{37}}+3392\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2+$$

$$\left(31534\,\mathrm{i}+5098\,\mathrm{i}\sqrt{37}-3256\sqrt{6+\sqrt{37}}-616\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$
$$\left(37\,\mathrm{i}\sqrt{6+\sqrt{37}}+7\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}-16\left(539+83\sqrt{37}\right)\right)\omega_0^4-$$
$$2\left(37\sqrt{6+\sqrt{37}}+7\sqrt{37\left(6+\sqrt{37}\right)}+37\,\mathrm{i}\left(7+\sqrt{37}\right)\right)\omega_0^5-$$
$$3\left(59+9\sqrt{37}\right)\omega_0^6-2\,\mathrm{i}\left(7+\sqrt{37}\right)\omega_0^7\Big)\Big)\Big/\Big(\left(7+\sqrt{37}\right)^{3/2}\left(2\,\mathrm{i}+\omega_0\right)^3$$
$$\left(-73-12\sqrt{37}-2\,\mathrm{i}\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,\mathrm{i}\,\omega_0^3+\omega_0^4+2\,\mathrm{i}\,\omega_0^5\right)$$
$$\left(-73-12\sqrt{37}-3\,\mathrm{i}\left(97+16\sqrt{37}\right)\omega_0+32\left(7+\sqrt{37}\right)\omega_0^2+108\,\mathrm{i}\,\omega_0^3+\omega_0^4+3\,\mathrm{i}\,\omega_0^5\right)\Big),$$

$$\Big(12\,\omega_0^3\Big(-296\left(6+\sqrt{37}\right)-8\left(185\,\mathrm{i}\left(6+\sqrt{37}\right)+12\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$
$$\left(9413+1619\sqrt{37}-740\,\mathrm{i}\sqrt{6+\sqrt{37}}-636\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2+$$
$$\left(-5873\,\mathrm{i}-695\,\mathrm{i}\sqrt{37}+1924\sqrt{6+\sqrt{37}}+1020\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$
$$\left(6892+716\sqrt{37}+96\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4+4\,\mathrm{i}\left(329+25\sqrt{37}\right)\omega_0^5+$$
$$\left(-41+\sqrt{37}\right)\omega_0^6+3\,\mathrm{i}\left(7+\sqrt{37}\right)\omega_0^7\Big)\Big)\Big/\Big(\left(7+\sqrt{37}\right)\left(2\,\mathrm{i}+\omega_0\right)^3$$
$$\left(-73-12\sqrt{37}-2\,\mathrm{i}\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,\mathrm{i}\,\omega_0^3+\omega_0^4+2\,\mathrm{i}\,\omega_0^5\right)$$
$$\left(-73-12\sqrt{37}-3\,\mathrm{i}\left(97+16\sqrt{37}\right)\omega_0+32\left(7+\sqrt{37}\right)\omega_0^2+108\,\mathrm{i}\,\omega_0^3+\omega_0^4+3\,\mathrm{i}\,\omega_0^5\right)\Big),$$

$$\Big(12\left(6+\sqrt{37}+6\,\mathrm{i}\,\omega_0\right)\omega_0\Big(-296\left(6+\sqrt{37}\right)-8\left(185\,\mathrm{i}\left(6+\sqrt{37}\right)+12\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$
$$\left(9413+1619\sqrt{37}-740\,\mathrm{i}\sqrt{6+\sqrt{37}}-636\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2+$$
$$\left(-5873\,\mathrm{i}-695\,\mathrm{i}\sqrt{37}+1924\sqrt{6+\sqrt{37}}+1020\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$
$$\left(6892+716\sqrt{37}+96\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4+4\,\mathrm{i}\left(329+25\sqrt{37}\right)\omega_0^5+$$
$$\left(-41+\sqrt{37}\right)\omega_0^6+3\,\mathrm{i}\left(7+\sqrt{37}\right)\omega_0^7\Big)\Big)\Big/\Big(\left(7+\sqrt{37}\right)\left(2\,\mathrm{i}+\omega_0\right)^3$$
$$\left(-73-12\sqrt{37}-2\,\mathrm{i}\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,\mathrm{i}\,\omega_0^3+\omega_0^4+2\,\mathrm{i}\,\omega_0^5\right)$$
$$\left(-73-12\sqrt{37}-3\,\mathrm{i}\left(97+16\sqrt{37}\right)\omega_0+32\left(7+\sqrt{37}\right)\omega_0^2+108\,\mathrm{i}\,\omega_0^3+\omega_0^4+3\,\mathrm{i}\,\omega_0^5\right)\Big)\Big\}$$

(* Cálculo do vetor complexo h30b *)

```
h30b = Simplify[ComplexExpand[Conjugate[h30]], ω0 ∈ Reals]
```

simplifica expande funções ⋯ conjugado números r

$$\Bigg\{\Bigg(48\,\omega_0^3\,\Big(249\,323\,020 + 40\,988\,452\sqrt{37} + 123\,813\,433\,i\sqrt{6+\sqrt{37}} +$$

$$20\,354\,803\,i\sqrt{37\left(6+\sqrt{37}\right)} + \Big(-412\,709\,507\sqrt{6+\sqrt{37}} -$$

$$67\,849\,025\sqrt{37\left(6+\sqrt{37}\right)} + 2\,i\left(641\,394\,979 + 105\,444\,685\sqrt{37}\right)\Big)\,\omega_0 +$$

$$\Big(-455\,314\,613 - 74\,853\,263\sqrt{37} + 501\,725\,624\,i\sqrt{6+\sqrt{37}} +$$

$$82\,483\,184\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^2 + \Big(6\,081\,705\,883\,i + 999\,826\,345\,i\sqrt{37} -$$

$$1\,603\,028\,848\sqrt{6+\sqrt{37}} - 263\,536\,360\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^3 +$$

$$\Big(1\,218\,824\,881\,i\sqrt{6+\sqrt{37}} + 200\,374\,939\,i\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$24\left(371\,529\,961 + 61\,079\,127\sqrt{37}\right)\Big)\,\omega_0^4 - \Big(2\,016\,138\,038\,i + 331\,448\,618\,i\sqrt{37} +$$

$$2\,199\,478\,875\sqrt{6+\sqrt{37}} + 361\,578\,681\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^5 +$$

$$\Big(-1\,591\,908\,661 - 261\,713\,791\sqrt{37} - 1\,038\,210\,676\,i\sqrt{6+\sqrt{37}} -$$

$$170\,620\,492\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^6 + \Big(132\,703\,164\sqrt{6+\sqrt{37}} +$$

$$21\,931\,236\sqrt{37\left(6+\sqrt{37}\right)} - i\left(892\,400\,081 + 146\,719\,235\sqrt{37}\right)\Big)\,\omega_0^7 +$$

$$\Big(32\,238\,316 + 5\,208\,220\sqrt{37} - 47\,115\,245\,i\sqrt{6+\sqrt{37}} - 7\,457\,711\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)$$

$$\omega_0^8 + \Big(15\,403\,951\sqrt{6+\sqrt{37}} + 2\,827\,045\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$2\,i\left(33\,359\,365 + 5\,539\,003\sqrt{37}\right)\Big)\,\omega_0^9 +$$

$$\Big(9\,196\,793 + 1\,215\,707\sqrt{37} - 1\,041\,328\,i\sqrt{6+\sqrt{37}} - 149\,512\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{10} +$$

$$\Big(1\,078\,328\sqrt{6+\sqrt{37}} + 209\,264\sqrt{37\left(6+\sqrt{37}\right)} - i\left(5\,720\,687 + 903\,989\sqrt{37}\right)\Big)\,\omega_0^{11} +$$

$$\Big(259\,i\sqrt{6+\sqrt{37}} + 289\,i\sqrt{37\left(6+\sqrt{37}\right)} + 624\left(2692 + 391\sqrt{37}\right)\Big)\,\omega_0^{12} +$$

$$3\left(14245\sqrt{6+\sqrt{37}}+2695\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(45651+6685\sqrt{37}\right)\right)\omega_0^{13}+$$

$$\left(85041+12915\sqrt{37}-148\,i\sqrt{6+\sqrt{37}}-28\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{14}+$$

$$\left(444\sqrt{6+\sqrt{37}}+84\sqrt{37\left(6+\sqrt{37}\right)}-i\left(8347+1201\sqrt{37}\right)\right)\omega_0^{15}+$$

$$8\left(124+19\sqrt{37}\right)\omega_0^{16}-12\,i\left(7+\sqrt{37}\right)\omega_0^{17}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^{3/2}\left(-2\,i+\omega_0\right)^3\Big(10657+1752\sqrt{37}+\left(52604+8648\sqrt{37}\right)\omega_0^2-$$
$$2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)$$
$$\Big(10657+1752\sqrt{37}+\left(108809+17888\sqrt{37}\right)\omega_0^2+\left(25062+3944\sqrt{37}\right)\omega_0^4+$$
$$\left(10366-224\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\Big)\Big),$$

$$\Big(12\,\omega_0^3\Big(-296\left(128766+21169\sqrt{37}\right)-$$

$$32\left(194472\sqrt{6+\sqrt{37}}+31971\sqrt{37\left(6+\sqrt{37}\right)}+185\,i\left(10657+1752\sqrt{37}\right)\right)\omega_0+$$

$$\Big(-507601283-83449133\sqrt{37}+7697924\,i\sqrt{6+\sqrt{37}}+$$

$$1265532\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^2-2\Big(406172937\,i+66774487\,i\sqrt{37}+$$

$$74776704\sqrt{6+\sqrt{37}}+12293168\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^3+\Big(-1357401451-$$

$$223156965\sqrt{37}-65256308\,i\sqrt{6+\sqrt{37}}-10731308\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^4-$$

$$2\Big(1180367605\,i+194052139\,i\sqrt{37}+312371316\sqrt{6+\sqrt{37}}+$$

$$51371900\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^5+\Big(735360741+120887483\sqrt{37}-$$

$$743286488\,i\sqrt{6+\sqrt{37}}-122265512\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^6+$$

$$\Big(368136976\sqrt{6+\sqrt{37}}+59913200\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$2\,i\left(171820271+28241897\sqrt{37}\right)\Big)\omega_0^7+\Big(289790021+47873419\sqrt{37}+$$

$$78871864\,i\sqrt{6+\sqrt{37}}+12662664\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^8+$$

$$\left(58\,886\,274\,i + 9\,754\,094\,i\,\sqrt{37} + 9\,857\,984\,\sqrt{6+\sqrt{37}} - 670\,560\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\omega_0^9 +$$

$$\left(2\,835\,423 + 1\,686\,513\,\sqrt{37} + 4\,321\,748\,i\,\sqrt{6+\sqrt{37}} + 1\,041\,388\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{10} + \left(5\,319\,010\,i + 480\,894\,i\,\sqrt{37} - 246\,864\,\sqrt{6+\sqrt{37}} - 239\,184\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{11} + \left(-98\,569 + 60\,025\,\sqrt{37} + 5180\,i\,\sqrt{6+\sqrt{37}} + 31\,140\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\omega_0^{12} +$$

$$\left(393\,226\,i + 28\,246\,i\,\sqrt{37} - 11\,544\,\sqrt{6+\sqrt{37}} - 5640\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\omega_0^{13} +$$

$$\left(6719 + 513\,\sqrt{37} + 576\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\omega_0^{14} +$$

$$2\,i\,\left(7111 + 769\,\sqrt{37}\right)\omega_0^{15} + 9\,\left(39+\sqrt{37}\right)\omega_0^{16} + 18\,i\,\left(7+\sqrt{37}\right)\omega_0^{17}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)\,\left(-2\,i+\omega_0\right)^3\,\Big(10\,657 + 1752\,\sqrt{37} + \left(52\,604 + 8648\,\sqrt{37}\right)\omega_0^2 -$$
$$2\,\left(89 + 28\,\sqrt{37}\right)\omega_0^4 - 104\,\left(-4+\sqrt{37}\right)\omega_0^6 + 129\,\omega_0^8 + 4\,\omega_0^{10}\Big)$$
$$\Big(10\,657 + 1752\,\sqrt{37} + \left(108\,809 + 17\,888\,\sqrt{37}\right)\omega_0^2 + \left(25\,062 + 3944\,\sqrt{37}\right)\omega_0^4 +$$
$$\left(10\,366 - 224\,\sqrt{37}\right)\omega_0^6 + 649\,\omega_0^8 + 9\,\omega_0^{10}\Big)\Big),$$

$$\Big(12\,\omega_0\,\Big(-296\,\left(1\,555\,849 + 255\,780\,\sqrt{37}\right) - 16\,\Big(4\,699\,518\,\sqrt{6+\sqrt{37}} +$$

$$772\,596\,\sqrt{37\,\left(6+\sqrt{37}\right)} + 259\,i\,\left(128\,766 + 21\,169\,\sqrt{37}\right)\Big)\,\omega_0 + \Big(-6\,511\,762\,259 -$$

$$1\,070\,527\,121\,\sqrt{37} + 130\,350\,852\,i\,\sqrt{6+\sqrt{37}} + 21\,429\,548\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^2 -$$

$$8\,\Big(846\,222\,448\,i + 139\,118\,115\,i\,\sqrt{37} + 220\,103\,417\,\sqrt{6+\sqrt{37}} +$$

$$36\,184\,779\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^3 + \Big(108\,724\,204\,i\,\sqrt{6+\sqrt{37}} +$$

$$5\,\left(-4\,255\,058\,331 - 699\,527\,417\,\sqrt{37} + 3\,574\,772\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\Big)\,\omega_0^4 -$$

$$8\,\Big(2\,547\,482\,605\,i + 418\,802\,386\,i\,\sqrt{37} + 992\,689\,280\,\sqrt{6+\sqrt{37}} +$$

$$163\,199\,160\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^5 + \Big(-5\,279\,409\,943 - 867\,940\,029\,\sqrt{37} -$$

$$5\,235\,087\,080\,i\,\sqrt{6+\sqrt{37}} - 860\,416\,760\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^6 -$$

$$4\left(2\,140\,977\,019\,i+351\,967\,051\,i\sqrt{37}+8\,527\,168\sqrt{6+\sqrt{37}}+1\,494\,224\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^7+\left(1\,448\,213\,377+238\,127\,771\sqrt{37}-1\,267\,072\,104\,i\sqrt{6+\sqrt{37}}-204\,631\,352\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^8+$$

$$4\left(126\,892\,092\sqrt{6+\sqrt{37}}+20\,452\,652\sqrt{37\left(6+\sqrt{37}\right)}-i\left(256\,130\,251+42\,457\,419\sqrt{37}\right)\right)\omega_0^9+\left(432\,731\,163+71\,479\,065\sqrt{37}+5\,313\,940\,i\sqrt{6+\sqrt{37}}+14\,593\,436\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{10}+$$

$$8\left(4\,086\,825\,i-239\,338\,i\sqrt{37}+1\,949\,937\sqrt{6+\sqrt{37}}+570\,795\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{11}+$$

$$\left(33\,543\,571+3\,146\,945\sqrt{37}+2\,664\,444\,i\sqrt{6+\sqrt{37}}+1\,627\,124\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{12}+8\left(499\,484\,i+25\,319\,i\sqrt{37}-30\,858\sqrt{6+\sqrt{37}}+17\,682\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{13}+$$

$$\left(90\,576\,i\sqrt{6+\sqrt{37}}+37\,296\,i\sqrt{37\left(6+\sqrt{37}\right)}+31\left(78\,021+5783\sqrt{37}\right)\right)\omega_0^{14}+4\left(25\,481\,i+5093\,i\sqrt{37}+864\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{15}+$$

$$\left(87\,771+9633\sqrt{37}\right)\omega_0^{16}+36\,i\left(-19+5\sqrt{37}\right)\omega_0^{17}+108\left(7+\sqrt{37}\right)\omega_0^{18}\Big)\Big)\Big/$$

$$\Big(\Big(7+\sqrt{37}\Big)\left(-2\,i+\omega_0\right)^3\Big(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)$$
$$\Big(10\,657+1752\sqrt{37}+\left(108\,809+17\,888\sqrt{37}\right)\omega_0^2+\left(25\,062+3944\sqrt{37}\right)\omega_0^4+\left(10\,366-224\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\Big)\Big)\}$$

(* Matriz D1 = $i\omega_0$I *)
unidad

```
D1 = ⅈ ω0 IdentityMatrix[3]
```
matriz identidade

$\{\{i\,\omega_0, 0, 0\}, \{0, i\,\omega_0, 0\}, \{0, 0, i\,\omega_0\}\}$

(* Matriz L = $i\omega_0$I-A *)
unidade ir

L = D1 - A

$$\left\{\left\{1+i\,\omega_0,\ 2\sqrt{7+\sqrt{37}},\ 0\right\},\ \left\{0,\ 3+\frac{\sqrt{37}}{2}+i\,\omega_0,\ -\frac{\omega_0^2}{2}\right\},\ \left\{-\sqrt{7+\sqrt{37}},\ -\frac{\omega_0^2}{2},\ 3+\frac{\sqrt{37}}{2}+i\,\omega_0\right\}\right\}$$

q

$$\left\{-\frac{2\left(-i\sqrt{37\left(6+\sqrt{37}\right)}-6\,i\,\omega_0+\omega_0^2\right)}{\sqrt{7+\sqrt{37}}\left(2\,i\,\omega_0+\omega_0^2\right)},\ \frac{\omega_0^2}{2\,i\,\omega_0+\omega_0^2},\ 1\right\}$$

pb

$$\left\{-\frac{i\sqrt{7+\sqrt{37}}\,\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(-i+\omega_0\right)\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},\right.$$
$$\left.\frac{\left(1+\frac{2\,i}{\omega_0}\right)\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},\ \frac{\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)}\right\}$$

$i\,\omega_0$ IdentityMatrix[3] - A
matriz identidade

$$\left\{\left\{1+i\,\omega_0,\ 2\sqrt{7+\sqrt{37}},\ 0\right\},\ \left\{0,\ 3+\frac{\sqrt{37}}{2}+i\,\omega_0,\ -\frac{\omega_0^2}{2}\right\},\ \left\{-\sqrt{7+\sqrt{37}},\ -\frac{\omega_0^2}{2},\ 3+\frac{\sqrt{37}}{2}+i\,\omega_0\right\}\right\}$$

(* Matriz L21 = $\left(\begin{pmatrix} i\omega_0\text{IdentityMatrix[3]-A} & q \\ pb & 0 \end{pmatrix}\right)$ ***)**

$$L21 = \left\{\left\{1 + i\,\omega_0,\ 2\sqrt{7+\sqrt{37}},\ 0,\ -\frac{2\left(-i\sqrt{37\left(6+\sqrt{37}\right)} - 6\,i\,\omega_0 + \omega_0^2\right)}{\sqrt{7+\sqrt{37}}\left(2\,i\,\omega_0 + \omega_0^2\right)}\right\},\right.$$
$$\left\{0,\ 3+\frac{\sqrt{37}}{2} + i\,\omega_0,\ -\frac{\omega_0^2}{2},\ \frac{\omega_0^2}{2\,i\,\omega_0+\omega_0^2}\right\},\ \left\{-\sqrt{7+\sqrt{37}},\ -\frac{\omega_0^2}{2},\ 3+\frac{\sqrt{37}}{2}+i\,\omega_0,\ 1\right\},$$
$$\left\{-\frac{i\sqrt{7+\sqrt{37}}\ \omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(-i+\omega_0\right)\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},\right.$$
$$\frac{\left(1+\frac{2\,i}{\omega_0}\right)\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},$$
$$\left.\left.\frac{\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},\ 0\right\}\right\}$$

$$\left\{\left\{1 + i\,\omega_0,\ 2\sqrt{7+\sqrt{37}},\ 0,\ -\frac{2\left(-i\sqrt{37\left(6+\sqrt{37}\right)} - 6\,i\,\omega_0 + \omega_0^2\right)}{\sqrt{7+\sqrt{37}}\left(2\,i\,\omega_0 + \omega_0^2\right)}\right\},\right.$$
$$\left\{0,\ 3+\frac{\sqrt{37}}{2} + i\,\omega_0,\ -\frac{\omega_0^2}{2},\ \frac{\omega_0^2}{2\,i\,\omega_0+\omega_0^2}\right\},\ \left\{-\sqrt{7+\sqrt{37}},\ -\frac{\omega_0^2}{2},\ 3+\frac{\sqrt{37}}{2}+i\,\omega_0,\ 1\right\},$$
$$\left\{-\frac{i\sqrt{7+\sqrt{37}}\ \omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(-i+\omega_0\right)\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},\right.$$
$$\frac{\left(1+\frac{2\,i}{\omega_0}\right)\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},$$
$$\left.\left.\frac{\omega_0\left(2+\omega_0\left(i+\omega_0\right)\right)}{2\left(\sqrt{37\left(6+\sqrt{37}\right)}+\omega_0\left(8+\omega_0\left(2\,i+\omega_0\right)\right)\right)},\ 0\right\}\right\}$$

```
(* Inversa da matriz L21 *)

L21I = Simplify[Inverse[L21]];
```

simplifica matriz inversa

```
(* Cálculo de R21 *)

{b11, b22, b33} = Simplify[cc[q, q, qb] + bb[qb, h20] + 2 bb[q, h11] - G21 q];
```

simplifica

```
(* Cálculo de H21 *)

H21 = {b11, b22, b33, 0};

(* Cálculo de h21 *)

{r21, r22, r23, S} = Simplify[L21I.H21];
                     simplifica

(* Cálculo do vetor complexo h21 *)

h21 = Simplify[{r21, r22, r23}]
      simplifica
```

$$\Big\{\Big(4\,\omega_0\,(-i+\omega_0)\,\Big(-10\,952\,i\,\big(128\,766+21\,169\sqrt{37}\big)+$$
$$592\,\Big(5\,552\,960+912\,901\sqrt{37}-842\,712\,i\sqrt{6+\sqrt{37}}-138\,541\,i\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0+$$
$$37\,\Big(47\,999\,025\,i+7\,890\,991\,i\sqrt{37}+22\,895\,156\sqrt{6+\sqrt{37}}+$$
$$3\,763\,940\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0^2+\Big(-215\,465\,430\,i\sqrt{6+\sqrt{37}}-$$
$$35\,422\,514\,i\sqrt{37\,\big(6+\sqrt{37}\big)}-296\,\big(10\,000\,235+1\,644\,029\sqrt{37}\big)\Big)\,\omega_0^3-$$
$$\Big(4\,769\,165\,831\,i+784\,046\,005\,i\sqrt{37}+98\,301\,785\sqrt{6+\sqrt{37}}+$$
$$16\,169\,335\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0^4+\Big(3\,226\,362\,044+530\,424\,476\sqrt{37}-$$
$$796\,906\,259\,i\sqrt{6+\sqrt{37}}-131\,028\,221\,i\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0^5+$$
$$\Big(332\,716\,730\,i+54\,737\,414\,i\sqrt{37}+548\,842\,571\sqrt{6+\sqrt{37}}+$$
$$90\,353\,737\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0^6+\Big(464\,089\,367+76\,233\,653\sqrt{37}+$$
$$160\,013\,863\,i\sqrt{6+\sqrt{37}}+26\,709\,453\,i\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0^7+$$
$$2\,\Big(87\,027\,709\,i+14\,244\,475\,i\sqrt{37}+7\,382\,795\sqrt{6+\sqrt{37}}+978\,325\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)$$
$$\omega_0^8+\Big(-11\,337\,461-1\,683\,955\sqrt{37}+$$
$$20\,295\,166\,i\sqrt{6+\sqrt{37}}+3\,255\,490\,i\sqrt{37\,\big(6+\sqrt{37}\big)}\Big)\,\omega_0^9+$$

$$\left(14201305\,i+2390071\,i\sqrt{37}-2714394\sqrt{6+\sqrt{37}}-533670\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{10}+\left(89244\,i\sqrt{6+\sqrt{37}}-6972\,i\sqrt{37\left(6+\sqrt{37}\right)}-754\left(6607+1045\sqrt{37}\right)\right)\omega_0^{11}-$$

$$\left(449127\,i+58133\,i\sqrt{37}+182521\sqrt{6+\sqrt{37}}+36999\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{12}+$$

$$\left(-42587\,i\sqrt{6+\sqrt{37}}-8197\,i\sqrt{37\left(6+\sqrt{37}\right)}-34\left(2801+419\sqrt{37}\right)\right)\omega_0^{13}-$$

$$\left(8325\sqrt{6+\sqrt{37}}+1607\sqrt{37\left(6+\sqrt{37}\right)}+4\,i\left(28479+4309\sqrt{37}\right)\right)\omega_0^{14}+$$

$$\left(4927+829\sqrt{37}-925\,i\sqrt{6+\sqrt{37}}-175\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{15}-$$

$$4\left(1139\,i+173\,i\sqrt{37}+37\sqrt{6+\sqrt{37}}+7\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{16}-$$

$$3\left(7+\sqrt{37}\right)\omega_0^{17}-4\,i\left(7+\sqrt{37}\right)\omega_0^{18}\Bigg)\Bigg)\Big/$$

$$\Bigg(\left(7+\sqrt{37}\right)^{3/2}\left(-2\,i+\omega_0\right)\left(2\,i+\omega_0\right)^2\left(\sqrt{37\left(6+\sqrt{37}\right)}+8\,\omega_0+2\,i\,\omega_0^2+\omega_0^3\right)$$

$$\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)$$

$$\left(-73-12\sqrt{37}-2\,i\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,i\,\omega_0^3+\omega_0^4+2\,i\,\omega_0^5\right)$$

$$\Bigg(-\sqrt{6+\sqrt{37}}\left(518+85\sqrt{37}\right)+$$

$$\left(-534-88\sqrt{37}-222\,i\sqrt{6+\sqrt{37}}-44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$

$$2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-i\,\omega_0^6\Bigg)\Bigg),$$

$$\Bigg(4\,\omega_0^3\Bigg(296\left(394309+64824\sqrt{37}+5365\,i\sqrt{6+\sqrt{37}}+882\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)+$$

$$592\left(403129\,i+66274\,i\sqrt{37}+59015\sqrt{6+\sqrt{37}}+9702\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$\Bigg(-71077777-11685155\sqrt{37}+43556141\,i\sqrt{6+\sqrt{37}}+$$

$$7160495\, i \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^2 + 4\,(-51321949\, i - 8437225\, i \sqrt{37} +$$

$$13401400 \sqrt{6+\sqrt{37}} + 2202702 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^3 +$$

$$2\,(160272933 + 26349843 \sqrt{37} + 4627183\, i \sqrt{6+\sqrt{37}} + 759969\, i \sqrt{37\,(6+\sqrt{37})}\,)$$

$$\omega_0^4 + (185414759\, i + 30489217\, i \sqrt{37} +$$

$$16812985 \sqrt{6+\sqrt{37}} + 2778067 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^5 +$$

$$2\, i\,(18716419\, i + 3081457\, i \sqrt{37} + 5635359 \sqrt{6+\sqrt{37}} + 959921 \sqrt{37\,(6+\sqrt{37})}\,)$$

$$\omega_0^6 - 4\,(324143\, i + 57175\, i \sqrt{37} + 406926 \sqrt{6+\sqrt{37}} + 91302 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^7 +$$

$$20\, i\,(90647\, i + 13567\, i \sqrt{37} + 80364 \sqrt{6+\sqrt{37}} + 11540 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^8 -$$

$$2\,(386261\, i + 58143\, i \sqrt{37} + 187257 \sqrt{6+\sqrt{37}} + 35075 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^9 +$$

$$(111239 + 20469 \sqrt{37} - 13431\, i \sqrt{6+\sqrt{37}} - 7285\, i \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^{10} -$$

$$8\,(-6570\, i - 1288\, i \sqrt{37} + 481 \sqrt{6+\sqrt{37}} + 104 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^{11} +$$

$$6\,(3503 + 561 \sqrt{37} - 1073\, i \sqrt{6+\sqrt{37}} - 211\, i \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^{12} +$$

$$(1555\, i + 349\, i \sqrt{37} + 185 \sqrt{6+\sqrt{37}} + 35 \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^{13} +$$

$$4\,(256 + 40 \sqrt{37} - 37\, i \sqrt{6+\sqrt{37}} - 7\, i \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0^{14} - 16\, i\,(7+\sqrt{37})\, \omega_0^{15}\,)\,)\, /$$

$$(\,(7+\sqrt{37})\,(-2\, i + \omega_0)\,(2\, i + \omega_0)^2\,(\sqrt{37\,(6+\sqrt{37})} + 8\,\omega_0 + 2\, i\, \omega_0^2 + \omega_0^3)$$

$$(73 + 12\sqrt{37} + 4\,(7+\sqrt{37})\,\omega_0^2 - \omega_0^4)$$

$$(-73 - 12\sqrt{37} - 2\, i\,(97 + 16\sqrt{37})\,\omega_0 + 12\,(7+\sqrt{37})\,\omega_0^2 + 32\, i\, \omega_0^3 + \omega_0^4 + 2\, i\, \omega_0^5)$$

$$(-\sqrt{6+\sqrt{37}}\,(518 + 85\sqrt{37}) +$$

$$(-534 - 88\sqrt{37} - 222\, i \sqrt{6+\sqrt{37}} - 44\, i \sqrt{37\,(6+\sqrt{37})}\,)\, \omega_0 +$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^3+\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-i\,\omega_0^6\Big)\Big),$$

$$\Big(4\,\omega_0\Big(10\,952\left(128\,766+21\,169\sqrt{37}\right)+592\Big(848\,077\sqrt{6+\sqrt{37}}+$$

$$139\,423\sqrt{37\left(6+\sqrt{37}\right)}+37\,i\left(139\,423+22\,921\sqrt{37}\right)\Big)\omega_0+37\,i$$

$$\left(38\,253\,369\,i+6\,288\,815\,i\sqrt{37}+20\,963\,756\sqrt{6+\sqrt{37}}+3\,446\,420\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^2+\Big(267\,640\,832\sqrt{6+\sqrt{37}}+43\,999\,920\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$74\,i\left(41\,304\,951+6\,790\,493\sqrt{37}\right)\Big)\omega_0^3+\Big(4\,429\,668\,016+728\,233\,360\sqrt{37}-$$

$$249\,069\,126\,i\sqrt{6+\sqrt{37}}-40\,951\,446\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^4+$$

$$\Big(761\,809\,391\sqrt{6+\sqrt{37}}+125\,257\,289\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$156\,i\left(17\,888\,193+2\,940\,859\sqrt{37}\right)\Big)\omega_0^5+\Big(-282\,433\,078-46\,458\,666\sqrt{37}+$$

$$505\,962\,235\,i\sqrt{6+\sqrt{37}}+83\,277\,665\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^6-$$

$$4\Big(38\,571\,353\sqrt{6+\sqrt{37}}+6\,411\,959\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\left(88\,385\,071+14\,517\,566\sqrt{37}\right)\Big)\omega_0^7+4\,i\Big(34\,803\,931\,i+$$

$$5\,695\,859\,i\sqrt{37}+1\,687\,570\sqrt{6+\sqrt{37}}+191\,806\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^8-$$

$$\left(6\,415\,009\,i+912\,575\,i\sqrt{37}+15\,452\,902\sqrt{6+\sqrt{37}}+2\,428\,682\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^9+$$

$$\left(-13\,933\,409-2\,351\,303\sqrt{37}-3\,572\,646\,i\sqrt{6+\sqrt{37}}-624\,170\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{10}+\left(258\,408\sqrt{6+\sqrt{37}}+62\,552\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(2\,215\,817+356\,291\sqrt{37}\right)\right)$$

$$\omega_0^{11}+2\left(221\,504+29\,136\sqrt{37}-80\,697\,i\sqrt{6+\sqrt{37}}-14\,025\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{12}+$$

$$\left(33559\sqrt{6+\sqrt{37}}+6497\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(94915+15641\sqrt{37}\right)\right)\omega_0^{13}+$$

$$\left(96008+14568\sqrt{37}-2257\,i\sqrt{6+\sqrt{37}}-411\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{14}+$$

$$4\left(331\,i+40\,i\sqrt{37}+111\sqrt{6+\sqrt{37}}+21\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{15}+$$

$$4\left(827+125\sqrt{37}\right)\omega_0^{16}+13\,i\left(7+\sqrt{37}\right)\omega_0^{17}+4\left(7+\sqrt{37}\right)\omega_0^{18}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)\left(-2\,i+\omega_0\right)\left(2\,i+\omega_0\right)^2\left(\sqrt{37\left(6+\sqrt{37}\right)}+8\,\omega_0+2\,i\,\omega_0^2+\omega_0^3\right)$$

$$\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)$$

$$\left(-73-12\sqrt{37}-2\,i\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,i\,\omega_0^3+\omega_0^4+2\,i\,\omega_0^5\right)$$

$$\Big(-\sqrt{6+\sqrt{37}}\left(518+85\sqrt{37}\right)+$$

$$\left(-534-88\sqrt{37}-222\,i\sqrt{6+\sqrt{37}}-44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$

$$2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-i\,\omega_0^6\Big)\Big)\Big\}$$

(* Cálculo do vetor complexo h21b *)

```
h21b = Simplify[ComplexExpand[Conjugate[h21]], ω0 ∈ Reals]
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simplifica | expande funções ⋯ | conjugado | números r

$$\Big\{-\Big(\Big(4\,\omega_0\left(i+\omega_0\right)\Big(-405224\,i\left(38650402549+6354087038\sqrt{37}\right)+$$

$$21904\Big(574970341742+94524541913\sqrt{37}-$$

$$391914755012\,i\sqrt{6+\sqrt{37}}-64430388834\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0+$$

$$1369\Big(8464738350068\sqrt{6+\sqrt{37}}+1391594412556\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\left(52610472785541+8649108447599\sqrt{37}\right)\Big)\omega_0^2+$$

$$148\Big(481976320612645+79236419013881\sqrt{37}-$$

$$200\,574\,819\,239\,166\,i\sqrt{6+\sqrt{37}}-32\,974\,297\,161\,874\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^3+$$

$$37\Big(1\,107\,133\,183\,595\,646\sqrt{6+\sqrt{37}}+182\,011\,574\,195\,266\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\left(272\,624\,910\,989\,527+44\,819\,259\,280\,889\sqrt{37}\right)\Big)\omega_0^4+$$

$$2\Big(77\,402\,452\,504\,857\,557+12\,724\,884\,806\,748\,859\sqrt{37}+7\,643\,067\,331\,326\,291$$

$$i\sqrt{6+\sqrt{37}}+1\,256\,512\,529\,180\,177\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^5+$$

$$\Big(31\,320\,254\,196\,901\,222\sqrt{6+\sqrt{37}}+5\,149\,018\,071\,890\,490\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$9\,i\left(4\,392\,240\,558\,191\,497+722\,079\,899\,771\,407\sqrt{37}\right)\Big)\omega_0^6+$$

$$2\Big(34\,064\,586\,885\,009\,406+5\,600\,183\,587\,165\,630\sqrt{37}+$$

$$5\,085\,257\,038\,910\,629\,i\sqrt{6+\sqrt{37}}+836\,011\,109\,246\,795\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^7+$$

$$2\Big(14\,877\,894\,655\,152\,961\,i+2\,445\,910\,808\,878\,849\,i\sqrt{37}+$$

$$4\,580\,213\,517\,943\,296\sqrt{6+\sqrt{37}}+752\,982\,466\,039\,104\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^8+$$

$$2\Big(5\,292\,715\,886\,289\,699+870\,117\,137\,900\,869\sqrt{37}+$$

$$2\,090\,704\,279\,990\,599\,i\sqrt{6+\sqrt{37}}+343\,709\,566\,168\,673\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^9+$$

$$\Big(6\,426\,632\,210\,816\,895\,i+1\,056\,531\,959\,710\,709\,i\sqrt{37}+$$

$$1\,447\,226\,541\,366\,158\sqrt{6+\sqrt{37}}+237\,922\,876\,423\,226\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^{10}+$$

$$2\Big(1\,053\,069\,733\,663\,445+173\,123\,493\,967\,943\sqrt{37}+$$

$$466\,143\,989\,609\,087\,i\sqrt{6+\sqrt{37}}+76\,633\,546\,593\,221\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^{11}+$$

$$\Big(1\,145\,570\,805\,548\,147\,i+188\,331\,526\,119\,657\,i\sqrt{37}+$$

$$162\,805\,091\,578\,806\sqrt{6+\sqrt{37}}+26\,764\,484\,509\,698\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{12}+$$

$$4\Bigg(76\,257\,209\,883\,323+12\,536\,596\,871\,771\sqrt{37}+28\,714\,527\,210\,260\,\mathrm{i}\sqrt{6+\sqrt{37}}+$$

$$4\,721\,868\,540\,500\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{13}+$$

$$\Bigg(192\,033\,272\,564\,677\,\mathrm{i}+31\,570\,833\,064\,915\,\mathrm{i}\sqrt{37}+27\,504\,991\,797\,900\sqrt{6+\sqrt{37}}+$$

$$4\,522\,929\,034\,452\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{14}+$$

$$2\Bigg(3\,139\,992\,652\,295+516\,050\,221\,897\sqrt{37}+7\,990\,640\,372\,322\,\mathrm{i}\sqrt{6+\sqrt{37}}+$$

$$1\,321\,147\,868\,814\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{15}+$$

$$4\Bigg(4\,258\,070\,749\,759\,\mathrm{i}+699\,327\,588\,741\,\mathrm{i}\sqrt{37}+271\,511\,285\,820\sqrt{6+\sqrt{37}}+$$

$$45\,808\,325\,100\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{16}+$$

$$4\,\mathrm{i}\Bigg(395\,166\,060\,645\,\mathrm{i}+65\,288\,228\,463\,\mathrm{i}\sqrt{37}+513\,272\,680\,607\sqrt{6+\sqrt{37}}+$$

$$86\,230\,885\,545\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{17}+$$

$$\Bigg(-404\,618\,847\,536\sqrt{6+\sqrt{37}}-64\,024\,425\,376\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$3\,\mathrm{i}\left(353\,258\,143\,891+57\,484\,479\,145\sqrt{37}\right)\Bigg)\,\omega_0^{18}+$$

$$4\,\mathrm{i}\Bigg(54\,525\,250\,942\,\mathrm{i}+9\,168\,199\,144\,\mathrm{i}\sqrt{37}+15\,037\,333\,281\sqrt{6+\sqrt{37}}+$$

$$2\,856\,389\,035\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{19}+\Bigg(27\,626\,824\,899\,\mathrm{i}+4\,144\,559\,749\,\mathrm{i}\sqrt{37}-$$

$$23\,949\,245\,374\sqrt{6+\sqrt{37}}-3\,394\,069\,170\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{20}-$$

$$2\,\mathrm{i}\Bigg(3\,765\,256\,493\sqrt{6+\sqrt{37}}+536\,891\,231\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$3\,\mathrm{i}\left(1\,272\,461\,349+245\,491\,283\sqrt{37}\right)\Bigg)\,\omega_0^{21}+$$

$$\left(2\,205\,252\,318 \sqrt{6+\sqrt{37}} + 423\,268\,898 \sqrt{37\left(6+\sqrt{37}\right)} - 3\,i\left(2\,481\,835\,903 + 421\,622\,057\sqrt{37}\right)\right)\omega_0^{22} +$$

$$2\left(980\,269\,820 + 146\,208\,592\sqrt{37} - 281\,401\,687\,i\sqrt{6+\sqrt{37}} - 41\,408\,937\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{23} + 2\left(122\,630\,728\sqrt{6+\sqrt{37}} + 22\,045\,768\sqrt{37\left(6+\sqrt{37}\right)} - i\left(453\,660\,207 + 75\,483\,103\sqrt{37}\right)\right)\omega_0^{24} +$$

$$2\left(112\,993\,319 + 17\,434\,433\sqrt{37} - 5\,806\,965\,i\sqrt{6+\sqrt{37}} - 794\,419\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{25} + \left(8\,634\,542\sqrt{6+\sqrt{37}} + 1\,549\,210\sqrt{37\left(6+\sqrt{37}\right)} - 3\,i\left(14\,150\,569 + 2\,324\,915\sqrt{37}\right)\right)\omega_0^{26} +$$

$$2\left(4\,541\,669 + 698\,111\sqrt{37} + 5883\,i\sqrt{6+\sqrt{37}} + 3817\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{27} +$$

$$\left(127\,058\sqrt{6+\sqrt{37}} + 23\,142\sqrt{37\left(6+\sqrt{37}\right)} - i\left(993\,539 + 159\,889\sqrt{37}\right)\right)\omega_0^{28} +$$

$$4\left(37\,722 + 5676\sqrt{37} + 481\,i\sqrt{6+\sqrt{37}} + 91\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{29} +$$

$$\left(592\sqrt{6+\sqrt{37}} + 112\sqrt{37\left(6+\sqrt{37}\right)} - i\left(11\,801 + 1823\sqrt{37}\right)\right)\omega_0^{30} +$$

$$114\left(7+\sqrt{37}\right)\omega_0^{31} - 8\,i\left(7+\sqrt{37}\right)\omega_0^{32}\Big)\Big) \Big/$$

$$\Big(\left(7+\sqrt{37}\right)^{3/2}\left(-2\,i+\omega_0\right)^2\left(2\,i+\omega_0\right)\left(73 + 12\sqrt{37} + 4\left(7+\sqrt{37}\right)\omega_0^2 - \omega_0^4\right)$$

$$\left(37\left(6+\sqrt{37}\right) + 16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0 + 64\,\omega_0^2 + 2\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^3 + 20\,\omega_0^4 + \omega_0^6\right)$$

$$\left(10\,657 + 1752\sqrt{37} + \left(52\,604 + 8648\sqrt{37}\right)\omega_0^2 - 2\left(89 + 28\sqrt{37}\right)\omega_0^4 - 104\left(-4+\sqrt{37}\right)\omega_0^6 + 129\,\omega_0^8 + 4\,\omega_0^{10}\right)$$

$$\Big(37\left(174\,922 + 28\,757\sqrt{37}\right) + 4\sqrt{6+\sqrt{37}}\left(276\,686 + 45\,487\sqrt{37}\right)\omega_0 +$$

$$4\left(314\,823 + 51\,764\sqrt{37}\right)\omega_0^2 + 56\sqrt{6+\sqrt{37}}\left(2664 + 455\sqrt{37}\right)\omega_0^3 +$$

$$\left(296\,509 + 48\,182\sqrt{37}\right)\omega_0^4 + 8\sqrt{6+\sqrt{37}}\left(4181 + 965\sqrt{37}\right)\omega_0^5 +$$

$$8\left(13\,319 + 1993\sqrt{37}\right)\omega_0^6 + 8\sqrt{6+\sqrt{37}}\left(296 + 107\sqrt{37}\right)\omega_0^7 +$$

$$\left(7776+881\sqrt{37}\right)\omega_0^8+28\sqrt{37\left(6+\sqrt{37}\right)}\;\omega_0^9+4\left(51+2\sqrt{37}\right)\omega_0^{10}+\omega_0^{12}\Big)\Big)\Big)\Big),$$

$$-\Bigg(\Bigg(\Bigg(4\,\omega_0^3\Bigg(10\,952\,i\left(1\,610\,357\,882\sqrt{6+\sqrt{37}}+264\,741\,205\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$

$$\left.37\,i\left(3\,198\,805\,112+525\,880\,321\sqrt{37}\right)\right)-21\,904\left(31\,739\,512\,235\sqrt{6+\sqrt{37}}+\right.$$

$$\left.5\,217\,943\,669\sqrt{37\left(6+\sqrt{37}\right)}+2\,i\left(31\,319\,325\,553+5\,148\,865\,404\sqrt{37}\right)\right)\omega_0-$$

$$37\,i\left(30\,950\,282\,042\,213\sqrt{6+\sqrt{37}}+5\,088\,195\,024\,555\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$

$$\left.37\,i\left(4\,105\,568\,619\,333+674\,951\,323\,331\sqrt{37}\right)\right)\omega_0^2-$$

$$74\left(101\,542\,327\,697\,811\,i+16\,693\,455\,842\,149\,i\sqrt{37}+\right.$$

$$\left.26\,927\,059\,164\,489\sqrt{6+\sqrt{37}}+4\,426\,781\,257\,755\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$

$$\left(-4\,044\,973\,468\,676\,203\,i\sqrt{6+\sqrt{37}}-664\,989\,541\,925\,797\,i\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$

$$\left.37\left(36\,638\,955\,043\,483+6\,023\,407\,105\,057\sqrt{37}\right)\right)\omega_0^4+$$

$$4\left(571\,637\,402\,829\,003\sqrt{6+\sqrt{37}}+93\,976\,610\,118\,427\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$

$$\left.i\left(3\,063\,753\,609\,778\,867+503\,677\,990\,800\,649\sqrt{37}\right)\right)\omega_0^5+$$

$$4\left(1\,683\,227\,702\,056\,429+276\,720\,929\,632\,951\sqrt{37}-\right.$$

$$\left.276\,434\,660\,462\,018\,i\sqrt{6+\sqrt{37}}-45\,445\,578\,069\,716\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^6+$$

$$2\left(492\,728\,705\,976\,725\sqrt{6+\sqrt{37}}+81\,004\,099\,144\,067\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$

$$\left.i\left(333\,138\,581\,758\,483+54\,767\,646\,682\,601\sqrt{37}\right)\right)\omega_0^7+$$

$$\left(1\,565\,381\,804\,451\,023+257\,347\,189\,415\,253\sqrt{37}-100\,862\,406\,501\,425\right.$$

$$\left.i\sqrt{6+\sqrt{37}}-16\,581\,722\,092\,283\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^8+$$

$$2\left(44\,679\,899\,204\,915\,i+7\,345\,344\,370\,617\,i\sqrt{37}+105\,206\,758\,858\,367\sqrt{6+\sqrt{37}}+17\,295\,918\,455\,765\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^9+$$
$$\left(239\,925\,009\,979\,869+39\,443\,409\,346\,451\sqrt{37}+12\,810\,342\,056\,963\,i\sqrt{6+\sqrt{37}}+2\,105\,967\,774\,009\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{10}+$$
$$\left(24\,927\,295\,311\,182\sqrt{6+\sqrt{37}}+4\,098\,089\,846\,378\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(14\,688\,960\,961\,475+2\,414\,764\,169\,741\sqrt{37}\right)\right)\omega_0^{11}+$$
$$10\left(5\,385\,275\,445\,399+885\,335\,045\,165\sqrt{37}+138\,551\,728\,433\,i\sqrt{6+\sqrt{37}}+22\,857\,074\,919\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{12}+$$
$$8\left(546\,008\,190\,921\sqrt{6+\sqrt{37}}+89\,843\,401\,599\sqrt{37\left(6+\sqrt{37}\right)}+8\,i\left(68\,031\,704\,559+11\,187\,601\,883\sqrt{37}\right)\right)\omega_0^{13}+$$
$$8\left(585\,992\,503\,970+96\,326\,890\,984\sqrt{37}+29\,593\,792\,917\,i\sqrt{6+\sqrt{37}}+5\,173\,848\,107\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{14}+$$
$$4\left(394\,096\,990\,607\,i+64\,696\,414\,469\,i\sqrt{37}+183\,771\,377\,445\sqrt{6+\sqrt{37}}+30\,511\,763\,479\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{15}+\left(222\,413\,145\,118+36\,347\,280\,498\sqrt{37}+215\,560\,409\,518\,i\sqrt{6+\sqrt{37}}+36\,497\,591\,170\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{16}+$$
$$4\left(44\,353\,361\,885\,i+7\,237\,799\,711\,i\sqrt{37}+1\,898\,177\,811\sqrt{6+\sqrt{37}}+447\,290\,309\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{17}+\left(2\,755\,929\,197+338\,829\,595\sqrt{37}+19\,953\,905\,565\,i\sqrt{6+\sqrt{37}}+3\,447\,133\,659\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{18}-$$

$$2\left(2\,022\,365\,943\sqrt{6+\sqrt{37}}+281\,606\,141\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$

$$\left.\mathrm{i}\left(5\,317\,774\,263+856\,506\,641\sqrt{37}\right)\right)\omega_0^{19}+$$

$$\left(-2\,826\,519\,901-489\,575\,415\sqrt{37}-35\,188\,295\,\mathrm{i}\sqrt{6+\sqrt{37}}+\right.$$

$$\left.4\,018\,279\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{20}-4\left(26\,302\,457\,\mathrm{i}+4\,674\,651\,\mathrm{i}\sqrt{37}+\right.$$

$$\left.32\,360\,681\sqrt{6+\sqrt{37}}+2\,963\,677\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{21}-$$

$$4\,\mathrm{i}\left(16\,768\,992\sqrt{6+\sqrt{37}}+2\,759\,218\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$

$$\left.3\,\mathrm{i}\left(22\,815\,543+3\,971\,761\sqrt{37}\right)\right)\omega_0^{22}+2\left(3\,195\,801\sqrt{6+\sqrt{37}}+\right.$$

$$\left.754\,743\sqrt{37\left(6+\sqrt{37}\right)}-\mathrm{i}\left(24\,068\,963+3\,886\,697\sqrt{37}\right)\right)\omega_0^{23}+\left(-8\,165\,101-\right.$$

$$\left.1\,489\,887\sqrt{37}-2\,977\,501\,\mathrm{i}\sqrt{6+\sqrt{37}}-512\,127\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{24}+$$

$$\left(368\,150\sqrt{6+\sqrt{37}}+71\,970\sqrt{37\left(6+\sqrt{37}\right)}-2\,\mathrm{i}\left(1\,089\,653+171\,711\sqrt{37}\right)\right)\omega_0^{25}+$$

$$\left(-42\,621-11\,187\sqrt{37}-50\,727\,\mathrm{i}\sqrt{6+\sqrt{37}}-9117\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{26}+$$

$$\left(4810\sqrt{6+\sqrt{37}}+910\sqrt{37\left(6+\sqrt{37}\right)}-2\,\mathrm{i}\left(18\,781+2851\sqrt{37}\right)\right)\omega_0^{27}+$$

$$8\left(122+14\sqrt{37}-37\,\mathrm{i}\sqrt{6+\sqrt{37}}-7\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{28}-$$

$$\left.\left.32\,\mathrm{i}\left(7+\sqrt{37}\right)\omega_0^{29}\right)\right)\Big/$$

$$\left(\left(7+\sqrt{37}\right)\left(-2\,\mathrm{i}+\omega_0\right)^2\left(2\,\mathrm{i}+\omega_0\right)\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)\right.$$

$$\left(37\left(6+\sqrt{37}\right)+16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0+64\,\omega_0^2+2\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^3+20\,\omega_0^4+\omega_0^6\right)$$

$$\left(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-\right.$$

$$\left.2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\right)$$

$$\left(37\left(174\,922+28\,757\sqrt{37}\right)+4\sqrt{6+\sqrt{37}}\left(276\,686+45\,487\sqrt{37}\right)\omega_0+\right.$$

$$4\left(314\,823+51\,764\sqrt{37}\right)\omega_0^2+56\sqrt{6+\sqrt{37}}\left(2664+455\sqrt{37}\right)\omega_0^3+$$

$$\left(296\,509+48\,182\sqrt{37}\right)\omega_0^4+8\sqrt{6+\sqrt{37}}\left(4181+965\sqrt{37}\right)\omega_0^5+$$
$$8\left(13\,319+1993\sqrt{37}\right)\omega_0^6+8\sqrt{6+\sqrt{37}}\left(296+107\sqrt{37}\right)\omega_0^7+$$
$$\left.\left.\left.\left(7776+881\sqrt{37}\right)\omega_0^8+28\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^9+4\left(51+2\sqrt{37}\right)\omega_0^{10}+\omega_0^{12}\right)\right)\right),$$

$$\left(4\,\omega_0\left(405\,224\left(38\,650\,402\,549+6\,354\,087\,038\sqrt{37}\right)+\right.\right.$$
$$21\,904\left(393\,525\,112\,894\sqrt{6+\sqrt{37}}+64\,695\,130\,039\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$
$$\left.74\,i\left(9\,369\,272\,039+1\,540\,298\,835\sqrt{37}\right)\right)\omega_0+$$
$$1369\left(49\,659\,189\,636\,997+8\,163\,920\,486\,727\sqrt{37}+9\,493\,285\,604\,644\,i\sqrt{6+\sqrt{37}}+\right.$$
$$\left.1\,560\,686\,539\,604\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2+$$
$$74\left(360\,804\,460\,814\,559\sqrt{6+\sqrt{37}}+59\,315\,887\,973\,169\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$
$$\left.37\,i\left(29\,657\,233\,497\,655+4\,875\,619\,153\,293\sqrt{37}\right)\right)\omega_0^3+$$
$$37\,i\left(285\,450\,438\,717\,038\,i+46\,927\,763\,050\,954\,i\sqrt{37}+\right.$$
$$\left.1\,183\,891\,138\,211\,389\sqrt{6+\sqrt{37}}+194\,630\,504\,201\,731\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4+$$
$$\left(-25\,368\,683\,978\,177\,174\sqrt{6+\sqrt{37}}-4\,170\,585\,955\,285\,818\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$
$$\left.74\,i\left(1\,953\,776\,893\,065\,667+321\,198\,942\,640\,601\sqrt{37}\right)\right)\omega_0^5+$$
$$\left(-62\,684\,552\,565\,345\,538-10\,305\,276\,961\,460\,746\sqrt{37}+\right.$$
$$\left.22\,702\,181\,505\,592\,995\,i\sqrt{6+\sqrt{37}}+3\,732\,215\,649\,017\,277\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^6-$$
$$2\left(5\,047\,720\,875\,100\,695\sqrt{6+\sqrt{37}}+829\,840\,201\,288\,805\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$
$$\left.4\,i\left(5\,301\,042\,214\,306\,489+871\,485\,971\,758\,132\sqrt{37}\right)\right)\omega_0^7+$$
$$2\,i\left(12\,177\,716\,414\,557\,069\,i+2\,002\,004\,242\,978\,149\,i\sqrt{37}+\right.$$

$$3\,041\,886\,785\,065\,810\,\sqrt{6+\sqrt{37}} + 500\,083\,111\,611\,538\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^8 +$$

$$\Bigg(-3\,397\,675\,961\,030\,598\,\sqrt{6+\sqrt{37}} - 558\,574\,378\,233\,778\,\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$2\,i\left(3\,394\,195\,500\,080\,193 + 558\,002\,301\,803\,015\,\sqrt{37}\right)\Bigg)\,\omega_0^9 +$$

$$\Bigg(-4\,682\,530\,809\,546\,212 - 769\,803\,392\,812\,988\,\sqrt{37} + 925\,537\,099\,431\,265\,i\,\sqrt{6+\sqrt{37}} +$$

$$152\,157\,480\,507\,883\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{10} -$$

$$2\Bigg(348\,126\,888\,693\,757\,\sqrt{6+\sqrt{37}} + 57\,231\,660\,363\,447\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i\left(857\,824\,622\,888\,491 + 141\,025\,428\,992\,109\,\sqrt{37}\right)\Bigg)\,\omega_0^{11} +$$

$$\Bigg(-964\,401\,639\,414\,178 - 158\,547\,173\,452\,170\,\sqrt{37} + 125\,760\,843\,013\,405\,i\,\sqrt{6+\sqrt{37}} +$$

$$20\,674\,272\,590\,951\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{12} -$$

$$2\Bigg(43\,579\,889\,480\,599\,\sqrt{6+\sqrt{37}} + 7\,166\,121\,408\,621\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i\left(83\,972\,704\,351\,181 + 13\,805\,079\,122\,151\,\sqrt{37}\right)\Bigg)\,\omega_0^{13} +$$

$$\Bigg(-129\,472\,459\,927\,135 - 21\,285\,469\,572\,241\,\sqrt{37} + 20\,154\,378\,027\,494\,i\,\sqrt{6+\sqrt{37}} +$$

$$3\,314\,005\,358\,058\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{14} -$$

$$2\Bigg(5\,569\,857\,265\,302\,\sqrt{6+\sqrt{37}} + 920\,383\,477\,562\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i\left(629\,067\,166\,423 + 103\,438\,354\,281\,\sqrt{37}\right)\Bigg)\,\omega_0^{15} -$$

$$4\,i\Bigg(36\,843\,883\,236\,\sqrt{6+\sqrt{37}} + 4\,867\,505\,644\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$5\,i\left(459\,578\,352\,121 + 75\,456\,195\,567\,\sqrt{37}\right)\Bigg)\,\omega_0^{16} -$$

$$4\Bigg(112\,275\,642\,597\,i + 18\,765\,454\,243\,i\,\sqrt{37} + 221\,721\,098\,403\,\sqrt{6+\sqrt{37}} +$$

$$37\,470\,326\,481\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{17}+$$

$$\Bigg(-482\,534\,391\,475-78\,203\,759\,249\,\sqrt{37}-204\,243\,860\,506\,\mathbb{i}\,\sqrt{6+\sqrt{37}}-$$

$$31\,105\,156\,678\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{18}-$$

$$2\,\Bigg(23\,099\,707\,311\,\mathbb{i}+4\,199\,628\,461\,\mathbb{i}\,\sqrt{37}+6\,324\,405\,375\,\sqrt{6+\sqrt{37}}+$$

$$1\,371\,063\,793\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{19}+$$

$$\Bigg(4\,094\,123\,963\,\mathbb{i}\,\sqrt{6+\sqrt{37}}+1\,179\,489\,053\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$70\,\left(51\,425\,695+5\,424\,337\,\sqrt{37}\right)\Bigg)\,\omega_0^{20}+$$

$$2\,\Bigg(4\,326\,910\,117\,\mathbb{i}+609\,608\,207\,\mathbb{i}\,\sqrt{37}+1\,707\,702\,255\,\sqrt{6+\sqrt{37}}+$$

$$259\,303\,369\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{21}+$$

$$\Bigg(4\,408\,568\,152+737\,893\,548\,\sqrt{37}+2\,428\,949\,471\,\mathbb{i}\,\sqrt{6+\sqrt{37}}+$$

$$450\,996\,593\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{22}+$$

$$\Bigg(299\,208\,714\,\sqrt{6+\sqrt{37}}+48\,889\,422\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$4\,\mathbb{i}\,\left(600\,725\,711+92\,260\,211\,\sqrt{37}\right)\Bigg)\,\omega_0^{23}+$$

$$2\,\Bigg(268\,629\,023+43\,879\,143\,\sqrt{37}+82\,701\,142\,\mathbb{i}\,\sqrt{6+\sqrt{37}}+$$

$$15\,017\,846\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{24}+$$

$$2\,\Bigg(97\,089\,457\,\mathbb{i}+15\,037\,623\,\mathbb{i}\,\sqrt{37}+6\,025\,265\,\sqrt{6+\sqrt{37}}+$$

$$1\,037\,035\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{25}+$$

$$\Bigg(29\,927\,994+4\,798\,014\,\sqrt{37}+4\,920\,741\,\mathbb{i}\,\sqrt{6+\sqrt{37}}+893\,143\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)$$

$$\omega_0^{26}+$$

$$2\left(3\,494\,637\,i + 537\,587\,i\sqrt{37} + 127\,465\sqrt{6+\sqrt{37}} + 23\,051\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{27} +$$

$$\left(875\,794 + 137\,298\sqrt{37} + 66\,711\,i\sqrt{6+\sqrt{37}} + 12\,205\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{28} +$$

$$2\left(55\,687\,i + 8389\,i\sqrt{37} + 1147\sqrt{6+\sqrt{37}} + 217\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{29} +$$

$$\left(12\,329 + 1871\sqrt{37} + 296\,i\sqrt{6+\sqrt{37}} + 56\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{30} +$$

$$82\,i\left(7+\sqrt{37}\right)\omega_0^{31} + 8\left(7+\sqrt{37}\right)\omega_0^{32}\Bigg)\Bigg) \Big/$$

$$\Bigg(\left(7+\sqrt{37}\right)\left(-2\,i+\omega_0\right)^2\left(2\,i+\omega_0\right)$$

$$\left(73 + 12\sqrt{37} + 4\left(7+\sqrt{37}\right)\omega_0^2 - \omega_0^4\right)$$

$$\Bigg(37\left(6+\sqrt{37}\right) +$$

$$16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0 + 64\,\omega_0^2 +$$

$$2\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^3 + 20\,\omega_0^4 + \omega_0^6\Bigg)$$

$$\Big(10\,657 + 1752\sqrt{37} + \left(52\,604 + 8648\sqrt{37}\right)\omega_0^2 - 2\left(89 + 28\sqrt{37}\right)\omega_0^4 -$$
$$104\left(-4+\sqrt{37}\right)\omega_0^6 + 129\,\omega_0^8 + 4\,\omega_0^{10}\Big)$$

$$\Bigg(37\left(174\,922 + 28\,757\sqrt{37}\right) + 4\sqrt{6+\sqrt{37}}\left(276\,686 + 45\,487\sqrt{37}\right)\omega_0 +$$

$$4\left(314\,823 + 51\,764\sqrt{37}\right)\omega_0^2 +$$

$$56\sqrt{6+\sqrt{37}}\left(2664 + 455\sqrt{37}\right)\omega_0^3 +$$

$$\left(296\,509 + 48\,182\sqrt{37}\right)\omega_0^4 +$$

$$8\sqrt{6+\sqrt{37}}\left(4181 + 965\sqrt{37}\right)\omega_0^5 +$$

$$8\left(13\,319 + 1993\sqrt{37}\right)\omega_0^6 +$$

$$8\sqrt{6+\sqrt{37}}\left(296 + 107\sqrt{37}\right)\omega_0^7 +$$

$$\left(7776 + 881\sqrt{37}\right)\omega_0^8 + 28\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^9 +$$

$$4\left(51 + 2\sqrt{37}\right)\omega_0^{10} + \omega_0^{12}\Bigg)\Bigg)\Big\}$$

(* Matriz 4iω_0I *)

unidad

```
D4 = Simplify[4 ⅈ ω0 IdentityMatrix[3]]
```
simplifica · matriz identidade

$$\{\{4\,i\,\omega_0,\ 0,\ 0\},\ \{0,\ 4\,i\,\omega_0,\ 0\},\ \{0,\ 0,\ 4\,i\,\omega_0\}\}$$

```
(* Matriz QA = 4ⅈω0I-A *)
```
unidade ir

```
QA = Simplify[D4 - A]
```
simplifica

$$\left\{\left\{1+4\,i\,\omega_0,\ 2\sqrt{7+\sqrt{37}},\ 0\right\},\ \left\{0,\ \frac{1}{2}\left(6+\sqrt{37}+8\,i\,\omega_0\right),\ -\frac{\omega_0^2}{2}\right\},\ \left\{-\sqrt{7+\sqrt{37}},\ -\frac{\omega_0^2}{2},\ \frac{1}{2}\left(6+\sqrt{37}+8\,i\,\omega_0\right)\right\}\right\}$$

```
(* Inversa da matriz QA *)
```

```
QAI = Simplify[Inverse[QA]]
```
simplifica · matriz inversa

$$\Big\{\Big\{\left(-73-12\sqrt{37}-16\,i\left(6+\sqrt{37}\right)\omega_0+64\,\omega_0^2+\omega_0^4\right)\Big/$$
$$\left(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\right),$$
$$\left(4\sqrt{7+\sqrt{37}}\left(6+\sqrt{37}+8\,i\,\omega_0\right)\right)\Big/$$
$$\left(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\right),$$
$$\left(4\sqrt{7+\sqrt{37}}\,\omega_0^2\right)\Big/\Big(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+$$
$$60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\Big)\Big\},\ \Big\{-\Big(\left(2\sqrt{7+\sqrt{37}}\,\omega_0^2\right)\Big/$$
$$\left(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\right)\Big),$$
$$\left(2\left(-i+4\,\omega_0\right)\left(-i\left(6+\sqrt{37}\right)+8\,\omega_0\right)\right)\Big/\Big(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+$$
$$60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\Big),\ \left(2\,i\left(i-4\,\omega_0\right)\omega_0^2\right)\Big/$$
$$\left(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\right)\Big\},$$
$$\Big\{-\Big(\left(2\sqrt{7+\sqrt{37}}\left(6+\sqrt{37}+8\,i\,\omega_0\right)\right)\Big/\Big(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+$$
$$60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\Big)\Big),\ \left(8\left(7+\sqrt{37}\right)-2\,\omega_0^2-8\,i\,\omega_0^3\right)\Big/$$
$$\left(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\right),$$
$$\left(2\left(-i+4\,\omega_0\right)\left(-i\left(6+\sqrt{37}\right)+8\,\omega_0\right)\right)\Big/$$
$$\left(-73-12\sqrt{37}-4\,i\left(97+16\sqrt{37}\right)\omega_0+60\left(7+\sqrt{37}\right)\omega_0^2+256\,i\,\omega_0^3+\omega_0^4+4\,i\,\omega_0^5\right)\Big\}\Big\}$$

```
(* Vetor complexo h40 *)
```

```
h40 = Simplify[ComplexExpand[
   QAI.(3 bb[h20, h20] + 4 bb[q, h30] + 6 cc[q, q, h20] + dd[q, q, q, q])]]
```
simplifica · expande funções complexas

$$\Big\{\Big(96\,\omega_0^4$$

$$\Bigl(148\,\Bigl(3\,740\,393\,909\,136\,443\,058+614\,916\,970\,785\,157\,827\,\sqrt{37}+415\,579\,824\,348\,194\,600$$

$$i\,\sqrt{6+\sqrt{37}}+68\,320\,902\,267\,381\,456\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Bigr)+16$$

$$\Bigl(32\,478\,822\,660\,827\,375\,432\,\sqrt{6+\sqrt{37}}+5\,339\,485\,554\,310\,310\,337\,\sqrt{37\,\left(6+\sqrt{37}\right)}-$$

$$37\,i\,\left(4\,877\,584\,015\,350\,072\,344+801\,869\,872\,620\,350\,175\,\sqrt{37}\right)\Bigr)\,\omega_0+$$

$$\Bigl(10\,762\,729\,667\,076\,204\,250\,835+1\,769\,381\,857\,908\,653\,182\,857\,\sqrt{37}+$$

$$372\,985\,767\,769\,504\,909\,800\,i\,\sqrt{6+\sqrt{37}}+61\,318\,482\,500\,617\,779\,504$$

$$i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Bigr)\,\omega_0^2+2\,\Bigl(6\,681\,282\,256\,316\,079\,610\,064\,\sqrt{6+\sqrt{37}}+$$

$$1\,098\,396\,036\,839\,616\,075\,816\,\sqrt{37\,\left(6+\sqrt{37}\right)}-$$

$$5\,i\,\left(6\,604\,768\,841\,856\,114\,242\,591+1\,085\,817\,308\,987\,122\,688\,545\,\sqrt{37}\right)\Bigr)\,\omega_0^3+$$

$$\Bigl(77\,654\,232\,120\,763\,560\,314\,089+12\,766\,277\,120\,628\,676\,773\,527\,\sqrt{37}-$$

$$19\,898\,894\,309\,379\,780\,323\,912\,i\,\sqrt{6+\sqrt{37}}-$$

$$3\,271\,358\,072\,958\,372\,901\,792\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Bigr)\,\omega_0^4+$$

$$2\,\Bigl(56\,337\,976\,533\,860\,447\,054\,104\,\sqrt{6+\sqrt{37}}+$$

$$9\,261\,906\,288\,999\,640\,471\,096\,\sqrt{37\,\left(6+\sqrt{37}\right)}-$$

$$i\,\left(299\,462\,083\,550\,189\,030\,607\,887+49\,231\,263\,272\,003\,370\,249\,249\,\sqrt{37}\right)\Bigr)\,\omega_0^5+$$

$$\Bigl(240\,908\,821\,965\,821\,116\,181\,479+39\,605\,166\,364\,107\,604\,869\,065\,\sqrt{37}-$$

$$283\,598\,327\,821\,868\,766\,442\,816\,i\,\sqrt{6+\sqrt{37}}-$$

$$46\,623\,277\,895\,407\,958\,168\,096\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Bigr)\,\omega_0^6+$$

$$\Bigl(326\,672\,355\,070\,511\,716\,870\,624\,\sqrt{6+\sqrt{37}}+$$

$$53\,704\,604\,354\,248\,224\,525\,408\,\sqrt{37\,\left(6+\sqrt{37}\right)}-$$

$$2\,i\left(1\,411\,786\,719\,753\,598\,769\,415\,319+232\,096\,307\,018\,643\,892\,533\,849\sqrt{37}\right)\Big)\,\omega_0^7+$$

$$\Big(-136\,157\,653\,910\,854\,892\,685\,687-22\,384\,180\,416\,817\,864\,721\,585\sqrt{37}-$$
$$1\,377\,414\,536\,681\,641\,437\,912\,256\,i\sqrt{6+\sqrt{37}}-$$
$$226\,445\,554\,930\,139\,762\,491\,968\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^8-$$

$$2\Big(3\,576\,074\,213\,040\,352\,764\,893\,283\,i+587\,902\,979\,152\,636\,511\,240\,389\,i\sqrt{37}+$$
$$82\,169\,027\,578\,480\,583\,564\,144\sqrt{6+\sqrt{37}}+$$
$$13\,508\,504\,921\,768\,182\,428\,880\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^9-$$

$$4\,i\Big(621\,796\,906\,968\,668\,938\,837\,704\sqrt{6+\sqrt{37}}+$$
$$102\,222\,781\,815\,252\,632\,198\,496\sqrt{37\left(6+\sqrt{37}\right)}-$$
$$i\left(935\,238\,940\,358\,630\,317\,643\,425+153\,752\,334\,683\,494\,268\,223\,323\sqrt{37}\right)\Big)\,\omega_0^{10}-$$

$$8\Big(908\,694\,329\,924\,444\,350\,869\,905\,i+149\,388\,427\,609\,698\,874\,595\,191\,i\sqrt{37}+$$
$$239\,492\,934\,636\,522\,394\,143\,636\sqrt{6+\sqrt{37}}+$$
$$39\,372\,395\,921\,032\,406\,348\,348\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{11}-$$

$$4\,i\Big(175\,028\,789\,699\,616\,625\,252\,664\sqrt{6+\sqrt{37}}+$$
$$28\,774\,555\,775\,899\,358\,415\,152\sqrt{37\left(6+\sqrt{37}\right)}-$$
$$i\left(2\,227\,355\,386\,567\,112\,461\,328\,747+366\,174\,969\,920\,765\,909\,296\,733\sqrt{37}\right)\Big)\,\omega_0^{12}-$$

$$8\Big(177\,954\,061\,743\,973\,067\,816\,020\sqrt{6+\sqrt{37}}+$$
$$29\,255\,467\,537\,583\,428\,321\,780\sqrt{37\left(6+\sqrt{37}\right)}-$$
$$i\left(335\,819\,208\,354\,324\,004\,915\,369+55\,208\,337\,771\,137\,693\,424\,287\sqrt{37}\right)\Big)\,\omega_0^{13}+$$

$$4\,i\Big(206\,690\,738\,982\,072\,927\,887\,509\,i+33\,979\,748\,174\,053\,870\,675\,779\,i\sqrt{37}+$$
$$136\,756\,110\,406\,456\,948\,105\,968\sqrt{6+\sqrt{37}}+$$

$$22\,482\,566\,058\,648\,551\,399\,760\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{14}-$$

$$8\Bigg(4\,592\,333\,510\,361\,546\,931\,460\sqrt{6+\sqrt{37}}+$$

$$754\,974\,978\,444\,257\,653\,412\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$\mathbb{i}\left(72\,802\,666\,962\,703\,663\,692\,769+11\,968\,684\,721\,782\,052\,325\,687\sqrt{37}\right)\Bigg)\,\omega_0^{15}+$$

$$4\,\mathbb{i}\Bigg(5\,175\,890\,394\,808\,545\,011\,543\,\mathbb{i}+850\,911\,139\,594\,033\,850\,875\,\mathbb{i}\sqrt{37}+$$

$$18\,821\,732\,509\,823\,785\,021\,472\sqrt{6+\sqrt{37}}+$$

$$3\,094\,273\,763\,590\,519\,889\,296\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{16}+$$

$$8\Bigg(7\,563\,338\,728\,219\,274\,083\,357\,\mathbb{i}+1\,243\,405\,227\,003\,053\,202\,975\,\mathbb{i}\sqrt{37}+$$

$$1\,359\,438\,589\,588\,851\,780\,020\sqrt{6+\sqrt{37}}+$$

$$223\,490\,327\,190\,110\,198\,992\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{17}+$$

$$2\,\mathbb{i}\Bigg(5\,221\,840\,449\,655\,409\,589\,651\,\mathbb{i}+858\,465\,293\,362\,775\,384\,281\,\mathbb{i}\sqrt{37}+$$

$$1\,062\,408\,465\,675\,137\,141\,368\sqrt{6+\sqrt{37}}+$$

$$174\,658\,825\,686\,526\,766\,512\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{18}-$$

$$4\Bigg(483\,375\,773\,444\,469\,357\,688\sqrt{6+\sqrt{37}}+$$

$$79\,466\,525\,608\,906\,460\,432\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$\mathbb{i}\left(3\,112\,144\,980\,052\,781\,057\,347+511\,633\,473\,436\,792\,282\,637\sqrt{37}\right)\Bigg)\,\omega_0^{19}+$$

$$2\,\mathbb{i}\Bigg(261\,448\,265\,349\,432\,266\,553\,\mathbb{i}+42\,981\,934\,471\,532\,533\,175\,\mathbb{i}\sqrt{37}+$$

$$761\,599\,284\,550\,397\,691\,048\sqrt{6+\sqrt{37}}+$$

$$125\,205\,213\,703\,280\,439\,552\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{20}-$$

$$4\Bigg(2\,162\,986\,203\,640\,025\,424\sqrt{6+\sqrt{37}}+356\,108\,700\,189\,008\,688\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$\mathbb{i}\left(485\,769\,173\,957\,419\,445\,719+79\,860\,018\,683\,252\,889\,417\sqrt{37}\right)\bigg)\,\omega_0^{21}+$$

$$2\left(62\,723\,631\,038\,361\,283\,041+10\,311\,313\,694\,931\,463\,359\sqrt{37}+121\,873\,032\,455\,346\,789\,344\,\mathbb{i}\sqrt{6+\sqrt{37}}+20\,027\,936\,241\,981\,221\,632\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{22}+$$

$$4\left(55\,913\,898\,114\,968\,164\,455\,\mathbb{i}+9\,193\,492\,868\,678\,623\,449\,\mathbb{i}\sqrt{37}+13\,325\,107\,854\,336\,714\,056\sqrt{6+\sqrt{37}}+2\,187\,912\,009\,095\,045\,768\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{23}+$$

$$2\left(14\,870\,809\,830\,528\,243\,503+2\,444\,355\,301\,751\,893\,825\sqrt{37}+5\,371\,565\,333\,108\,875\,296\,\mathbb{i}\sqrt{6+\sqrt{37}}+851\,300\,951\,672\,963\,104\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{24}+$$

$$4\left(5\,190\,635\,237\,046\,095\,171\,\mathbb{i}+860\,336\,448\,286\,085\,701\,\mathbb{i}\sqrt{37}+984\,191\,824\,577\,653\,656\sqrt{6+\sqrt{37}}+157\,163\,477\,911\,595\,432\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{25}+4\left(984\,462\,697\,808\,906\,607+161\,279\,007\,687\,299\,221\sqrt{37}+294\,214\,366\,578\,664\,696\,\mathbb{i}\sqrt{6+\sqrt{37}}+21\,376\,598\,608\,225\,472\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{26}+8\left(218\,197\,307\,636\,617\,417\,\mathbb{i}+42\,639\,575\,241\,993\,807\,\mathbb{i}\sqrt{37}+25\,220\,725\,157\,147\,428\sqrt{6+\sqrt{37}}+5\,194\,248\,750\,050\,396\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{27}+$$

$$4\left(119\,927\,654\,779\,788\,789+15\,834\,281\,029\,526\,963\sqrt{37}+68\,864\,729\,813\,615\,944\,\mathbb{i}\sqrt{6+\sqrt{37}}+2\,935\,846\,081\,216\,816\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{28}+$$

$$8\left(18\,796\,816\,913\,746\,561\,\mathbb{i}+5\,080\,776\,666\,671\,639\,\mathbb{i}\sqrt{37}+5\,015\,975\,928\,063\,604\sqrt{6+\sqrt{37}}+1\,337\,039\,715\,408\,020\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{29}+$$

$$12 \left(5\,833\,938\,425\,776\,589 + 516\,523\,452\,114\,267 \sqrt{37} + 2\,159\,109\,372\,641\,456\, i \sqrt{6+\sqrt{37}} - 43\,695\,480\,359\,152\, i \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{30} +$$

$$8 \left(1\,026\,439\,159\,705\,789\, i + 424\,988\,024\,210\,987\, i \sqrt{37} + 547\,608\,955\,103\,636 \sqrt{6+\sqrt{37}} + 172\,638\,663\,187\,316 \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{31} +$$

$$4 \left(1\,994\,180\,814\,434\,651 + 130\,748\,216\,106\,054 \sqrt{37} + 260\,088\,561\,226\,456\, i \sqrt{6+\sqrt{37}} - 53\,299\,853\,066\,720\, i \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{32} +$$

$$8 \left(13\,670\,361\,120\,489\, i + 19\,548\,526\,059\,137\, i \sqrt{37} + 27\,719\,996\,911\,788 \sqrt{6+\sqrt{37}} + 11\,561\,175\,018\,990 \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{33} +$$

$$\left(618\,989\,981\,828\,779 + 36\,595\,884\,535\,217 \sqrt{37} + 8\,669\,010\,577\,512\, i \sqrt{6+\sqrt{37}} - 17\,227\,914\,181\,648\, i \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{34} +$$

$$\left(-9\,268\,848\,019\,206\, i + 3\,272\,756\,982\,598\, i \sqrt{37} + 5\,736\,955\,046\,848 \sqrt{6+\sqrt{37}} + 3\,742\,540\,413\,680 \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{35} +$$

$$\left(32\,875\,546\,490\,129 + 2\,144\,577\,043\,791 \sqrt{37} - 708\,154\,147\,656\, i \sqrt{6+\sqrt{37}} - 670\,302\,596\,704\, i \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{36} +$$

$$2 \left(37\,617\,275\,144 \sqrt{6+\sqrt{37}} + 48\,450\,294\,184 \sqrt{37\left(6+\sqrt{37}\right)} - i \left(156\,258\,826\,263 + 505\,158\,905 \sqrt{37}\right)\right) \omega_0^{37} +$$

$$\left(1\,171\,133\,837\,199 + 91\,717\,863\,169 \sqrt{37} - 25\,264\,556\,672\, i \sqrt{6+\sqrt{37}} - 13\,881\,439\,712\, i \sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{38} +$$

$$2\left(13\,i\left(192418645-45703461\sqrt{37}\right)+223967808\sqrt{6+\sqrt{37}}+786907520\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{39}+$$

$$\left(26756104225+2509454919\sqrt{37}-327016064\,i\sqrt{6+\sqrt{37}}-145472768\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{40}+$$

$$2\left(191915909\,i-7044717\,i\sqrt{37}+516224\sqrt{6+\sqrt{37}}+7203968\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{41}+$$

$$16\left(22959623+2517881\sqrt{37}-92352\,i\sqrt{6+\sqrt{37}}-37344\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{42}+$$

$$16\left(395479\,i+2161\,i\sqrt{37}+3456\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{43}+$$

$$16\left(169061+21251\sqrt{37}\right)\omega_0^{44}+$$

$$864\,i\left(39+\sqrt{37}\right)\omega_0^{45}+1152\left(7+\sqrt{37}\right)\omega_0^{46}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)\left(2\,i+\omega_0\right)^4\left(10657+1752\sqrt{37}+\left(52604+8648\sqrt{37}\right)\omega_0^2-2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\right)^3$$

$$\left(10657+1752\sqrt{37}+\left(108809+17888\sqrt{37}\right)\omega_0^2+\left(25062+3944\sqrt{37}\right)\omega_0^4+\left(10366-224\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\right)$$

$$\left(10657+1752\sqrt{37}+8\left(23437+3853\sqrt{37}\right)\omega_0^2+2\left(55399+8804\sqrt{37}\right)\omega_0^4-8\left(-7909+49\sqrt{37}\right)\omega_0^6+2049\,\omega_0^8+16\,\omega_0^{10}\right)\Big),$$

$$-\Big(\Big(192\,\omega_0^4\Big(296\,i\sqrt{6+\sqrt{37}}\left(51947478043524325+8540112783422682\sqrt{37}\right)+$$

$$5328\left(8540112783422682\,i+1403985893068225\,i\sqrt{37}+8598609450986568\sqrt{6+\sqrt{37}}+1413602685976466\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$4\,i\Big(148695106356946869839\sqrt{6+\sqrt{37}}+3\Big(8148441634114820121\sqrt{37\left(6+\sqrt{37}\right)}+74\,i\left(79327557752930552+13041370159989027\sqrt{37}\right)\Big)\Big)\omega_0^2+2\Big(927815354308038962851$$

$$\sqrt{6+\sqrt{37}}\ + 152\,531\,904\,654\,602\,872\,633\ \sqrt{37\ \left(6+\sqrt{37}\right)}\ +$$

$$8\ \mathbb{i}\ (177\,818\,317\,928\,629\,942\,957 + 29\,233\,151\,391\,808\,820\,398\ \sqrt{37}\,))\ \omega_0^3 +$$

$$(1\,000\,717\,458\,361\,564\,773\,139 + 164\,516\,936\,733\,432\,298\,471\ \sqrt{37}\ +$$

$$8\,100\,394\,922\,459\,923\,362\,606\ \mathbb{i}\ \sqrt{6+\sqrt{37}}\ +$$

$$1\,331\,696\,722\,025\,869\,581\,386\ \mathbb{i}\ \sqrt{37\ \left(6+\sqrt{37}\right)}\,)\ \omega_0^4 +$$

$$(54\,470\,970\,775\,035\,904\,599\,242\ \mathbb{i} + 8\,954\,972\,432\,955\,615\,563\,918\ \mathbb{i}\ \sqrt{37}\ +$$

$$27\,836\,039\,391\,081\,083\,430\,592\ \sqrt{6+\sqrt{37}}\ +$$

$$4\,576\,216\,686\,485\,764\,472\,800\ \sqrt{37\ \left(6+\sqrt{37}\right)}\,)\ \omega_0^5 +$$

$$(67\,928\,939\,708\,705\,255\,938\,533 + 11\,167\,448\,896\,837\,816\,024\,621\ \sqrt{37}\ +$$

$$46\,014\,831\,723\,850\,974\,578\,868\ \mathbb{i}\ \sqrt{6+\sqrt{37}}\ + 7\,564\,791\,736\,427\,528\,223\,740$$

$$\mathbb{i}\ \sqrt{37\ \left(6+\sqrt{37}\right)}\,)\ \omega_0^6 + 4\ (49\,051\,523\,698\,202\,034\,094\,643\ \sqrt{6+\sqrt{37}}\ +$$

$$8\,064\,020\,821\,769\,148\,418\,997\ \sqrt{37\ \left(6+\sqrt{37}\right)}\ +$$

$$12\ \mathbb{i}\ (8\,678\,583\,251\,463\,642\,856\,793 + 1\,426\,750\,297\,785\,857\,806\,076\ \sqrt{37}\,))\ \omega_0^7 +$$

$$4\ (198\,667\,219\,092\,490\,141\,570\,801 + 32\,660\,689\,629\,576\,921\,995\,501\ \sqrt{37}\ +$$

$$19\,372\,884\,839\,811\,127\,335\,387\ \mathbb{i}\ \sqrt{6+\sqrt{37}}\ +$$

$$3\,184\,882\,648\,848\,258\,197\,925\ \mathbb{i}\ \sqrt{37\ \left(6+\sqrt{37}\right)}\,)\ \omega_0^8 +$$

$$12\ (94\,599\,111\,464\,268\,185\,761\,747\ \mathbb{i} + 15\,551\,998\,124\,712\,304\,643\,909\ \mathbb{i}\ \sqrt{37}\ +$$

$$56\,165\,330\,429\,937\,572\,769\,183\ \sqrt{6+\sqrt{37}}\ +$$

$$9\,233\,523\,444\,352\,504\,466\,365\ \sqrt{37\ \left(6+\sqrt{37}\right)}\,)\ \omega_0^9 +$$

$$4\ (866\,048\,517\,543\,601\,789\,493\,775 + 142\,377\,499\,241\,474\,089\,889\,507\ \sqrt{37}\ -$$

$$55\,698\,104\,268\,947\,971\,324\,703\ \mathbb{i}\ \sqrt{6+\sqrt{37}}\ - 9\,156\,711\,936\,643\,244\,197\,709$$

$$i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{10} + 4 \Bigg(249\,069\,195\,950\,300\,084\,354\,093 \sqrt{6+\sqrt{37}} +$$

$$40\,946\,723\,583\,188\,931\,641\,651 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$9\,i \left(17\,974\,597\,661\,440\,164\,091\,709 + 2\,955\,005\,652\,762\,006\,334\,927 \sqrt{37}\right)\Bigg) \omega_0^{11} +$$

$$4 \Bigg(1\,160\,916\,840\,161\,286\,720\,060\,604 + 190\,853\,552\,868\,250\,869\,266\,244 \sqrt{37} -$$

$$224\,038\,051\,056\,411\,022\,556\,603\,i \sqrt{6+\sqrt{37}} - 36\,831\,628\,711\,539\,908\,968\,261$$

$$i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{12} + 4 \Bigg(64\,330\,289\,798\,458\,898\,547\,945 \sqrt{6+\sqrt{37}} +$$

$$10\,575\,834\,495\,927\,221\,205\,435 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$i \left(1\,347\,004\,037\,229\,567\,186\,694\,421 + 221\,446\,099\,616\,768\,615\,111\,987 \sqrt{37}\right)\Bigg)$$

$$\omega_0^{13} - 4\,i \Bigg(185\,603\,786\,606\,259\,230\,733\,187 \sqrt{6+\sqrt{37}} +$$

$$30\,513\,074\,558\,105\,145\,318\,417 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$4\,i \left(145\,918\,097\,038\,631\,313\,441\,737 + 23\,988\,787\,382\,676\,183\,690\,512 \sqrt{37}\right)\Bigg)$$

$$\omega_0^{14} - 4 \Bigg(106\,597\,532\,662\,336\,352\,059\,057 \sqrt{6+\sqrt{37}} +$$

$$17\,524\,526\,418\,931\,061\,326\,567 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$3\,i \left(10\,535\,785\,147\,091\,489\,410\,565 + 1\,732\,072\,408\,645\,122\,152\,551 \sqrt{37}\right)\Bigg) \omega_0^{15} +$$

$$4\,i \Bigg(60\,985\,303\,331\,879\,148\,217\,571\,i + 10\,025\,922\,108\,330\,942\,809\,423\,i \sqrt{37} +$$

$$23\,084\,324\,537\,027\,185\,287\,235 \sqrt{6+\sqrt{37}} + 3\,795\,039\,576\,538\,613\,001\,081$$

$$\sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{16} - 4 \Bigg(1\,593\,701\,330\,413\,609\,291\,799 \sqrt{6+\sqrt{37}} +$$

$$262\,002\,885\,046\,161\,512\,213 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$3\,i \left(6\,086\,231\,362\,282\,517\,551\,971 + 1\,000\,570\,272\,308\,540\,565\,685 \sqrt{37}\right)\Bigg) \omega_0^{17} +$$

$$4 \Bigg(1\,575\,264\,322\,065\,157\,991\,747 + 258\,971\,858\,275\,828\,756\,035 \sqrt{37} +$$

$$1\,470\,746\,134\,994\,975\,295\,603\, i \sqrt{6+\sqrt{37}} + 241\,789\,171\,237\,790\,151\,861$$

$$i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{18} + 12 \Bigg(105\,417\,932\,522\,813\,820\,128 \sqrt{6+\sqrt{37}} +$$

$$17\,330\,598\,737\,812\,510\,948 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$3\, i \left(241\,738\,240\,627\,864\,974\,577 + 39\,741\,522\,725\,225\,344\,979 \sqrt{37}\right) \Bigg) \omega_0^{19} -$$

$$2\, i \Bigg(739\,200\,320\,594\,433\,117\,224 \sqrt{6+\sqrt{37}} +$$

$$121\,523\,983\,587\,495\,947\,136 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$i \left(3\,144\,654\,784\,280\,492\,607\,473 + 516\,978\,077\,631\,583\,547\,037 \sqrt{37}\right) \Bigg)$$

$$\omega_0^{20} - 4 \Bigg(260\,979\,904\,256\,567\,944\,123 \sqrt{6+\sqrt{37}} +$$

$$42\,904\,905\,378\,901\,622\,881 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$4\, i \left(41\,945\,776\,998\,423\,225\,688 + 6\,895\,834\,144\,823\,510\,413 \sqrt{37}\right) \Bigg) \omega_0^{21} +$$

$$2\, i \Bigg(374\,344\,337\,797\,741\,166\,471\, i + 61\,541\,889\,547\,293\,573\,559\, i \sqrt{37} +$$

$$124\,394\,286\,019\,176\,499\,574 \sqrt{6+\sqrt{37}} +$$

$$20\,448\,334\,218\,485\,667\,378 \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{22} - 4 \Bigg(8\,616\,092\,787\,436\,950\,071$$

$$\sqrt{6+\sqrt{37}} + 1\,416\,705\,038\,486\,477\,585 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$i \left(63\,939\,340\,882\,396\,863\,691 + 10\,511\,610\,500\,366\,414\,593 \sqrt{37}\right) \Bigg) \omega_0^{23} +$$

$$4\, i \Bigg(2\,439\,465\,399\,386\,541\,577\, i + 401\,138\,769\,619\,614\,877\, i \sqrt{37} +$$

$$6\,956\,101\,788\,291\,824\,697 \sqrt{6+\sqrt{37}} +$$

$$1\,139\,137\,474\,722\,514\,775 \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{24} + 4 \Bigg(2\,469\,367\,402\,553\,337\,047\, i +$$

$$407\,021\,853\,115\,062\,569\, i \sqrt{37} + 815\,387\,544\,880\,046\,103 \sqrt{6+\sqrt{37}} +$$

$$134\,372\,271\,724\,539\,237 \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{25} +$$

$$\Bigg(-4\,103\,273\,178\,819\,187\,804 - 676\,648\,184\,702\,991\,276 \sqrt{37} + 160\,311\,606\,926\,810\,924$$

$$\mathbb{i} \sqrt{6+\sqrt{37}} - 6\,269\,179\,774\,550\,524\,\mathbb{i} \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{26} -$$

$$4\Bigg(122\,781\,886\,199\,706\,525\sqrt{6+\sqrt{37}} + 17\,805\,705\,243\,897\,171\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$\mathbb{i}\left(349\,220\,632\,176\,433\,057 + 60\,151\,555\,782\,735\,451\sqrt{37}\right)\Bigg)\,\omega_0^{27} +$$

$$4\,\mathbb{i}\Bigg(53\,867\,395\,024\,823\,835\sqrt{6+\sqrt{37}} + 6\,458\,729\,553\,951\,765\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$2\,\mathbb{i}\left(66\,090\,736\,108\,344\,371 + 11\,640\,426\,694\,488\,115\sqrt{37}\right)\Bigg)\,\omega_0^{28} -$$

$$4\Bigg(10\,069\,529\,317\,526\,325\sqrt{6+\sqrt{37}} + 837\,852\,994\,246\,623\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$\mathbb{i}\left(65\,646\,931\,973\,615\,813 + 11\,459\,465\,816\,254\,355\sqrt{37}\right)\Bigg)\,\omega_0^{29} +$$

$$4\,\mathbb{i}\Bigg(2\,021\,922\,763\,628\,002\,\mathbb{i} + 839\,360\,614\,235\,750\,\mathbb{i}\sqrt{37} +$$

$$7\,702\,913\,085\,143\,407\sqrt{6+\sqrt{37}} + 960\,486\,341\,322\,293\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{30} +$$

$$4\Bigg(201\,204\,578\,772\,613\sqrt{6+\sqrt{37}} + 153\,692\,520\,467\,251\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$9\,\mathbb{i}\left(582\,209\,743\,339\,669 + 101\,134\,951\,606\,007\sqrt{37}\right)\Bigg)\,\omega_0^{31} +$$

$$4\Bigg(304\,463\,476\,616\,439 - 22\,659\,745\,853\,709\sqrt{37} + 437\,980\,769\,762\,863\,\mathbb{i}\sqrt{6+\sqrt{37}} +$$

$$51\,941\,840\,177\,519\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{32} +$$

$$12\Bigg(121\,899\,514\,985\,577\,\mathbb{i} + 19\,452\,310\,726\,475\,\mathbb{i}\sqrt{37} +$$

$$17\,287\,859\,285\,809\sqrt{6+\sqrt{37}} + 6\,086\,733\,301\,035\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{33} +$$

$$4\Bigg(42\,369\,307\,087\,865 + 1\,256\,294\,684\,675\sqrt{37} + 14\,581\,035\,936\,210\,\mathbb{i}\sqrt{6+\sqrt{37}} +$$

$$1\,752\,705\,236\,876\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{34} +$$

$$2\Bigg(50\,150\,425\,021\,778\,\mathbb{i} + 7\,234\,003\,873\,214\,\mathbb{i}\sqrt{37} + 6\,884\,571\,316\,221\sqrt{6+\sqrt{37}} +$$

$$2\,055\,439\,284\,399\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{35}+$$

$$\Big(14\,872\,389\,508\,087+1\,398\,537\,459\,803\,\sqrt{37}+1\,254\,929\,257\,890\,i\,\sqrt{6+\sqrt{37}}+$$

$$188\,037\,197\,174\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{36}+$$

$$6\,\Big(88\,780\,683\,706\,\sqrt{6+\sqrt{37}}+23\,192\,004\,830\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$11\,i\left(73\,496\,671\,003+9\,988\,552\,561\,\sqrt{37}\right)\Big)\,\omega_0^{37}+$$

$$\Big(704\,939\,027\,649+88\,167\,837\,529\,\sqrt{37}+16\,144\,455\,088\,i\,\sqrt{6+\sqrt{37}}+$$

$$3\,838\,576\,144\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{38}+$$

$$12\,\Big(12\,138\,769\,535\,i+1\,602\,917\,081\,i\,\sqrt{37}+980\,658\,064\,\sqrt{6+\sqrt{37}}+$$

$$227\,861\,808\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{39}+$$

$$16\,\Big(1\,085\,072\,697+153\,969\,179\,\sqrt{37}+6\,353\,714\,i\,\sqrt{6+\sqrt{37}}+$$

$$3\,011\,174\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{40}+16\,\Big(8\,266\,540\,\sqrt{6+\sqrt{37}}+$$

$$1\,729\,828\,\sqrt{37\left(6+\sqrt{37}\right)}+3\,i\left(54\,493\,771+7\,191\,943\,\sqrt{37}\right)\Big)\,\omega_0^{41}+$$

$$16\,\Big(13\,146\,367+2\,017\,167\,\sqrt{37}+15\,392\,i\,\sqrt{6+\sqrt{37}}+16\,736\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)$$

$$\omega_0^{42}+384\,\Big(67\,293\,i+9140\,i\,\sqrt{37}+1480\,\sqrt{6+\sqrt{37}}+280\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{43}+$$

$$1024\left(964+157\,\sqrt{37}\right)\omega_0^{44}+15\,360\,i\left(7+\sqrt{37}\right)\omega_0^{45}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^{3/2}\left(2\,i+\omega_0\right)^4\Big(10\,657+1752\,\sqrt{37}+\left(52\,604+8648\,\sqrt{37}\right)\omega_0^2-$$

$$2\left(89+28\,\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)^3$$

$$\Big(10\,657+1752\,\sqrt{37}+\left(108\,809+17\,888\,\sqrt{37}\right)\omega_0^2+\left(25\,062+3944\,\sqrt{37}\right)\omega_0^4+$$

$$\left(10\,366-224\,\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\Big)$$

$$\Big(10\,657+1752\,\sqrt{37}+8\left(23\,437+3853\,\sqrt{37}\right)\omega_0^2+2\left(55\,399+8804\,\sqrt{37}\right)\omega_0^4-$$

$$8\left(-7909+49\,\sqrt{37}\right)\omega_0^6+2049\,\omega_0^8+16\,\omega_0^{10}\Big)\Big),$$

$$\left(192\,\omega_0^2\left(-296\,\mathrm{i}\,\sqrt{6+\sqrt{37}}\,\left(627\,669\,041\,247\,785\,184+103\,188\,154\,744\,060\,417\,\sqrt{37}\right)-\right.\right.$$

$$592\left(928\,693\,392\,696\,543\,753\,\mathrm{i}+152\,676\,253\,276\,488\,288\,\mathrm{i}\,\sqrt{37}+\right.$$

$$\left.727\,264\,692\,609\,340\,550\,\sqrt{6+\sqrt{37}}+119\,561\,578\,967\,917\,548\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$4\left(303\,789\,563\,241\,205\,462\,434+49\,942\,696\,550\,790\,972\,108\,\sqrt{37}-\right.$$

$$1\,888\,274\,441\,838\,139\,121\,073\,\mathrm{i}\,\sqrt{6+\sqrt{37}}-$$

$$\left.310\,430\,405\,992\,778\,853\,713\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2-$$

$$2\left(8\,831\,450\,896\,357\,390\,147\,103\,\sqrt{6+\sqrt{37}}+\right.$$

$$1\,451\,881\,583\,798\,144\,832\,841\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$\left.8\,\mathrm{i}\left(2\,113\,315\,073\,426\,404\,847\,380+347\,426\,857\,928\,447\,737\,357\,\sqrt{37}\right)\right)\omega_0^3+$$

$$\left(10\,669\,313\,285\,558\,249\,016\,235+1\,754\,024\,299\,389\,370\,446\,979\,\sqrt{37}-\right.$$

$$112\,720\,193\,918\,645\,338\,092\,534\,\mathrm{i}\,\sqrt{6+\sqrt{37}}-$$

$$\left.18\,531\,085\,729\,088\,786\,813\,050\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4-$$

$$2\left(333\,082\,772\,168\,232\,860\,822\,765\,\mathrm{i}+54\,758\,470\,433\,318\,528\,185\,259\,\mathrm{i}\,\sqrt{37}+\right.$$

$$135\,766\,547\,183\,390\,199\,588\,152\,\sqrt{6+\sqrt{37}}+$$

$$\left.22\,319\,882\,866\,894\,356\,808\,152\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^5+$$

$$\left(-385\,001\,481\,234\,943\,491\,748\,239-63\,293\,853\,626\,087\,227\,574\,915\,\sqrt{37}-\right.$$

$$778\,674\,599\,719\,573\,059\,196\,324\,\mathrm{i}\,\sqrt{6+\sqrt{37}}-$$

$$\left.128\,013\,315\,634\,302\,259\,703\,708\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^6-$$

$$4\left(500\,648\,249\,146\,968\,746\,913\,011\,\sqrt{6+\sqrt{37}}+\right.$$

$$82\,306\,065\,155\,961\,868\,161\,145\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$2\,i\left(697\,096\,502\,869\,856\,831\,731\,953+114\,601\,959\,125\,910\,554\,184\,115\sqrt{37}\right)\Big)\,\omega_0^7-$

$4\,i\Bigg(626\,490\,156\,631\,868\,590\,092\,691\sqrt{6+\sqrt{37}}+$

$102\,994\,347\,307\,053\,863\,874\,913\sqrt{37\left(6+\sqrt{37}\right)}-$

$i\left(1\,567\,304\,838\,708\,777\,249\,006\,215+257\,663\,328\,282\,509\,324\,160\,511\sqrt{37}\right)\Bigg)\,\omega_0^8-$

$4\Bigg(5\,018\,393\,550\,939\,814\,291\,751\,753\,i+825\,018\,817\,674\,241\,416\,839\,611\,i\sqrt{37}+$

$1\,880\,913\,971\,343\,515\,286\,928\,713\sqrt{6+\sqrt{37}}+$

$309\,220\,352\,097\,371\,733\,118\,719\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^9-$

$4\,i\Bigg(674\,980\,963\,049\,013\,883\,196\,941\sqrt{6+\sqrt{37}}+$

$110\,966\,186\,775\,652\,670\,681\,803\sqrt{37\left(6+\sqrt{37}\right)}-i$

$\left(8\,193\,879\,902\,053\,715\,604\,592\,481+1\,347\,065\,557\,999\,351\,017\,377\,001\sqrt{37}\right)\Bigg)\,\omega_0^{10}-$

$4\Bigg(4\,973\,742\,984\,261\,297\,345\,467\,223\,i+817\,678\,309\,729\,682\,900\,204\,617\,i\sqrt{37}+$

$3\,455\,028\,782\,431\,374\,747\,463\,269\sqrt{6+\sqrt{37}}+$

$568\,003\,232\,942\,579\,627\,785\,671\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{11}+$

$4\,i\Bigg(714\,445\,001\,063\,042\,092\,332\,531\sqrt{6+\sqrt{37}}+$

$117\,454\,034\,660\,139\,023\,232\,961\sqrt{37\left(6+\sqrt{37}\right)}+$

$4\,i\left(3\,830\,313\,382\,179\,173\,574\,454\,425+629\,699\,641\,092\,414\,097\,943\,203\sqrt{37}\right)\Bigg)$

$\omega_0^{12}-4\Bigg(2\,569\,592\,023\,591\,348\,756\,341\,589\sqrt{6+\sqrt{37}}+$

$422\,438\,326\,466\,341\,497\,526\,643\sqrt{37\left(6+\sqrt{37}\right)}-$

$i\left(6\,988\,195\,187\,907\,548\,118\,825\,213+1\,148\,852\,211\,984\,171\,923\,236\,391\sqrt{37}\right)\Bigg)$

$\omega_0^{13}+4\,i\Bigg(1\,727\,964\,159\,899\,774\,572\,796\,991\sqrt{6+\sqrt{37}}+$

$284\,075\,557\,987\,472\,333\,000\,209\,\sqrt{37\,\left(6+\sqrt{37}\right)}\,+$

$4\,\mathbb{i}\,\left(930\,914\,359\,068\,327\,696\,189\,476+153\,041\,377\,898\,848\,814\,639\,165\,\sqrt{37}\right)\Big)\,\omega_0^{14}\,-$

$4\,\Big(196\,837\,619\,375\,606\,464\,428\,175\,\sqrt{6+\sqrt{37}}\,+$

$32\,359\,905\,288\,918\,442\,528\,877\,\sqrt{37\,\left(6+\sqrt{37}\right)}\,-$

$\mathbb{i}\,\left(4\,287\,474\,935\,228\,946\,661\,812\,253+704\,856\,537\,448\,751\,211\,118\,771\,\sqrt{37}\right)\Big)\,\omega_0^{15}\,+$

$4\,\Big(989\,729\,781\,529\,715\,519\,107\,637+162\,710\,573\,789\,347\,736\,735\,333\,\sqrt{37}\,+$

$573\,857\,849\,744\,599\,023\,709\,049\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\,+$

$94\,341\,649\,355\,189\,627\,318\,815\,\mathbb{i}\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{16}\,+$

$4\,\Big(267\,266\,961\,907\,699\,867\,014\,055\,\mathbb{i}+43\,938\,417\,878\,246\,259\,637\,141\,\mathbb{i}\,\sqrt{37}\,+$

$203\,930\,911\,025\,407\,114\,000\,555\,\sqrt{6+\sqrt{37}}\,+$

$33\,526\,035\,252\,999\,482\,373\,725\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{17}\,+$

$4\,\Big(127\,036\,008\,006\,183\,809\,323\,527+20\,884\,591\,063\,684\,843\,048\,483\,\sqrt{37}\,-$

$5\,021\,065\,502\,459\,213\,058\,083\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\,-$

$825\,458\,082\,052\,424\,109\,065\,\mathbb{i}\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{18}\,+$

$4\,\Big(13\,685\,677\,484\,883\,484\,571\,189\,\mathbb{i}+2\,249\,911\,526\,606\,323\,351\,779\,\mathbb{i}\,\sqrt{37}\,+$

$7\,944\,749\,834\,651\,964\,887\,292\,\sqrt{6+\sqrt{37}}\,+$

$1\,306\,108\,795\,053\,254\,557\,440\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{19}\,+$

$2\,\Big(3\,185\,810\,927\,638\,990\,546\,119+523\,743\,977\,637\,544\,212\,719\,\sqrt{37}\,+$

$3\,871\,528\,555\,208\,885\,381\,232\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\,+$

$636\,475\,482\,704\,408\,274\,536\,\mathbb{i}\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{20}\,+$

$4\,\Big(196\,559\,642\,181\,035\,242\,439\,\sqrt{6+\sqrt{37}}\,+$

$$32\,313\,402\,179\,993\,892\,865 \sqrt{37 \left(6+\sqrt{37}\right)} +$$

$$4\, i \left(2\,637\,834\,258\,931\,483\,368\,064 + 433\,657\,295\,764\,219\,258\,871 \sqrt{37}\right)\Bigg) \omega_0^{21} +$$

$$2 \Big(7\,207\,645\,667\,935\,395\,664\,541 + 1\,184\,929\,060\,350\,207\,274\,257 \sqrt{37} +$$

$$2\,672\,724\,385\,906\,058\,415\,538\, i \sqrt{6+\sqrt{37}} +$$

$$439\,394\,194\,732\,335\,462\,254\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big) \omega_0^{22} +$$

$$4 \Big(724\,811\,717\,383\,026\,143\,797\, i + 119\,158\,554\,304\,578\,942\,987\, i \sqrt{37} +$$

$$601\,691\,787\,225\,327\,369\,367 \sqrt{6+\sqrt{37}} +$$

$$98\,909\,659\,892\,298\,485\,093 \sqrt{37 \left(6+\sqrt{37}\right)}\Big) \omega_0^{23} +$$

$$4 \Big(540\,993\,653\,931\,419\,909\,439 + 88\,939\,182\,020\,035\,547\,583 \sqrt{37} -$$

$$14\,955\,954\,994\,988\,394\,289\, i \sqrt{6+\sqrt{37}} -$$

$$2\,457\,286\,328\,735\,092\,667\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big) \omega_0^{24} +$$

$$4 \Big(45\,784\,714\,983\,246\,369\,189 \sqrt{6+\sqrt{37}} + 7\,491\,478\,622\,552\,836\,675 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$3\, i \left(3\,453\,429\,928\,495\,001\,573 + 567\,462\,788\,095\,597\,815 \sqrt{37}\right)\Big) \omega_0^{25} +$$

$$4 \Big(32\,168\,844\,697\,158\,147\,385 + 5\,296\,965\,396\,679\,784\,417 \sqrt{37} -$$

$$6\,705\,577\,856\,515\,992\,863\, i \sqrt{6+\sqrt{37}} -$$

$$1\,105\,652\,305\,866\,190\,841\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big) \omega_0^{26} +$$

$$4 \Big(3\,885\,615\,000\,618\,565\,579\, i + 643\,166\,402\,533\,136\,789\, i \sqrt{37} +$$

$$1\,716\,125\,625\,076\,056\,325 \sqrt{6+\sqrt{37}} + 217\,077\,758\,113\,988\,503 \sqrt{37 \left(6+\sqrt{37}\right)}\Big)$$

$$\omega_0^{27} + 4 \Big(4\,448\,245\,466\,103\,717\,418 + 753\,079\,038\,812\,429\,730 \sqrt{37} +$$

$$420\,077\,725\,952\,493\,885\, i \sqrt{6+\sqrt{37}} + 49\,825\,869\,602\,642\,943\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big)$$

$$\omega_0^{28} + 4 \left(239\,569\,950\,690\,403\,923\,\mathbb{i} + 51\,843\,100\,240\,667\,897\,\mathbb{i}\,\sqrt{37} + 522\,356\,896\,890\,873\,681\,\sqrt{6+\sqrt{37}} + 66\,766\,483\,714\,620\,183\,\sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{29} +$$

$$4 \left(568\,363\,335\,097\,417\,266 + 98\,733\,812\,979\,077\,342\,\sqrt{37} - 1\,199\,238\,599\,574\,683\,\mathbb{i}\,\sqrt{6+\sqrt{37}} - 6\,763\,007\,179\,104\,181\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{30} +$$

$$4 \left(54\,729\,453\,951\,223\,291\,\sqrt{6+\sqrt{37}} + 6\,560\,531\,029\,002\,225\,\sqrt{37\left(6+\sqrt{37}\right)} - \mathbb{i}\left(48\,941\,882\,916\,118\,441 + 3\,986\,290\,162\,895\,399\,\sqrt{37}\right)\right) \omega_0^{31} +$$

$$4 \left(40\,930\,731\,257\,344\,767 + 7\,113\,211\,514\,138\,319\,\sqrt{37} - 6\,159\,369\,335\,326\,285\,\mathbb{i}\,\sqrt{6+\sqrt{37}} - 1\,979\,171\,974\,565\,985\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{32} +$$

$$4 \left(2\,517\,037\,294\,543\,457\,\sqrt{6+\sqrt{37}} + 254\,109\,944\,144\,095\,\sqrt{37\left(6+\sqrt{37}\right)} - 3\,\mathbb{i}\left(2\,263\,035\,191\,103\,541 + 178\,187\,390\,401\,203\,\sqrt{37}\right)\right) \omega_0^{33} +$$

$$4 \left(2\,624\,889\,613\,793\,683 + 416\,948\,382\,239\,485\,\sqrt{37} - 567\,244\,932\,241\,088\,\mathbb{i}\,\sqrt{6+\sqrt{37}} - 171\,178\,866\,582\,306\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{34} +$$

$$2 \left(115\,937\,893\,559\,271\,\sqrt{6+\sqrt{37}} + 8\,826\,076\,767\,401\,\sqrt{37\left(6+\sqrt{37}\right)} - 2\,\mathbb{i}\left(623\,234\,803\,422\,713 + 56\,827\,581\,607\,931\,\sqrt{37}\right)\right) \omega_0^{35} +$$

$$\left(661\,426\,577\,287\,215 + 92\,480\,447\,704\,519\,\sqrt{37} - 124\,640\,092\,902\,314\,\mathbb{i}\,\sqrt{6+\sqrt{37}} - 35\,270\,180\,991\,318\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right) \omega_0^{36} +$$

$$2 \left(847\,352\,188\,722\,\sqrt{6+\sqrt{37}} + 68\,350\,650\,638\,\sqrt{37\left(6+\sqrt{37}\right)} - \mathbb{i}\left(86\,237\,921\,567\,923 + 9\,997\,273\,389\,389\,\sqrt{37}\right)\right) \omega_0^{37} +$$

$$\left(31\,314\,398\,135\,117 + 4\,040\,009\,699\,385\sqrt{37} - 4\,500\,366\,865\,744\,i\sqrt{6+\sqrt{37}} - 1\,152\,392\,143\,792\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{38} +$$

$$4\left(-10\,655\,595\,664\sqrt{6+\sqrt{37}} + 633\,665\,552\sqrt{37\left(6+\sqrt{37}\right)} - i\left(1\,806\,299\,702\,919 + 241\,604\,491\,121\sqrt{37}\right)\right)\omega_0^{39} +$$

$$16\left(60\,625\,321\,405 + 7\,608\,614\,715\sqrt{37} - 6\,033\,484\,106\,i\sqrt{6+\sqrt{37}} - 1\,391\,591\,606\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{40} +$$

$$16\left(-62\,773\,164\sqrt{6+\sqrt{37}} + 5\,443\,884\sqrt{37\left(6+\sqrt{37}\right)} - i\left(10\,459\,775\,127 + 1\,524\,689\,719\sqrt{37}\right)\right)\omega_0^{41} +$$

$$16\left(1\,154\,337\,123 + 147\,357\,263\sqrt{37} - 66\,843\,904\,i\sqrt{6+\sqrt{37}} - 13\,954\,432\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{42} + 128$$

$$\left(-42\,328\sqrt{6+\sqrt{37}} + 7256\sqrt{37\left(6+\sqrt{37}\right)} - i\left(15\,372\,181 + 2\,383\,566\sqrt{37}\right)\right)\omega_0^{43} +$$

$$2048\left(95\,143 + 12\,757\sqrt{37} - 2220\,i\sqrt{6+\sqrt{37}} - 420\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{44} -$$

$$1024\,i\left(8897 + 1451\sqrt{37}\right)\omega_0^{45} +$$

$$122\,880\left(7+\sqrt{37}\right)\omega_0^{46}\Bigg)\Bigg)\Big/$$

$$\Bigg(\left(7+\sqrt{37}\right)^{3/2}\left(2\,i+\omega_0\right)^4\Big(10\,657 + 1752\sqrt{37} + \left(52\,604 + 8648\sqrt{37}\right)\omega_0^2 - 2\left(89 + 28\sqrt{37}\right)\omega_0^4 - 104\left(-4+\sqrt{37}\right)\omega_0^6 + 129\,\omega_0^8 + 4\,\omega_0^{10}\Big)^3$$

$$\Big(10\,657 + 1752\sqrt{37} + \left(108\,809 + 17\,888\sqrt{37}\right)\omega_0^2 + \left(25\,062 + 3944\sqrt{37}\right)\omega_0^4 + \left(10\,366 - 224\sqrt{37}\right)\omega_0^6 + 649\,\omega_0^8 + 9\,\omega_0^{10}\Big)$$

$$\Big(10\,657 + 1752\sqrt{37} + 8\left(23\,437 + 3853\sqrt{37}\right)\omega_0^2 + 2\left(55\,399 + 8804\sqrt{37}\right)\omega_0^4 - 8\left(-7909 + 49\sqrt{37}\right)\omega_0^6 + 2049\,\omega_0^8 + 16\,\omega_0^{10}\Big)\Bigg)\Big\}$$

(* Vetor complexo h40b *)

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h40b = Simplify[ComplexExpand[Conjugate[h40]], ω0 ∈ Reals]
       simplifica  expande funções ⋯  conjugado        números r
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$$\{\Big(96\,\omega_0^4$$

$$\Big(148\,\Big(3\,740\,393\,909\,136\,443\,058 + 614\,916\,970\,785\,157\,827\,\sqrt{37} - 415\,579\,824\,348\,194\,600$$

$$i\,\sqrt{6+\sqrt{37}} - 68\,320\,902\,267\,381\,456\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big) + 16$$

$$\Big(32\,478\,822\,660\,827\,375\,432\,\sqrt{6+\sqrt{37}} + 5\,339\,485\,554\,310\,310\,337\,\sqrt{37\,\left(6+\sqrt{37}\right)} +$$

$$37\,i\,\left(4\,877\,584\,015\,350\,072\,344 + 801\,869\,872\,620\,350\,175\,\sqrt{37}\right)\Big)\,\omega_0 +$$

$$\Big(10\,762\,729\,667\,076\,204\,250\,835 + 1\,769\,381\,857\,908\,653\,182\,857\,\sqrt{37} -$$

$$372\,985\,767\,769\,504\,909\,800\,i\,\sqrt{6+\sqrt{37}} - 61\,318\,482\,500\,617\,779\,504$$

$$i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^2 + 2\,\Big(6\,681\,282\,256\,316\,079\,610\,064\,\sqrt{6+\sqrt{37}} +$$

$$1\,098\,396\,036\,839\,616\,075\,816\,\sqrt{37\,\left(6+\sqrt{37}\right)} +$$

$$5\,i\,\left(6\,604\,768\,841\,856\,114\,242\,591 + 1\,085\,817\,308\,987\,122\,688\,545\,\sqrt{37}\right)\Big)\,\omega_0^3 +$$

$$\Big(77\,654\,232\,120\,763\,560\,314\,089 + 12\,766\,277\,120\,628\,676\,773\,527\,\sqrt{37} +$$

$$19\,898\,894\,309\,379\,780\,323\,912\,i\,\sqrt{6+\sqrt{37}} +$$

$$3\,271\,358\,072\,958\,372\,901\,792\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^4 +$$

$$2\,\Big(299\,462\,083\,550\,189\,030\,607\,887\,i + 49\,231\,263\,272\,003\,370\,249\,249\,i\,\sqrt{37} +$$

$$56\,337\,976\,533\,860\,447\,054\,104\,\sqrt{6+\sqrt{37}} +$$

$$9\,261\,906\,288\,999\,640\,471\,096\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^5 +$$

$$\Big(240\,908\,821\,965\,821\,116\,181\,479 + 39\,605\,166\,364\,107\,604\,869\,065\,\sqrt{37} +$$

$$283\,598\,327\,821\,868\,766\,442\,816\,i\,\sqrt{6+\sqrt{37}} +$$

$$46\,623\,277\,895\,407\,958\,168\,096\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^6 +$$

$$\Big(2\,823\,573\,439\,507\,197\,538\,830\,638\,i + 464\,192\,614\,037\,287\,785\,067\,698\,i\,\sqrt{37} +$$

$$326\,672\,355\,070\,511\,716\,870\,624\,\sqrt{6+\sqrt{37}}\;+$$

$$53\,704\,604\,354\,248\,224\,525\,408\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^7+$$

$$\Bigg(-136\,157\,653\,910\,854\,892\,685\,687-22\,384\,180\,416\,817\,864\,721\,585\,\sqrt{37}\;+$$

$$1\,377\,414\,536\,681\,641\,437\,912\,256\,\mathrm{i}\,\sqrt{6+\sqrt{37}}\;+$$

$$226\,445\,554\,930\,139\,762\,491\,968\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^8-$$

$$2\,\Bigg(82\,169\,027\,578\,480\,583\,564\,144\,\sqrt{6+\sqrt{37}}\;+$$

$$13\,508\,504\,921\,768\,182\,428\,880\,\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$\mathrm{i}\,\left(3\,576\,074\,213\,040\,352\,764\,893\,283+587\,902\,979\,152\,636\,511\,240\,389\,\sqrt{37}\right)\Bigg)\,\omega_0^9+$$

$$4\,\mathrm{i}\,\Bigg(935\,238\,940\,358\,630\,317\,643\,425\,\mathrm{i}+153\,752\,334\,683\,494\,268\,223\,323\,\mathrm{i}\,\sqrt{37}\;+$$

$$621\,796\,906\,968\,668\,938\,837\,704\,\sqrt{6+\sqrt{37}}\;+$$

$$102\,222\,781\,815\,252\,632\,198\,496\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{10}-$$

$$8\,\Bigg(239\,492\,934\,636\,522\,394\,143\,636\,\sqrt{6+\sqrt{37}}\;+$$

$$39\,372\,395\,921\,032\,406\,348\,348\,\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$\mathrm{i}\,\left(908\,694\,329\,924\,444\,350\,869\,905+149\,388\,427\,609\,698\,874\,595\,191\,\sqrt{37}\right)\Bigg)\,\omega_0^{11}+$$

$$4\,\mathrm{i}\,\Bigg(2\,227\,355\,386\,567\,112\,461\,328\,747\,\mathrm{i}+366\,174\,969\,920\,765\,909\,296\,733\,\mathrm{i}\,\sqrt{37}\;+$$

$$175\,028\,789\,699\,616\,625\,252\,664\,\sqrt{6+\sqrt{37}}\;+$$

$$28\,774\,555\,775\,899\,358\,415\,152\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{12}-$$

$$8\,\Bigg(335\,819\,208\,354\,324\,004\,915\,369\,\mathrm{i}+55\,208\,337\,771\,137\,693\,424\,287\,\mathrm{i}\,\sqrt{37}\;+$$

$$177\,954\,061\,743\,973\,067\,816\,020\,\sqrt{6+\sqrt{37}}\;+$$

$$29\,255\,467\,537\,583\,428\,321\,780\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{13}+$$

$$\Bigg(-826\,762\,955\,928\,291\,711\,550\,036-135\,918\,992\,696\,215\,482\,703\,116\,\sqrt{37}\;-$$

$547\,024\,441\,625\,827\,792\,423\,872\,i\sqrt{6+\sqrt{37}}\;-$

$89\,930\,264\,234\,594\,205\,599\,040\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{14}\;-$

$8\Big(72\,802\,666\,962\,703\,663\,692\,769\,i+11\,968\,684\,721\,782\,052\,325\,687\,i\sqrt{37}\;+$

$4\,592\,333\,510\,361\,546\,931\,460\sqrt{6+\sqrt{37}}\;+$

$754\,974\,978\,444\,257\,653\,412\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{15}\;-$

$4\,i\Big(18\,821\,732\,509\,823\,785\,021\,472\sqrt{6+\sqrt{37}}\;+$

$3\,094\,273\,763\,590\,519\,889\,296\sqrt{37\left(6+\sqrt{37}\right)}\;-$

$i\left(5\,175\,890\,394\,808\,545\,011\,543+850\,911\,139\,594\,033\,850\,875\sqrt{37}\right)\Bigg)\omega_0^{16}\;+$

$8\Big(1\,359\,438\,589\,588\,851\,780\,020\sqrt{6+\sqrt{37}}\;+$

$223\,490\,327\,190\,110\,198\,992\sqrt{37\left(6+\sqrt{37}\right)}\;-$

$i\left(7\,563\,338\,728\,219\,274\,083\,357+1\,243\,405\,227\,003\,053\,202\,975\sqrt{37}\right)\Bigg)\omega_0^{17}\;-$

$2\,i\Big(1\,062\,408\,465\,675\,137\,141\,368\sqrt{6+\sqrt{37}}\;+$

$174\,658\,825\,686\,526\,766\,512\sqrt{37\left(6+\sqrt{37}\right)}\;-$

$i\left(5\,221\,840\,449\,655\,409\,589\,651+858\,465\,293\,362\,775\,384\,281\sqrt{37}\right)\Bigg)\omega_0^{18}\;-$

$4\Big(3\,112\,144\,980\,052\,781\,057\,347\,i+511\,633\,473\,436\,792\,282\,637\,i\sqrt{37}\;+$

$483\,375\,773\,444\,469\,357\,688\sqrt{6+\sqrt{37}}\;+$

$79\,466\,525\,608\,906\,460\,432\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{19}\;+$

$\Big(-522\,896\,530\,698\,864\,533\,106-85\,963\,868\,943\,065\,066\,350\sqrt{37}\;-$

$1\,523\,198\,569\,100\,795\,382\,096\,i\sqrt{6+\sqrt{37}}\;-$

$250\,410\,427\,406\,560\,879\,104\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{20}\;-$

$4\Big(485\,769\,173\,957\,419\,445\,719\,i+79\,860\,018\,683\,252\,889\,417\,i\sqrt{37}\;+$

$$2\,162\,986\,203\,640\,025\,424 \sqrt{6+\sqrt{37}} + 356\,108\,700\,189\,008\,688 \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)$$

$$\omega_0^{21} + 2\Bigg(62\,723\,631\,038\,361\,283\,041 + 10\,311\,313\,694\,931\,463\,359 \sqrt{37} -$$

$$121\,873\,032\,455\,346\,789\,344\, i \sqrt{6+\sqrt{37}} -$$

$$20\,027\,936\,241\,981\,221\,632\, i \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{22} +$$

$$4\Bigg(13\,325\,107\,854\,336\,714\,056 \sqrt{6+\sqrt{37}} + 2\,187\,912\,009\,095\,045\,768 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$3\, i \left(18\,637\,966\,038\,322\,721\,485 + 3\,064\,497\,622\,892\,874\,483 \sqrt{37}\right)\Bigg)\omega_0^{23} +$$

$$2\Bigg(14\,870\,809\,830\,528\,243\,503 + 2\,444\,355\,301\,751\,893\,825 \sqrt{37} -$$

$$5\,371\,565\,333\,108\,875\,296\, i \sqrt{6+\sqrt{37}} -$$

$$851\,300\,951\,672\,963\,104\, i \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{24} +$$

$$4\Bigg(984\,191\,824\,577\,653\,656 \sqrt{6+\sqrt{37}} + 157\,163\,477\,911\,595\,432 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i \left(5\,190\,635\,237\,046\,095\,171 + 860\,336\,448\,286\,085\,701 \sqrt{37}\right)\Bigg)\omega_0^{25} +$$

$$4\Bigg(984\,462\,697\,808\,906\,607 + 161\,279\,007\,687\,299\,221 \sqrt{37} - 294\,214\,366\,578\,664\,696$$

$$i \sqrt{6+\sqrt{37}} - 21\,376\,598\,608\,225\,472\, i \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{26} +$$

$$8\Bigg(25\,220\,725\,157\,147\,428 \sqrt{6+\sqrt{37}} + 5\,194\,248\,750\,050\,396 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i \left(218\,197\,307\,636\,617\,417 + 42\,639\,575\,241\,993\,807 \sqrt{37}\right)\Bigg)\omega_0^{27} +$$

$$4\Bigg(119\,927\,654\,779\,788\,789 + 15\,834\,281\,029\,526\,963 \sqrt{37} - 68\,864\,729\,813\,615\,944$$

$$i \sqrt{6+\sqrt{37}} - 2\,935\,846\,081\,216\,816\, i \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{28} +$$

$$8\Bigg(5\,015\,975\,928\,063\,604 \sqrt{6+\sqrt{37}} + 1\,337\,039\,715\,408\,020 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i \left(18\,796\,816\,913\,746\,561 + 5\,080\,776\,666\,671\,639 \sqrt{37}\right)\Bigg)\omega_0^{29} +$$

$$12\left(5\,833\,938\,425\,776\,589+516\,523\,452\,114\,267\sqrt{37}-2\,159\,109\,372\,641\,456\,i\sqrt{6+\sqrt{37}}+43\,695\,480\,359\,152\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{30}+$$

$$8\left(547\,608\,955\,103\,636\sqrt{6+\sqrt{37}}+172\,638\,663\,187\,316\sqrt{37\left(6+\sqrt{37}\right)}-i\left(1\,026\,439\,159\,705\,789+424\,988\,024\,210\,987\sqrt{37}\right)\right)\omega_0^{31}+$$

$$4\left(1\,994\,180\,814\,434\,651+130\,748\,216\,106\,054\sqrt{37}-260\,088\,561\,226\,456\,i\sqrt{6+\sqrt{37}}+53\,299\,853\,066\,720\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{32}+$$

$$8\left(27\,719\,996\,911\,788\sqrt{6+\sqrt{37}}+11\,561\,175\,018\,990\sqrt{37\left(6+\sqrt{37}\right)}-i\left(13\,670\,361\,120\,489+19\,548\,526\,059\,137\sqrt{37}\right)\right)\omega_0^{33}+$$

$$\left(618\,989\,981\,828\,779+36\,595\,884\,535\,217\sqrt{37}-8\,669\,010\,577\,512\,i\sqrt{6+\sqrt{37}}+17\,227\,914\,181\,648\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{34}+$$

$$\left(9\,268\,848\,019\,206\,i-3\,272\,756\,982\,598\,i\sqrt{37}+5\,736\,955\,046\,848\sqrt{6+\sqrt{37}}+3\,742\,540\,413\,680\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{35}+$$

$$\left(32\,875\,546\,490\,129+2\,144\,577\,043\,791\sqrt{37}+708\,154\,147\,656\,i\sqrt{6+\sqrt{37}}+670\,302\,596\,704\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{36}+$$

$$2\left(156\,258\,826\,263\,i+505\,158\,905\,i\sqrt{37}+37\,617\,275\,144\sqrt{6+\sqrt{37}}+48\,450\,294\,184\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{37}+$$

$$\left(1\,171\,133\,837\,199+91\,717\,863\,169\sqrt{37}+25\,264\,556\,672\,i\sqrt{6+\sqrt{37}}+13\,881\,439\,712\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{38}+$$

$$2\left(223\,967\,808\sqrt{6+\sqrt{37}}+786\,907\,520\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$
$$\left.13\,i\left(-192\,418\,645+45\,703\,461\sqrt{37}\right)\right)\omega_0^{39}+$$
$$\left(26\,756\,104\,225+2\,509\,454\,919\sqrt{37}+327\,016\,064\,i\sqrt{6+\sqrt{37}}+\right.$$
$$\left.145\,472\,768\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{40}+$$
$$2\left(-191\,915\,909\,i+7\,044\,717\,i\sqrt{37}+516\,224\sqrt{6+\sqrt{37}}+7\,203\,968\sqrt{37\left(6+\sqrt{37}\right)}\right)$$
$$\omega_0^{41}+$$
$$16\left(22\,959\,623+2\,517\,881\sqrt{37}+92\,352\,i\sqrt{6+\sqrt{37}}+37\,344\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{42}+$$
$$16\left(3456\sqrt{37\left(6+\sqrt{37}\right)}-i\left(395\,479+2161\sqrt{37}\right)\right)\omega_0^{43}+$$
$$16\left(169\,061+21\,251\sqrt{37}\right)\omega_0^{44}-$$
$$\left.\left.864\,i\left(39+\sqrt{37}\right)\omega_0^{45}+1152\left(7+\sqrt{37}\right)\omega_0^{46}\right)\right)\Big/$$
$$\left(\left(7+\sqrt{37}\right)\left(-2\,i+\omega_0\right)^4\left(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-\right.\right.$$
$$\left.2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\right)^3$$
$$\left(10\,657+1752\sqrt{37}+\left(108\,809+17\,888\sqrt{37}\right)\omega_0^2+\left(25\,062+3944\sqrt{37}\right)\omega_0^4+\right.$$
$$\left.\left(10\,366-224\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\right)$$
$$\left(10\,657+1752\sqrt{37}+8\left(23\,437+3853\sqrt{37}\right)\omega_0^2+2\left(55\,399+8804\sqrt{37}\right)\omega_0^4-\right.$$
$$\left.\left.8\left(-7909+49\sqrt{37}\right)\omega_0^6+2049\,\omega_0^8+16\,\omega_0^{10}\right)\right),$$
$$-\left(\left(192\,\omega_0^4\left(-296\,i\sqrt{6+\sqrt{37}}\left(51\,947\,478\,043\,524\,325+8\,540\,112\,783\,422\,682\sqrt{37}\right)+\right.\right.\right.$$
$$5328\left(8\,598\,609\,450\,986\,568\sqrt{6+\sqrt{37}}+1\,413\,602\,685\,976\,466\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$
$$\left.i\left(8\,540\,112\,783\,422\,682+1\,403\,985\,893\,068\,225\sqrt{37}\right)\right)\omega_0+$$
$$\left(-594\,780\,425\,427\,787\,479\,356\,i\sqrt{6+\sqrt{37}}+\right.$$
$$12\left(-8\,148\,441\,634\,114\,820\,121\,i\sqrt{37\left(6+\sqrt{37}\right)}-74\left(79\,327\,557\,752\,930\,552+\right.\right.$$
$$\left.\left.\left.13\,041\,370\,159\,989\,027\sqrt{37}\right)\right)\right)\omega_0^2+2\left(927\,815\,354\,308\,038\,962\,851\right.$$

$$\sqrt{6+\sqrt{37}} + 152\,531\,904\,654\,602\,872\,633\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$8\,i\left(177\,818\,317\,928\,629\,942\,957 + 29\,233\,151\,391\,808\,820\,398\sqrt{37}\right)\Bigg)\omega_0^3 +$$

$$\Bigg(1\,000\,717\,458\,361\,564\,773\,139 + 164\,516\,936\,733\,432\,298\,471\sqrt{37} -$$

$$8\,100\,394\,922\,459\,923\,362\,606\,i\sqrt{6+\sqrt{37}} - 1\,331\,696\,722\,025\,869\,581\,386$$

$$i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^4 + \Bigg(27\,836\,039\,391\,081\,083\,430\,592\sqrt{6+\sqrt{37}} +$$

$$4\,576\,216\,686\,485\,764\,472\,800\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$2\,i\left(27\,235\,485\,387\,517\,952\,299\,621 + 4\,477\,486\,216\,477\,807\,781\,959\sqrt{37}\right)\Bigg)\omega_0^5 +$$

$$\Bigg(67\,928\,939\,708\,705\,255\,938\,533 + 11\,167\,448\,896\,837\,816\,024\,621\sqrt{37} -$$

$$46\,014\,831\,723\,850\,974\,578\,868\,i\sqrt{6+\sqrt{37}} - 7\,564\,791\,736\,427\,528\,223\,740$$

$$i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^6 + 4\Bigg(49\,051\,523\,698\,202\,034\,094\,643\sqrt{6+\sqrt{37}} +$$

$$8\,064\,020\,821\,769\,148\,418\,997\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$12\,i\left(8\,678\,583\,251\,463\,642\,856\,793 + 1\,426\,750\,297\,785\,857\,806\,076\sqrt{37}\right)\Bigg)\omega_0^7 +$$

$$4\Bigg(198\,667\,219\,092\,490\,141\,570\,801 + 32\,660\,689\,629\,576\,921\,995\,501\sqrt{37} -$$

$$19\,372\,884\,839\,811\,127\,335\,387\,i\sqrt{6+\sqrt{37}} - 3\,184\,882\,648\,848\,258\,197\,925$$

$$i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^8 + 12\Bigg(56\,165\,330\,429\,937\,572\,769\,183\sqrt{6+\sqrt{37}} +$$

$$9\,233\,523\,444\,352\,504\,466\,365\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i\left(94\,599\,111\,464\,268\,185\,761\,747 + 15\,551\,998\,124\,712\,304\,643\,909\sqrt{37}\right)\Bigg)\omega_0^9 +$$

$$4\Bigg(866\,048\,517\,543\,601\,789\,493\,775 + 142\,377\,499\,241\,474\,089\,889\,507\sqrt{37} +$$

$$55\,698\,104\,268\,947\,971\,324\,703\,i\sqrt{6+\sqrt{37}} + 9\,156\,711\,936\,643\,244\,197\,709$$

$$i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{10} + 4\Bigg(249\,069\,195\,950\,300\,084\,354\,093\sqrt{6+\sqrt{37}} +$$

$$40\,946\,723\,583\,188\,931\,641\,651\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$9\,\mathrm{i}\,\left(17\,974\,597\,661\,440\,164\,091\,709+2\,955\,005\,652\,762\,006\,334\,927\,\sqrt{37}\right)\right)\,\omega_0^{11}+$$

$$4\,\left(1\,160\,916\,840\,161\,286\,720\,060\,604+190\,853\,552\,868\,250\,869\,266\,244\,\sqrt{37}+\right.$$

$$224\,038\,051\,056\,411\,022\,556\,603\,\mathrm{i}\,\sqrt{6+\sqrt{37}}+$$

$$\left.36\,831\,628\,711\,539\,908\,968\,261\,\mathrm{i}\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\,\omega_0^{12}+$$

$$4\,\left(1\,347\,004\,037\,229\,567\,186\,694\,421\,\mathrm{i}+221\,446\,099\,616\,768\,615\,111\,987\,\mathrm{i}\,\sqrt{37}+\right.$$

$$64\,330\,289\,798\,458\,898\,547\,945\,\sqrt{6+\sqrt{37}}+$$

$$\left.10\,575\,834\,495\,927\,221\,205\,435\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\,\omega_0^{13}+$$

$$4\,\mathrm{i}\,\left(583\,672\,388\,154\,525\,253\,766\,948\,\mathrm{i}+95\,955\,149\,530\,704\,734\,762\,048\,\mathrm{i}\,\sqrt{37}+\right.$$

$$185\,603\,786\,606\,259\,230\,733\,187\,\sqrt{6+\sqrt{37}}+$$

$$\left.30\,513\,074\,558\,105\,145\,318\,417\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\,\omega_0^{14}-$$

$$4\,\left(31\,607\,355\,441\,274\,468\,231\,695\,\mathrm{i}+5\,196\,217\,225\,935\,366\,457\,653\,\mathrm{i}\,\sqrt{37}+\right.$$

$$106\,597\,532\,662\,336\,352\,059\,057\,\sqrt{6+\sqrt{37}}+17\,524\,526\,418\,931\,061\,326\,567$$

$$\left.\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\,\omega_0^{15}-4\,\mathrm{i}\,\left(23\,084\,324\,537\,027\,185\,287\,235\,\sqrt{6+\sqrt{37}}+\right.$$

$$3\,795\,039\,576\,538\,613\,001\,081\,\sqrt{37\,\left(6+\sqrt{37}\right)}-$$

$$\left.\mathrm{i}\,\left(60\,985\,303\,331\,879\,148\,217\,571+10\,025\,922\,108\,330\,942\,809\,423\,\sqrt{37}\right)\right)\,\omega_0^{16}-$$

$$4\,\left(1\,593\,701\,330\,413\,609\,291\,799\,\sqrt{6+\sqrt{37}}+\right.$$

$$262\,002\,885\,046\,161\,512\,213\,\sqrt{37\,\left(6+\sqrt{37}\right)}+$$

$$\left.3\,\mathrm{i}\,\left(6\,086\,231\,362\,282\,517\,551\,971+1\,000\,570\,272\,308\,540\,565\,685\,\sqrt{37}\right)\right)\,\omega_0^{17}+$$

$$4\,\left(1\,575\,264\,322\,065\,157\,991\,747+258\,971\,858\,275\,828\,756\,035\,\sqrt{37}-\right.$$

$$1\,470\,746\,134\,994\,975\,295\,603\,\mathrm{i}\,\sqrt{6+\sqrt{37}}-241\,789\,171\,237\,790\,151\,861$$

$$\left.\mathrm{i}\,\sqrt{37\,\left(6+\sqrt{37}\right)}\right)\,\omega_0^{18}+12\,\left(105\,417\,932\,522\,813\,820\,128\,\sqrt{6+\sqrt{37}}+\right.$$

$17\,330\,598\,737\,812\,510\,948\,\sqrt{37\,\left(6+\sqrt{37}\right)}\,+$

$3\,i\,\left(241\,738\,240\,627\,864\,974\,577+39\,741\,522\,725\,225\,344\,979\,\sqrt{37}\right)\Big)\,\omega_0^{19}+$

$2\,i\,\Big(3\,144\,654\,784\,280\,492\,607\,473\,i+516\,978\,077\,631\,583\,547\,037\,i\,\sqrt{37}\,+$

$739\,200\,320\,594\,433\,117\,224\,\sqrt{6+\sqrt{37}}\,+121\,523\,983\,587\,495\,947\,136$

$\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{20}-4\,\Big(260\,979\,904\,256\,567\,944\,123\,\sqrt{6+\sqrt{37}}\,+$

$42\,904\,905\,378\,901\,622\,881\,\sqrt{37\,\left(6+\sqrt{37}\right)}\,+$

$4\,i\,\left(41\,945\,776\,998\,423\,225\,688+6\,895\,834\,144\,823\,510\,413\,\sqrt{37}\right)\Big)\,\omega_0^{21}+$

$\Big(-748\,688\,675\,595\,482\,332\,942-123\,083\,779\,094\,587\,147\,118\,\sqrt{37}\,-$

$248\,788\,572\,038\,352\,999\,148\,i\,\sqrt{6+\sqrt{37}}\,-$

$40\,896\,668\,436\,971\,334\,756\,i\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{22}-$

$4\,\Big(63\,939\,340\,882\,396\,863\,691\,i+10\,511\,610\,500\,366\,414\,593\,i\,\sqrt{37}\,+$

$8\,616\,092\,787\,436\,950\,071\,\sqrt{6+\sqrt{37}}\,+$

$1\,416\,705\,038\,486\,477\,585\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{23}-4\,i\,\Big(6\,956\,101\,788\,291\,824\,697$

$\sqrt{6+\sqrt{37}}\,+1\,139\,137\,474\,722\,514\,775\,\sqrt{37\,\left(6+\sqrt{37}\right)}\,-$

$i\,\left(2\,439\,465\,399\,386\,541\,577+401\,138\,769\,619\,614\,877\,\sqrt{37}\right)\Big)\,\omega_0^{24}+$

$4\,\Big(815\,387\,544\,880\,046\,103\,\sqrt{6+\sqrt{37}}\,+134\,372\,271\,724\,539\,237\,\sqrt{37\,\left(6+\sqrt{37}\right)}\,-$

$i\,\left(2\,469\,367\,402\,553\,337\,047+407\,021\,853\,115\,062\,569\,\sqrt{37}\right)\Big)\,\omega_0^{25}+$

$4\,i\,\Big(1\,025\,818\,294\,704\,796\,951\,i+169\,162\,046\,175\,747\,819\,i\,\sqrt{37}\,-$

$40\,077\,901\,731\,702\,731\,\sqrt{6+\sqrt{37}}\,+1\,567\,294\,943\,637\,631\,\sqrt{37\,\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{26}-$

$4\,\Big(349\,220\,632\,176\,433\,057\,i+60\,151\,555\,782\,735\,451\,i\,\sqrt{37}\,+$

$$122\,781\,886\,199\,706\,525\sqrt{6+\sqrt{37}}+17\,805\,705\,243\,897\,171\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)$$

$$\omega_0^{27}-4\,i\Bigg(53\,867\,395\,024\,823\,835\sqrt{6+\sqrt{37}}+6\,458\,729\,553\,951\,765$$

$$\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(66\,090\,736\,108\,344\,371+11\,640\,426\,694\,488\,115\sqrt{37}\right)\Bigg)$$

$$\omega_0^{28}-4\Bigg(65\,646\,931\,973\,615\,813\,i+11\,459\,465\,816\,254\,355\,i\sqrt{37}+$$

$$10\,069\,529\,317\,526\,325\sqrt{6+\sqrt{37}}+837\,852\,994\,246\,623\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{29}-$$

$$4\,i\Bigg(7\,702\,913\,085\,143\,407\sqrt{6+\sqrt{37}}+960\,486\,341\,322\,293\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$2\,i\left(1\,010\,961\,381\,814\,001+419\,680\,307\,117\,875\sqrt{37}\right)\Bigg)\omega_0^{30}+$$

$$4\Bigg(201\,204\,578\,772\,613\sqrt{6+\sqrt{37}}+153\,692\,520\,467\,251\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$9\,i\left(582\,209\,743\,339\,669+101\,134\,951\,606\,007\sqrt{37}\right)\Bigg)\omega_0^{31}+$$

$$4\Bigg(304\,463\,476\,616\,439-22\,659\,745\,853\,709\sqrt{37}-437\,980\,769\,762\,863\,i\sqrt{6+\sqrt{37}}-$$

$$51\,941\,840\,177\,519\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{32}+$$

$$12\Bigg(17\,287\,859\,285\,809\sqrt{6+\sqrt{37}}+6\,086\,733\,301\,035\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\left(121\,899\,514\,985\,577+19\,452\,310\,726\,475\sqrt{37}\right)\Bigg)\omega_0^{33}+$$

$$4\Bigg(42\,369\,307\,087\,865+1\,256\,294\,684\,675\sqrt{37}-14\,581\,035\,936\,210\,i\sqrt{6+\sqrt{37}}-$$

$$1\,752\,705\,236\,876\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{34}+$$

$$2\Bigg(6\,884\,571\,316\,221\sqrt{6+\sqrt{37}}+2\,055\,439\,284\,399\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$2\,i\left(25\,075\,212\,510\,889+3\,617\,001\,936\,607\sqrt{37}\right)\Bigg)\omega_0^{35}+$$

$$\Bigg(14\,872\,389\,508\,087+1\,398\,537\,459\,803\sqrt{37}-1\,254\,929\,257\,890\,i\sqrt{6+\sqrt{37}}-$$

$$188\,037\,197\,174\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{36}+$$

$$6\left(88\,780\,683\,706\,\sqrt{6+\sqrt{37}}+23\,192\,004\,830\,\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$

$$\left.11\,\mathbb{i}\left(73\,496\,671\,003+9\,988\,552\,561\,\sqrt{37}\right)\right)\omega_0^{37}+$$

$$\left(704\,939\,027\,649+88\,167\,837\,529\,\sqrt{37}-16\,144\,455\,088\,\mathbb{i}\,\sqrt{6+\sqrt{37}}-\right.$$

$$\left.3\,838\,576\,144\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{38}+12\left(980\,658\,064\,\sqrt{6+\sqrt{37}}+\right.$$

$$\left.227\,861\,808\,\sqrt{37\left(6+\sqrt{37}\right)}-\mathbb{i}\left(12\,138\,769\,535+1\,602\,917\,081\,\sqrt{37}\right)\right)\omega_0^{39}+$$

$$16\left(1\,085\,072\,697+153\,969\,179\,\sqrt{37}-6\,353\,714\,\mathbb{i}\,\sqrt{6+\sqrt{37}}-\right.$$

$$\left.3\,011\,174\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{40}+16\left(8\,266\,540\,\sqrt{6+\sqrt{37}}+\right.$$

$$\left.1\,729\,828\,\sqrt{37\left(6+\sqrt{37}\right)}-3\,\mathbb{i}\left(54\,493\,771+7\,191\,943\,\sqrt{37}\right)\right)\omega_0^{41}+$$

$$16\left(13\,146\,367+2\,017\,167\,\sqrt{37}-15\,392\,\mathbb{i}\,\sqrt{6+\sqrt{37}}-16\,736\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{42}+384\left(1480\,\sqrt{6+\sqrt{37}}+280\,\sqrt{37\left(6+\sqrt{37}\right)}-\mathbb{i}\left(67\,293+9140\,\sqrt{37}\right)\right)\omega_0^{43}+$$

$$1024\left(964+157\,\sqrt{37}\right)\omega_0^{44}-15\,360\,\mathbb{i}\left(7+\sqrt{37}\right)\omega_0^{45}\Bigg)\Bigg)\Bigg/$$

$$\Bigg(\left(7+\sqrt{37}\right)^{3/2}\left(-2\,\mathbb{i}+\omega_0\right)^4\left(10\,657+1752\,\sqrt{37}+\left(52\,604+8648\,\sqrt{37}\right)\omega_0^2-\right.$$

$$\left.2\left(89+28\,\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\right)^3$$

$$\left(10\,657+1752\,\sqrt{37}+\left(108\,809+17\,888\,\sqrt{37}\right)\omega_0^2+\left(25\,062+3944\,\sqrt{37}\right)\omega_0^4+\right.$$

$$\left.\left(10\,366-224\,\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\right)$$

$$\left(10\,657+1752\,\sqrt{37}+8\left(23\,437+3853\,\sqrt{37}\right)\omega_0^2+2\left(55\,399+8804\,\sqrt{37}\right)\omega_0^4-\right.$$

$$\left.8\left(-7909+49\,\sqrt{37}\right)\omega_0^6+2049\,\omega_0^8+16\,\omega_0^{10}\right)\Bigg)\Bigg),$$

$$\Bigg(192\,\omega_0^2\Bigg(296\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\left(627\,669\,041\,247\,785\,184+103\,188\,154\,744\,060\,417\,\sqrt{37}\right)-$$

$$592\left(-928\,693\,392\,696\,543\,753\,\mathbb{i}-152\,676\,253\,276\,488\,288\,\mathbb{i}\,\sqrt{37}+\right.$$

$$\left.727\,264\,692\,609\,340\,550\,\sqrt{6+\sqrt{37}}+119\,561\,578\,967\,917\,548\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$4\left(303\,789\,563\,241\,205\,462\,434+49\,942\,696\,550\,790\,972\,108\sqrt{37}+1\,888\,274\,441\,838\,139\,121\,073\,i\sqrt{6+\sqrt{37}}+310\,430\,405\,992\,778\,853\,713\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2+$$

$$\left(-17\,662\,901\,792\,714\,780\,294\,206\sqrt{6+\sqrt{37}}-2\,903\,763\,167\,596\,289\,665\,682\sqrt{37\left(6+\sqrt{37}\right)}+16\,i\left(2\,113\,315\,073\,426\,404\,847\,380+347\,426\,857\,928\,447\,737\,357\sqrt{37}\right)\right)\omega_0^3+$$

$$\left(10\,669\,313\,285\,558\,249\,016\,235+1\,754\,024\,299\,389\,370\,446\,979\sqrt{37}+112\,720\,193\,918\,645\,338\,092\,534\,i\sqrt{6+\sqrt{37}}+18\,531\,085\,729\,088\,786\,813\,050\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4+$$

$$\left(-271\,533\,094\,366\,780\,399\,176\,304\sqrt{6+\sqrt{37}}-44\,639\,765\,733\,788\,713\,616\,304\sqrt{37\left(6+\sqrt{37}\right)}+2\,i\left(333\,082\,772\,168\,232\,860\,822\,765+54\,758\,470\,433\,318\,528\,185\,259\sqrt{37}\right)\right)\omega_0^5+$$

$$\left(-385\,001\,481\,234\,943\,491\,748\,239-63\,293\,853\,626\,087\,227\,574\,915\sqrt{37}+778\,674\,599\,719\,573\,059\,196\,324\,i\sqrt{6+\sqrt{37}}+128\,013\,315\,634\,302\,259\,703\,708\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^6-$$

$$4\left(500\,648\,249\,146\,968\,746\,913\,011\sqrt{6+\sqrt{37}}+82\,306\,065\,155\,961\,868\,161\,145\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(697\,096\,502\,869\,856\,831\,731\,953+114\,601\,959\,125\,910\,554\,184\,115\sqrt{37}\right)\right)\omega_0^7+$$

$$4\,i\left(1\,567\,304\,838\,708\,777\,249\,006\,215\,i+257\,663\,328\,282\,509\,324\,160\,511\,i\sqrt{37}+626\,490\,156\,631\,868\,590\,092\,691\sqrt{6+\sqrt{37}}+102\,994\,347\,307\,053\,863\,874\,913\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^8-$$

$$4 \left(1\,880\,913\,971\,343\,515\,286\,928\,713 \sqrt{6+\sqrt{37}} + 309\,220\,352\,097\,371\,733\,118\,719 \sqrt{37 \left(6+\sqrt{37}\right)} - \mathrm{i} \left(5\,018\,393\,550\,939\,814\,291\,751\,753 + 825\,018\,817\,674\,241\,416\,839\,611 \sqrt{37}\right)\right) \omega_0^9 +$$

$$4\,\mathrm{i} \left(8\,193\,879\,902\,053\,715\,604\,592\,481\,\mathrm{i} + 1\,347\,065\,557\,999\,351\,017\,377\,001\,\mathrm{i} \sqrt{37} + 674\,980\,963\,049\,013\,883\,196\,941 \sqrt{6+\sqrt{37}} + 110\,966\,186\,775\,652\,670\,681\,803 \sqrt{37 \left(6+\sqrt{37}\right)}\right) \omega_0^{10} -$$

$$4 \left(3\,455\,028\,782\,431\,374\,747\,463\,269 \sqrt{6+\sqrt{37}} + 568\,003\,232\,942\,579\,627\,785\,671 \sqrt{37 \left(6+\sqrt{37}\right)} - \mathrm{i} \left(4\,973\,742\,984\,261\,297\,345\,467\,223 + 817\,678\,309\,729\,682\,900\,204\,617 \sqrt{37}\right)\right) \omega_0^{11} -$$

$$4\,\mathrm{i} \left(714\,445\,001\,063\,042\,092\,332\,531 \sqrt{6+\sqrt{37}} + 117\,454\,034\,660\,139\,023\,232\,961 \sqrt{37 \left(6+\sqrt{37}\right)} - 4\,\mathrm{i} \left(3\,830\,313\,382\,179\,173\,574\,454\,425 + 629\,699\,641\,092\,414\,097\,943\,203 \sqrt{37}\right)\right) \omega_0^{12} -$$

$$4 \left(6\,988\,195\,187\,907\,548\,118\,825\,213\,\mathrm{i} + 1\,148\,852\,211\,984\,171\,923\,236\,391\,\mathrm{i} \sqrt{37} + 2\,569\,592\,023\,591\,348\,756\,341\,589 \sqrt{6+\sqrt{37}} + 422\,438\,326\,466\,341\,497\,526\,643 \sqrt{37 \left(6+\sqrt{37}\right)}\right) \omega_0^{13} -$$

$$4\,\mathrm{i} \left(1\,727\,964\,159\,899\,774\,572\,796\,991 \sqrt{6+\sqrt{37}} + 284\,075\,557\,987\,472\,333\,000\,209 \sqrt{37 \left(6+\sqrt{37}\right)} - 4\,\mathrm{i} \left(930\,914\,359\,068\,327\,696\,189\,476 + 153\,041\,377\,898\,848\,814\,639\,165 \sqrt{37}\right)\right) \omega_0^{14} -$$

$$4 \left(4\,287\,474\,935\,228\,946\,661\,812\,253\,\mathrm{i} + 704\,856\,537\,448\,751\,211\,118\,771\,\mathrm{i} \sqrt{37} + 196\,837\,619\,375\,606\,464\,428\,175 \sqrt{6+\sqrt{37}} + \right.$$

$32\,359\,905\,288\,918\,442\,528\,877 \sqrt{37 \left(6+\sqrt{37}\right)}\Big)\, \omega_0^{15} +$

$4 \Big(989\,729\,781\,529\,715\,519\,107\,637 + 162\,710\,573\,789\,347\,736\,735\,333 \sqrt{37} -$

$573\,857\,849\,744\,599\,023\,709\,049\, i \sqrt{6+\sqrt{37}} -$

$94\,341\,649\,355\,189\,627\,318\,815\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big)\, \omega_0^{16} +$

$4 \Big(203\,930\,911\,025\,407\,114\,000\,555 \sqrt{6+\sqrt{37}} +$

$33\,526\,035\,252\,999\,482\,373\,725 \sqrt{37 \left(6+\sqrt{37}\right)} -$

$i \left(267\,266\,961\,907\,699\,867\,014\,055 + 43\,938\,417\,878\,246\,259\,637\,141 \sqrt{37}\right)\Big)\, \omega_0^{17} +$

$4 \Big(127\,036\,008\,006\,183\,809\,323\,527 + 20\,884\,591\,063\,684\,843\,048\,483 \sqrt{37} +$

$5\,021\,065\,502\,459\,213\,058\,083\, i \sqrt{6+\sqrt{37}} +$

$825\,458\,082\,052\,424\,109\,065\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big)\, \omega_0^{18} +$

$4 \Big(7\,944\,749\,834\,651\,964\,887\,292 \sqrt{6+\sqrt{37}} +$

$1\,306\,108\,795\,053\,254\,557\,440 \sqrt{37 \left(6+\sqrt{37}\right)} -$

$i \left(13\,685\,677\,484\,883\,484\,571\,189 + 2\,249\,911\,526\,606\,323\,351\,779 \sqrt{37}\right)\Big)\, \omega_0^{19} +$

$2 \Big(3\,185\,810\,927\,638\,990\,546\,119 + 523\,743\,977\,637\,544\,212\,719 \sqrt{37} -$

$3\,871\,528\,555\,208\,885\,381\,232\, i \sqrt{6+\sqrt{37}} -$

$636\,475\,482\,704\,408\,274\,536\, i \sqrt{37 \left(6+\sqrt{37}\right)}\Big)\, \omega_0^{20} +$

$4 \Big(196\,559\,642\,181\,035\,242\,439 \sqrt{6+\sqrt{37}} +$

$32\,313\,402\,179\,993\,892\,865 \sqrt{37 \left(6+\sqrt{37}\right)} -$

$4\, i \left(2\,637\,834\,258\,931\,483\,368\,064 + 433\,657\,295\,764\,219\,258\,871 \sqrt{37}\right)\Big)\, \omega_0^{21} +$

$2 \Big(7\,207\,645\,667\,935\,395\,664\,541 + 1\,184\,929\,060\,350\,207\,274\,257 \sqrt{37} -$

$2\,672\,724\,385\,906\,058\,415\,538\, i \sqrt{6+\sqrt{37}} -$

$$439\,394\,194\,732\,335\,462\,254\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{22}+$$

$$4\Bigg(601\,691\,787\,225\,327\,369\,367\,\sqrt{6+\sqrt{37}}+$$

$$98\,909\,659\,892\,298\,485\,093\,\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$\mathrm{i}\left(724\,811\,717\,383\,026\,143\,797+119\,158\,554\,304\,578\,942\,987\,\sqrt{37}\right)\Bigg)\,\omega_0^{23}+$$

$$4\Bigg(540\,993\,653\,931\,419\,909\,439+88\,939\,182\,020\,035\,547\,583\,\sqrt{37}+$$

$$14\,955\,954\,994\,988\,394\,289\,\mathrm{i}\,\sqrt{6+\sqrt{37}}+$$

$$2\,457\,286\,328\,735\,092\,667\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{24}+$$

$$4\Bigg(45\,784\,714\,983\,246\,369\,189\,\sqrt{6+\sqrt{37}}+7\,491\,478\,622\,552\,836\,675\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$3\,\mathrm{i}\left(3\,453\,429\,928\,495\,001\,573+567\,462\,788\,095\,597\,815\,\sqrt{37}\right)\Bigg)\,\omega_0^{25}+$$

$$4\Bigg(32\,168\,844\,697\,158\,147\,385+5\,296\,965\,396\,679\,784\,417\,\sqrt{37}+$$

$$6\,705\,577\,856\,515\,992\,863\,\mathrm{i}\,\sqrt{6+\sqrt{37}}+$$

$$1\,105\,652\,305\,866\,190\,841\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{26}+$$

$$4\Bigg(1\,716\,125\,625\,076\,056\,325\,\sqrt{6+\sqrt{37}}+217\,077\,758\,113\,988\,503\,\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$\mathrm{i}\left(3\,885\,615\,000\,618\,565\,579+643\,166\,402\,533\,136\,789\,\sqrt{37}\right)\Bigg)\,\omega_0^{27}+$$

$$4\Bigg(4\,448\,245\,466\,103\,717\,418+753\,079\,038\,812\,429\,730\,\sqrt{37}-420\,077\,725\,952\,493\,885$$

$$\mathrm{i}\,\sqrt{6+\sqrt{37}}-49\,825\,869\,602\,642\,943\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{28}+$$

$$4\Bigg(522\,356\,896\,890\,873\,681\,\sqrt{6+\sqrt{37}}+66\,766\,483\,714\,620\,183\,\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$\mathrm{i}\left(239\,569\,950\,690\,403\,923+51\,843\,100\,240\,667\,897\,\sqrt{37}\right)\Bigg)\,\omega_0^{29}+$$

$$4\Bigg(568\,363\,335\,097\,417\,266+98\,733\,812\,979\,077\,342\,\sqrt{37}+$$

$$1\,199\,238\,599\,574\,683\,i\,\sqrt{6+\sqrt{37}}+6\,763\,007\,179\,104\,181\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{30}+$$

$$4\,\Bigg(48\,941\,882\,916\,118\,441\,i+3\,986\,290\,162\,895\,399\,i\,\sqrt{37}+$$

$$54\,729\,453\,951\,223\,291\,\sqrt{6+\sqrt{37}}+6\,560\,531\,029\,002\,225\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{31}+$$

$$4\,\Bigg(40\,930\,731\,257\,344\,767+7\,113\,211\,514\,138\,319\,\sqrt{37}+$$

$$6\,159\,369\,335\,326\,285\,i\,\sqrt{6+\sqrt{37}}+1\,979\,171\,974\,565\,985\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{32}+$$

$$4\,\Bigg(6\,789\,105\,573\,310\,623\,i+534\,562\,171\,203\,609\,i\,\sqrt{37}+$$

$$2\,517\,037\,294\,543\,457\,\sqrt{6+\sqrt{37}}+254\,109\,944\,144\,095\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{33}+$$

$$4\,\Bigg(2\,624\,889\,613\,793\,683+416\,948\,382\,239\,485\,\sqrt{37}+567\,244\,932\,241\,088\,i\,\sqrt{6+\sqrt{37}}+$$

$$171\,178\,866\,582\,306\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{34}+$$

$$2\,\Bigg(115\,937\,893\,559\,271\,\sqrt{6+\sqrt{37}}+8\,826\,076\,767\,401\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$2\,i\,\left(623\,234\,803\,422\,713+56\,827\,581\,607\,931\,\sqrt{37}\right)\Bigg)\,\omega_0^{35}+$$

$$\Bigg(661\,426\,577\,287\,215+92\,480\,447\,704\,519\,\sqrt{37}+124\,640\,092\,902\,314\,i\,\sqrt{6+\sqrt{37}}+$$

$$35\,270\,180\,991\,318\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{36}+$$

$$2\,\Bigg(86\,237\,921\,567\,923\,i+9\,997\,273\,389\,389\,i\,\sqrt{37}+847\,352\,188\,722\,\sqrt{6+\sqrt{37}}+$$

$$68\,350\,650\,638\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{37}+$$

$$\Bigg(31\,314\,398\,135\,117+4\,040\,009\,699\,385\,\sqrt{37}+4\,500\,366\,865\,744\,i\,\sqrt{6+\sqrt{37}}+$$

$$1\,152\,392\,143\,792\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{38}+$$

$$4\,\Bigg(1\,806\,299\,702\,919\,i+241\,604\,491\,121\,i\,\sqrt{37}-10\,655\,595\,664\,\sqrt{6+\sqrt{37}}+$$

$$633\,665\,552\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{39}+$$

$$16\Big(60\,625\,321\,405+7\,608\,614\,715\sqrt{37}+6\,033\,484\,106\,i\sqrt{6+\sqrt{37}}+$$

$$1\,391\,591\,606\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{40}+$$

$$16\Big(10\,459\,775\,127\,i+1\,524\,689\,719\,i\sqrt{37}-62\,773\,164\sqrt{6+\sqrt{37}}+$$

$$5\,443\,884\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{41}+$$

$$16\Big(1\,154\,337\,123+147\,357\,263\sqrt{37}+66\,843\,904\,i\sqrt{6+\sqrt{37}}+$$

$$13\,954\,432\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{42}+$$

$$128\left(15\,372\,181\,i+2\,383\,566\,i\sqrt{37}-42\,328\sqrt{6+\sqrt{37}}+7256\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{43}+$$

$$2048\left(95\,143+12\,757\sqrt{37}+2220\,i\sqrt{6+\sqrt{37}}+420\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{44}+$$

$$1024\,i\left(8897+1451\sqrt{37}\right)\omega_0^{45}+$$

$$122\,880\left(7+\sqrt{37}\right)\omega_0^{46}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^{3/2}\left(-2\,i+\omega_0\right)^4\Big(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-$$

$$2\left(89+28\sqrt{37}\right)\omega_0^4-104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)^3$$

$$\Big(10\,657+1752\sqrt{37}+\left(108\,809+17\,888\sqrt{37}\right)\omega_0^2+\left(25\,062+3944\sqrt{37}\right)\omega_0^4+$$

$$\left(10\,366-224\sqrt{37}\right)\omega_0^6+649\,\omega_0^8+9\,\omega_0^{10}\Big)$$

$$\Big(10\,657+1752\sqrt{37}+8\left(23\,437+3853\sqrt{37}\right)\omega_0^2+$$

$$2\left(55\,399+8804\sqrt{37}\right)\omega_0^4-$$

$$8\left(-7909+49\sqrt{37}\right)\omega_0^6+2049\,\omega_0^8+16\,\omega_0^{10}\Big)\Big)\Big\}$$

(* Cálculo do vetor complexo h31 *)

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h31 = Simplify[DAI.(3 bb[h20, h11] + bb[qb, h30] + 3 bb[q, h21] - 3 G21 h20)]
      simplifica
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$$\Big\{\Big(12\,\omega_0^4$$

$$\Big(148\Big(4\,736\,792\,749\,770\,090\,152+778\,723\,931\,137\,561\,496\sqrt{37}-1\,304\,643\,213\,596\,305\,787$$

$$i\sqrt{6+\sqrt{37}}-214\,482\,023\,110\,039\,613\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)+$$

$$296 \left(23\,888\,491\,913\,340\,640\,649\,i + 3\,927\,243\,878\,805\,418\,235\,i\,\sqrt{37} + 8\,183\,095\,385\,792\,264\,567\,\sqrt{6+\sqrt{37}} + 1\,345\,292\,594\,447\,390\,309\,\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0 + 3\,i \left(4\,319\,831\,501\,160\,044\,568\,237\,\sqrt{6+\sqrt{37}} + 710\,175\,924\,120\,492\,687\,557\,\sqrt{37\left(6+\sqrt{37}\right)} + 296\,i\left(31\,714\,851\,122\,117\,957\,398 + 5\,213\,889\,407\,016\,366\,447\,\sqrt{37}\right)\right)\omega_0^2 -$$

$$2\left(28\,539\,796\,875\,936\,553\,017\,641\,i + 4\,691\,913\,704\,304\,568\,991\,933\,i\,\sqrt{37} + 19\,604\,487\,717\,095\,319\,322\,360\,\sqrt{6+\sqrt{37}} + 3\,222\,957\,927\,330\,786\,298\,444\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3 +$$

$$\left(66\,761\,807\,772\,849\,420\,197\,075 + 10\,975\,573\,588\,531\,375\,406\,975\,\sqrt{37} - 76\,082\,196\,332\,811\,509\,101\,483\,i\,\sqrt{6+\sqrt{37}} - 12\,507\,836\,029\,081\,580\,163\,355\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4 +$$

$$2\left(31\,900\,780\,836\,821\,592\,499\,495\,i + 5\,244\,456\,063\,823\,618\,056\,511\,i\,\sqrt{37} + 52\,525\,325\,139\,883\,708\,103\,595\,\sqrt{6+\sqrt{37}} + 8\,635\,110\,260\,881\,505\,792\,847\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^5 +$$

$$\left(-84\,995\,209\,273\,834\,202\,216\,071 - 13\,973\,126\,330\,425\,255\,122\,715\,\sqrt{37} + 117\,098\,954\,531\,145\,395\,213\,396\,i\,\sqrt{6+\sqrt{37}} + 19\,250\,949\,539\,436\,382\,210\,892\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^6 -$$

$$6\left(17\,727\,221\,181\,366\,695\,529\,793\,i + 2\,914\,337\,209\,954\,764\,220\,209\,i\,\sqrt{37} + 19\,700\,946\,403\,340\,667\,490\,838\,\sqrt{6+\sqrt{37}} + 3\,238\,815\,637\,666\,326\,743\,506\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^7 +$$

$$4\left(17\,045\,655\,756\,364\,524\,946\,161 + 2\,802\,288\,544\,302\,061\,960\,323\,\sqrt{37} -\right.$$

$$26\,914\,915\,723\,439\,147\,173\,141\, i \sqrt{6+\sqrt{37}} -$$
$$4\,424\,784\,888\,342\,433\,498\,399\, i \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^8 +$$
$$4\Bigg(19\,402\,888\,808\,729\,991\,023\,395 \sqrt{6+\sqrt{37}} +$$
$$3\,189\,815\,270\,953\,658\,789\,641 \sqrt{37\left(6+\sqrt{37}\right)} -$$
$$i\left(4\,363\,617\,088\,889\,847\,134\,479 + 717\,374\,230\,401\,842\,272\,453\sqrt{37}\right)\Bigg)\,\omega_0^9 +$$
$$4\Bigg(19\,187\,734\,923\,419\,563\,289\,507 + 3\,154\,444\,190\,093\,819\,105\,261\sqrt{37} +$$
$$8\,696\,390\,082\,390\,470\,851\,406\, i \sqrt{6+\sqrt{37}} +$$
$$1\,429\,677\,722\,757\,365\,704\,916\, i \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{10} +$$
$$4\Bigg(18\,039\,701\,135\,957\,247\,052\,759\, i + 2\,965\,708\,598\,042\,642\,163\,097\, i\sqrt{37} +$$
$$61\,404\,340\,536\,748\,390\,259 \sqrt{6+\sqrt{37}} +$$
$$10\,094\,811\,400\,260\,222\,785 \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{11} +$$
$$8\, i\Bigg(1\,869\,180\,220\,614\,838\,508\,100 \sqrt{6+\sqrt{37}} +$$
$$307\,291\,335\,360\,276\,399\,411 \sqrt{37\left(6+\sqrt{37}\right)} +$$
$$5\, i\left(813\,548\,302\,578\,128\,362\,565 + 133\,746\,517\,067\,792\,199\,902\sqrt{37}\right)\Bigg)\,\omega_0^{12} -$$
$$4\Bigg(130\,378\,359\,568\,711\,486\,493\, i + 21\,434\,070\,279\,766\,465\,571\, i\sqrt{37} +$$
$$3\,413\,551\,737\,041\,399\,604\,915 \sqrt{6+\sqrt{37}} +$$
$$561\,184\,448\,685\,372\,400\,581 \sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{13} -$$
$$4\, i\Bigg(1\,812\,938\,928\,219\,127\,428\,173 \sqrt{6+\sqrt{37}} +$$
$$298\,045\,323\,867\,885\,402\,407 \sqrt{37\left(6+\sqrt{37}\right)} -$$
$$2\, i\left(1\,231\,304\,215\,684\,115\,435\,672 + 202\,425\,166\,116\,322\,165\,481\sqrt{37}\right)\Bigg)\,\omega_0^{14} +$$
$$12\Bigg(207\,238\,783\,809\,575\,866\,481 \sqrt{6+\sqrt{37}} +$$

$$34\,069\,846\,256\,571\,215\,931\,\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$13\,\mathbb{i}\left(47\,718\,714\,522\,532\,412\,503+7\,844\,908\,341\,256\,325\,569\,\sqrt{37}\right)\Bigg)\,\omega_0^{15}\;+$$

$$12\,\Bigg(252\,230\,396\,450\,501\,171\,861+41\,466\,421\,678\,481\,929\,291\,\sqrt{37}\;+$$

$$38\,111\,470\,250\,688\,934\,839\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\;+$$

$$6\,265\,487\,407\,877\,077\,861\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{16}\;+$$

$$4\,\Bigg(9\,692\,575\,032\,667\,911\,259\,\sqrt{6+\sqrt{37}}\;+1\,593\,447\,462\,112\,682\,353\,\sqrt{37\left(6+\sqrt{37}\right)}\;+$$

$$3\,\mathbb{i}\left(53\,689\,589\,017\,183\,273\,123+8\,826\,513\,912\,992\,944\,545\,\sqrt{37}\right)\Bigg)\,\omega_0^{17}\;+$$

$$\Bigg(31\,547\,023\,314\,102\,408\,236+5\,186\,302\,116\,004\,670\,084\,\sqrt{37}\;+$$

$$57\,205\,632\,695\,333\,178\,638\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\;+$$

$$9\,404\,537\,206\,350\,106\,454\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{18}\;-$$

$$4\,\Bigg(4\,733\,251\,076\,890\,879\,793\,\sqrt{6+\sqrt{37}}\;+778\,139\,675\,190\,647\,183\,\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$18\,\mathbb{i}\left(1\,052\,579\,107\,623\,120\,119+173\,043\,004\,588\,200\,002\,\sqrt{37}\right)\Bigg)\,\omega_0^{19}\;+$$

$$\Bigg(-2\,663\,504\,050\,815\,501\,758\,\mathbb{i}\,\sqrt{6+\sqrt{37}}\;-437\,874\,152\,386\,875\,590\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$6\left(4\,442\,278\,082\,444\,888\,383+730\,306\,691\,149\,298\,071\,\sqrt{37}\right)\Bigg)\,\omega_0^{20}\;-$$

$$8\,\Bigg(462\,176\,045\,352\,008\,279\,\mathbb{i}+75\,981\,526\,490\,881\,754\,\mathbb{i}\,\sqrt{37}\;+$$

$$38\,436\,007\,412\,094\,900\,\sqrt{6+\sqrt{37}}\;+6\,318\,911\,446\,191\,879\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{21}\;+$$

$$\Bigg(-593\,460\,339\,384\,694\,646-97\,563\,780\,361\,104\,038\,\sqrt{37}\;-233\,043\,804\,956\,616\,012$$

$$\mathbb{i}\,\sqrt{6+\sqrt{37}}\;-38\,311\,828\,367\,169\,060\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{22}\;+$$

$$4\,\Bigg(11\,758\,582\,593\,205\,293\,\sqrt{6+\sqrt{37}}\;+1\,933\,008\,868\,535\,583\,\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$26\,i\left(3\,799\,853\,333\,075\,662+624\,693\,299\,844\,625\sqrt{37}\right)\Big)\,\omega_0^{23}+$$

$$4\Big(18\,556\,846\,019\,027\,409+3\,050\,777\,503\,174\,431\sqrt{37}+$$

$$387\,514\,985\,816\,369\,i\sqrt{6+\sqrt{37}}+63\,681\,521\,638\,211\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{24}+$$

$$4\Big(349\,508\,762\,088\,217\sqrt{6+\sqrt{37}}+57\,458\,095\,970\,539\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\left(191\,789\,195\,992\,285+31\,510\,829\,307\,103\sqrt{37}\right)\Big)\,\omega_0^{25}+$$

$$4\Big(836\,892\,209\,854\,065+137\,583\,142\,599\,039\sqrt{37}+83\,488\,434\,090\,200\,i\sqrt{6+\sqrt{37}}+$$

$$13\,722\,865\,967\,162\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{26}-$$

$$4\Big(5\,748\,588\,924\,695\sqrt{6+\sqrt{37}}+943\,839\,944\,765\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$3\,i\left(54\,100\,362\,110\,443+8\,894\,729\,239\,613\sqrt{37}\right)\Big)\,\omega_0^{27}+$$

$$8\,i\Big(547\,175\,376\,641\sqrt{6+\sqrt{37}}+90\,002\,799\,902\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$11\,i\left(73\,300\,698\,088+12\,100\,851\,433\sqrt{37}\right)\Big)\,\omega_0^{28}-$$

$$4\Big(305\,840\,825\,213\sqrt{6+\sqrt{37}}+50\,163\,881\,315\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\left(4\,555\,266\,416\,053+748\,837\,356\,739\sqrt{37}\right)\Big)\,\omega_0^{29}-$$

$$4\,i\Big(25\,459\,484\,327\sqrt{6+\sqrt{37}}+4\,166\,049\,221\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$2\,i\left(361\,911\,929\,423+59\,557\,468\,022\sqrt{37}\right)\Big)\,\omega_0^{30}-$$

$$4\Big(1\,383\,602\,975\sqrt{6+\sqrt{37}}+222\,949\,205\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$3\,i\left(1\,171\,289\,627+187\,183\,925\sqrt{37}\right)\Big)\,\omega_0^{31}+$$

$$\Big(-53\,843\,286\,444-8\,877\,395\,220\sqrt{37}-2\,615\,685\,992\,i\sqrt{6+\sqrt{37}}-$$

$$426\,526\,088\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{32}+4\Big(73\,867\,651\sqrt{6+\sqrt{37}}+$$

$$12\,213\,865\sqrt{37\left(6+\sqrt{37}\right)}-i\left(1\,546\,518\,167+255\,240\,269\sqrt{37}\right)\Big)\,\omega_0^{33}+$$

$$\Big(131\,912\,276+20\,849\,492\sqrt{37}-10\,138\,925\,i\sqrt{6+\sqrt{37}}-$$

$$1\,594\,205\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{34}+$$

$$2\left(1\,695\,266\sqrt{6+\sqrt{37}}+278\,498\sqrt{37\left(6+\sqrt{37}\right)}-i\left(41\,703\,491+6\,916\,523\sqrt{37}\right)\right)$$

$$\omega_0^{35}+\left(14\,141\,359+2\,306\,899\sqrt{37}+278\,721\,i\sqrt{6+\sqrt{37}}+46\,809\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{36}-2\left(3323\,i+1559\,i\sqrt{37}+10\,619\sqrt{6+\sqrt{37}}+1763\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{37}+$$

$$3\left(47\,335+7699\sqrt{37}+2072\,i\sqrt{6+\sqrt{37}}+344\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{38}+$$

$$226\,i\left(43+7\sqrt{37}\right)\omega_0^{39}+24\left(43+7\sqrt{37}\right)\omega_0^{40}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^2\left(-2\,i+\omega_0\right)\left(2\,i+\omega_0\right)^3\left(\sqrt{37\left(6+\sqrt{37}\right)}+8\,\omega_0+2\,i\,\omega_0^2+\omega_0^3\right)$$

$$\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)^2$$

$$\left(-73-12\sqrt{37}-2\,i\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,i\,\omega_0^3+\omega_0^4+2\,i\,\omega_0^5\right)^4$$

$$\Big(-73-12\sqrt{37}-3\,i\left(97+16\sqrt{37}\right)\omega_0+32\left(7+\sqrt{37}\right)\omega_0^2+$$

$$108\,i\,\omega_0^3+\omega_0^4+3\,i\,\omega_0^5\Big)\Big(-\sqrt{6+\sqrt{37}}\left(518+85\sqrt{37}\right)+$$

$$\left(-534-88\sqrt{37}-222\,i\sqrt{6+\sqrt{37}}-44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$

$$2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-i\,\omega_0^6\Big)\Big),$$

$$\Big(48\,\omega_0^4\Big(5476\,i\Big(1\,668\,623\,731\,468\,136\sqrt{6+\sqrt{37}}+274\,320\,051\,647\,048\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$15\,i\left(297\,324\,670\,041\,031+48\,879\,874\,655\,645\sqrt{37}\right)\Big)-$$

$$296\left(430\,106\,183\,778\,700\,393\sqrt{6+\sqrt{37}}+70\,709\,021\,046\,990\,235\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$

$$\left.37\,i\left(27\,960\,039\,245\,995\,921+4\,596\,602\,137\,059\,775\sqrt{37}\right)\right)\omega_0+$$

$$37\left(43\,721\,469\,778\,342\,218\,551+7\,187\,765\,355\,061\,244\,747\sqrt{37}-\right.$$

$$20\,726\,457\,136\,963\,249\,076\,i\sqrt{6+\sqrt{37}}-$$

$$\left.3\,407\,408\,563\,744\,653\,164\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^2+\left(2\,628\,851\,013\,360\,024\,943\,546\right.$$

$$\sqrt{6+\sqrt{37}}+432\,180\,444\,373\,050\,385\,354\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$\left.296\,i\left(16\,172\,443\,014\,286\,060\,877+2\,658\,733\,253\,802\,228\,290\sqrt{37}\right)\right)\omega_0^3+$$

$$\left(-9\,353\,005\,835\,161\,128\,732\,814-1\,537\,624\,687\,561\,587\,050\,266\sqrt{37}+\right.$$

$$5\,716\,564\,570\,569\,606\,523\,221\,i\sqrt{6+\sqrt{37}}+$$

$$\left.939\,797\,426\,267\,327\,984\,451\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^4-$$

$$3\left(2\,828\,625\,244\,328\,231\,914\,417\sqrt{6+\sqrt{37}}+\right.$$

$$465\,023\,125\,633\,930\,158\,125\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$\left.6\,i\left(841\,696\,189\,823\,395\,850\,863+138\,374\,001\,225\,744\,053\,377\sqrt{37}\right)\right)\omega_0^5+$$

$$\left(25\,006\,408\,118\,728\,902\,152\,929+4\,111\,028\,170\,863\,496\,229\,599\sqrt{37}-\right.$$

$$9\,581\,894\,646\,215\,852\,106\,212\,i\sqrt{6+\sqrt{37}}-$$

$$\left.1\,575\,253\,776\,304\,510\,841\,234\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^6+$$

$$\left(38\,101\,386\,526\,999\,817\,017\,171\,i+6\,263\,829\,359\,968\,754\,159\,623\,i\sqrt{37}+\right.$$

$$10\,038\,472\,668\,802\,759\,335\,986\sqrt{6+\sqrt{37}}+$$

$$\left.1\,650\,314\,740\,843\,680\,943\,994\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^7+$$

$$\left(-43\,570\,126\,498\,812\,134\,597\,008-7\,162\,884\,673\,171\,027\,349\,110\sqrt{37}+\right.$$

$$11\,318\,986\,157\,920\,703\,992\,353\,i\sqrt{6+\sqrt{37}}+$$

$$1\,860\,829\,861\,685\,520\,625\,299\, i \sqrt{37 \left(6+\sqrt{37}\right)}\;)\; \omega_0^8 -$$

$$(12\,195\,787\,001\,143\,619\,104\,507 \sqrt{6+\sqrt{37}} +$$

$$2\,004\,975\,032\,379\,850\,105\,843 \sqrt{37 \left(6+\sqrt{37}\right)} +$$

$$10\, i \left(3\,254\,903\,074\,243\,291\,698\,559 + 535\,102\,769\,182\,691\,322\,013 \sqrt{37}\right)\;)\; \omega_0^9 +$$

$$(11\,050\,079\,286\,300\,578\,972\,489 + 1\,816\,621\,844\,311\,717\,160\,123 \sqrt{37} -$$

$$10\,262\,778\,827\,918\,887\,167\,776\, i \sqrt{6+\sqrt{37}} - 1\,687\,190\,446\,248\,736\,035\,626$$

$$i \sqrt{37 \left(6+\sqrt{37}\right)}\;)\; \omega_0^{10} + (5\,618\,943\,485\,542\,115\,574\,418 \sqrt{6+\sqrt{37}} +$$

$$923\,748\,618\,749\,232\,209\,014 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$11\, i \left(568\,246\,008\,903\,924\,378\,923 + 93\,419\,068\,404\,115\,170\,875 \sqrt{37}\right)\;)\; \omega_0^{11} +$$

$$(11\,312\,428\,318\,486\,010\,272\,252 + 1\,859\,751\,759\,523\,296\,100\,042 \sqrt{37} +$$

$$968\,715\,779\,358\,806\,650\,793\, i \sqrt{6+\sqrt{37}} + 159\,255\,893\,113\,363\,419\,911$$

$$i \sqrt{37 \left(6+\sqrt{37}\right)}\;)\; \omega_0^{12} + (1\,462\,930\,965\,719\,513\,854\,625 \sqrt{6+\sqrt{37}} +$$

$$240\,504\,369\,262\,036\,104\,245 \sqrt{37 \left(6+\sqrt{37}\right)} +$$

$$2\, i \left(3\,807\,461\,686\,493\,040\,380\,867 + 625\,942\,845\,463\,236\,066\,515 \sqrt{37}\right)\;)\; \omega_0^{13} +$$

$$(1\,695\,157\,129\,436\,142\,019\,604\, i \sqrt{6+\sqrt{37}} +$$

$$278\,682\,115\,409\,158\,737\,302\, i \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$7 \left(361\,887\,854\,746\,872\,462\,365 + 59\,493\,996\,839\,299\,335\,227 \sqrt{37}\right)\;)\; \omega_0^{14} +$$

$$(134\,239\,826\,939\,670\,612\,343\, i + 22\,068\,891\,569\,860\,740\,883\, i \sqrt{37} -$$

$$1\,004\,831\,176\,930\,283\,533\,942 \sqrt{6+\sqrt{37}} -$$

$$165\,193\,227\,990\,007\,682\,422 \sqrt{37 \left(6+\sqrt{37}\right)}\;)\; \omega_0^{15} +$$

$$\left(-634\,882\,715\,448\,317\,289\,232-104\,374\,075\,359\,251\,876\,662\sqrt{37}-380\,941\,709\,658\,674\,559\,667\,i\sqrt{6+\sqrt{37}}-62\,626\,431\,778\,760\,809\,873\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{16}+$$

$$\left(84\,238\,636\,768\,935\,764\,793\sqrt{6+\sqrt{37}}+13\,848\,748\,016\,764\,878\,297\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(181\,486\,834\,247\,514\,108\,043+29\,836\,251\,603\,006\,941\,245\sqrt{37}\right)\right)\omega_0^{17}+$$

$$\left(108\,503\,762\,143\,988\,469\,891+17\,837\,907\,921\,386\,422\,221\sqrt{37}+1\,014\,775\,590\,591\,127\,936\,i\sqrt{6+\sqrt{37}}+166\,830\,690\,579\,505\,702\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{18}+$$

$$\left(11\,012\,595\,018\,410\,394\,299\,i+1\,810\,458\,284\,375\,369\,843\,i\sqrt{37}+7\,002\,461\,304\,026\,857\,230\sqrt{6+\sqrt{37}}+1\,151\,194\,500\,836\,104\,122\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{19}+$$

$$\left(5\,545\,585\,711\,927\,638\,088+911\,690\,104\,799\,485\,726\sqrt{37}+2\,745\,198\,716\,653\,270\,487\,i\sqrt{6+\sqrt{37}}+451\,305\,638\,010\,712\,385\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{20}+$$

$$\left(2\,991\,676\,201\,180\,199\,738\,i+491\,829\,672\,459\,176\,522\,i\sqrt{37}-476\,282\,490\,560\,130\,393\sqrt{6+\sqrt{37}}-78\,299\,431\,142\,741\,733\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{21}+$$

$$\left(-631\,148\,051\,377\,958\,021-103\,760\,617\,763\,469\,995\sqrt{37}+11\,673\,282\,653\,976\,772\,i\sqrt{6+\sqrt{37}}+1\,919\,151\,845\,528\,554\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{22}-$$

$$\left(9\,360\,291\,579\,641\,815\,i+1\,538\,889\,097\,265\,131\,i\sqrt{37}+27\,337\,297\,417\,685\,642\sqrt{6+\sqrt{37}}+4\,494\,098\,747\,333\,474\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{23}+$$

$$\left(-32\,172\,611\,059\,484\,960-5\,289\,205\,106\,629\,346\sqrt{37}-\right.$$

$$6\,403\,062\,255\,390\,253\,\mathrm{i}\sqrt{6+\sqrt{37}} - 1\,052\,566\,667\,983\,399\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{24} +$$

$$\Bigg(436\,141\,178\,145\,839\sqrt{6+\sqrt{37}} + 71\,673\,378\,198\,551\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$6\,\mathrm{i}\left(1\,450\,499\,330\,716\,851 + 238\,468\,723\,839\,505\sqrt{37}\right)\Bigg)\,\omega_0^{25} +$$

$$\Bigg(641\,842\,402\,767\,331 + 105\,534\,868\,161\,289\sqrt{37} - 113\,570\,072\,312\,080\,\mathrm{i}\sqrt{6+\sqrt{37}} -$$

$$18\,666\,658\,699\,006\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{26} +$$

$$\Bigg(32\,978\,595\,939\,046\sqrt{6+\sqrt{37}} + 5\,416\,857\,224\,290\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$\mathrm{i}\left(193\,262\,460\,013\,915 + 31\,773\,595\,511\,899\sqrt{37}\right)\Bigg)\,\omega_0^{27} +$$

$$\Bigg(58\,081\,877\,033\,844 + 9\,551\,488\,284\,654\sqrt{37} + 3\,127\,502\,514\,963\,\mathrm{i}\sqrt{6+\sqrt{37}} +$$

$$513\,667\,659\,597\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{28} +$$

$$\Bigg(128\,175\,189\,803\sqrt{6+\sqrt{37}} + 20\,813\,395\,223\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$6\,\mathrm{i}\left(732\,057\,375\,491 + 120\,434\,179\,675\sqrt{37}\right)\Bigg)\,\omega_0^{29} +$$

$$\Bigg(786\,350\,482\,551 + 129\,436\,874\,529\sqrt{37} + 67\,595\,636\,108\,\mathrm{i}\sqrt{6+\sqrt{37}} +$$

$$11\,064\,797\,330\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{30} +$$

$$\Bigg(203\,476\,539\,757\,\mathrm{i} + 33\,507\,141\,937\,\mathrm{i}\sqrt{37} - 7\,122\,321\,586\sqrt{6+\sqrt{37}} -$$

$$1\,177\,370\,530\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{31} +$$

$$\Bigg(-300\,582\,561\,\mathrm{i}\sqrt{6+\sqrt{37}} - 50\,929\,851\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$2\left(5\,104\,068\,958 + 837\,245\,443\sqrt{37}\right)\Bigg)\,\omega_0^{32} +$$

$$\Bigg(-11\,523\,909\sqrt{6+\sqrt{37}} - 2\,020\,485\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$2\,i\left(874\,660\,931+145\,108\,061\sqrt{37}\right)\Big)\,\omega_0^{33}+$$

$$\left(-294\,065\,890-48\,306\,856\sqrt{37}-8\,315\,084\,i\sqrt{6+\sqrt{37}}-1\,385\,954\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{34}-$$

$$\left(30\,396\,367\,i+4\,927\,903\,i\sqrt{37}+73\,408\sqrt{6+\sqrt{37}}+15\,196\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{35}-$$

$$2\,i\left(38\,036\sqrt{6+\sqrt{37}}+6284\sqrt{37\left(6+\sqrt{37}\right)}-3\,i\left(238\,151+39\,050\sqrt{37}\right)\right)\omega_0^{36}-$$

$$4\left(211\,585\,i+34\,528\,i\sqrt{37}+6216\sqrt{6+\sqrt{37}}+1032\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{37}-$$

$$8\left(1657+271\sqrt{37}\right)\omega_0^{38}-96\,i\left(43+7\sqrt{37}\right)\omega_0^{39}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^{5/2}\left(-2\,i+\omega_0\right)\left(2\,i+\omega_0\right)^3\left(\sqrt{37\left(6+\sqrt{37}\right)}+8\,\omega_0+2\,i\,\omega_0^2+\omega_0^3\right)$$

$$\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)^2$$

$$\left(-73-12\sqrt{37}-2\,i\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,i\,\omega_0^3+\omega_0^4+2\,i\,\omega_0^5\right)^4$$

$$\left(-73-12\sqrt{37}-3\,i\left(97+16\sqrt{37}\right)\omega_0+32\left(7+\sqrt{37}\right)\omega_0^2+108\,i\,\omega_0^3+\omega_0^4+3\,i\,\omega_0^5\right)$$

$$\Big(-\sqrt{6+\sqrt{37}}\left(518+85\sqrt{37}\right)+$$

$$\left(-534-88\sqrt{37}-222\,i\sqrt{6+\sqrt{37}}-44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$

$$2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-i\,\omega_0^6\Big)\Big),$$

$$\Big(48\,\omega_0^2\Big(5476\,i\Big(20\,161\,584\,299\,749\,592\sqrt{6+\sqrt{37}}+3\,314\,544\,041\,350\,424\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$9\,i\left(3\,592\,503\,382\,505\,051+590\,603\,917\,974\,901\sqrt{37}\right)\Big)-$$

$$296\Big(5\,188\,591\,971\,409\,488\,049\sqrt{6+\sqrt{37}}+852\,999\,265\,640\,427\,175\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$37\, i \left(187\,350\,368\,533\,193\,081 + 30\,800\,210\,858\,142\,419 \sqrt{37}\right)\Big)\, \omega_0 +$$

$$37 \Big(257\,760\,160\,594\,174\,829\,311 + 42\,375\,509\,369\,348\,603\,863 \sqrt{37} -$$

$$249\,632\,326\,472\,327\,418\,980\, i \sqrt{6 + \sqrt{37}} -$$

$$41\,039\,301\,670\,730\,961\,644\, i \sqrt{37 \left(6 + \sqrt{37}\right)}\Big)\, \omega_0^2 +$$

$$6 \Big(5\,281\,849\,808\,790\,237\,377\,501 \sqrt{6 + \sqrt{37}} +$$

$$868\,330\,759\,664\,110\,058\,297 \sqrt{37 \left(6 + \sqrt{37}\right)} +$$

$$74\, i \left(50\,842\,864\,815\,714\,061\,373 + 8\,358\,515\,487\,406\,572\,731 \sqrt{37}\right)\Big)\, \omega_0^3 +$$

$$\Big(-30\,723\,512\,737\,238\,597\,999\,108 - 5\,050\,914\,380\,465\,271\,243\,032 \sqrt{37} +$$

$$69\,541\,790\,641\,249\,999\,635\,849\, i \sqrt{6 + \sqrt{37}} +$$

$$11\,432\,599\,956\,822\,673\,008\,783\, i \sqrt{37 \left(6 + \sqrt{37}\right)}\Big)\, \omega_0^4 -$$

$$\Big(42\,220\,082\,041\,642\,736\,419\,690\, i + 6\,940\,938\,731\,595\,167\,472\,334\, i \sqrt{37} +$$

$$106\,046\,220\,588\,404\,786\,561\,249 \sqrt{6 + \sqrt{37}} +$$

$$17\,433\,891\,272\,294\,277\,272\,729 \sqrt{37 \left(6 + \sqrt{37}\right)}\Big)\, \omega_0^5 +$$

$$\Big(108\,080\,140\,636\,554\,223\,055\,497 + 17\,768\,265\,668\,470\,108\,168\,735 \sqrt{37} -$$

$$125\,625\,748\,766\,220\,996\,630\,330\, i \sqrt{6 + \sqrt{37}} -$$

$$20\,652\,745\,876\,643\,969\,722\,356\, i \sqrt{37 \left(6 + \sqrt{37}\right)}\Big)\, \omega_0^6 +$$

$$3 \Big(78\,213\,556\,482\,654\,699\,791\,131\, i + 12\,858\,229\,479\,298\,795\,278\,523\, i \sqrt{37} +$$

$$45\,077\,986\,597\,678\,778\,000\,354 \sqrt{6 + \sqrt{37}} +$$

$$7\,410\,775\,346\,422\,859\,430\,714 \sqrt{37 \left(6 + \sqrt{37}\right)}\Big)\, \omega_0^7 +$$

$$\Big(-313\,304\,935\,170\,673\,903\,401\,818 - 51\,507\,014\,059\,829\,407\,425\,020 \sqrt{37} +$$

$$144\,587\,533\,445\,251\,599\,750\,169\, i \sqrt{6 + \sqrt{37}} +$$

$$23\,770\,044\,075\,378\,840\,209\,827\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^8 -$$

$$\Bigg(250\,872\,281\,445\,394\,588\,873\,406\,\mathbb{i} + 41\,243\,149\,012\,607\,446\,042\,034\,\mathbb{i}\,\sqrt{37} +$$

$$142\,073\,713\,083\,639\,589\,660\,357\,\sqrt{6+\sqrt{37}} +$$

$$23\,356\,774\,553\,662\,238\,478\,625\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^9 +$$

$$\Bigg(91\,898\,602\,808\,933\,912\,474\,957 + 15\,108\,037\,236\,565\,997\,755\,175\,\sqrt{37} -$$

$$112\,165\,783\,700\,938\,484\,546\,194\,\mathbb{i}\,\sqrt{6+\sqrt{37}} -$$

$$18\,439\,941\,250\,746\,034\,236\,904\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{10} +$$

$$\Bigg(61\,459\,847\,870\,335\,514\,886\,394\,\sqrt{6+\sqrt{37}} +$$

$$10\,103\,936\,749\,824\,494\,591\,902\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$3\,\mathbb{i}\left(15\,115\,087\,073\,145\,028\,232\,965 + 2\,484\,905\,007\,857\,344\,368\,353\,\sqrt{37}\right)\Bigg)\,\omega_0^{11} +$$

$$\Bigg(92\,196\,529\,930\,740\,552\,817\,678 + 15\,157\,016\,153\,681\,790\,127\,264\,\sqrt{37} +$$

$$13\,976\,651\,276\,256\,733\,787\,045\,\mathbb{i}\,\sqrt{6+\sqrt{37}} +$$

$$2\,297\,747\,315\,736\,844\,438\,523\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{12} +$$

$$\Bigg(12\,211\,108\,537\,146\,374\,294\,723\,\sqrt{6+\sqrt{37}} +$$

$$2\,007\,493\,877\,381\,876\,498\,051\,\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$2\,\mathbb{i}\left(33\,573\,165\,012\,808\,242\,951\,709 + 5\,519\,394\,328\,740\,864\,779\,329\,\sqrt{37}\right)\Bigg)\,\omega_0^{13} +$$

$$\Bigg(17\,065\,053\,266\,041\,622\,645\,326\,\mathbb{i}\,\sqrt{6+\sqrt{37}} +$$

$$2\,805\,477\,475\,236\,981\,002\,452\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$9\left(2\,781\,623\,161\,866\,024\,992\,011 + 457\,296\,030\,875\,448\,563\,317\,\sqrt{37}\right)\Bigg)\,\omega_0^{14} +$$

$$\Bigg(162\,010\,754\,740\,775\,542\,805\,\mathbb{i} + 26\,634\,404\,018\,617\,051\,373\,\mathbb{i}\,\sqrt{37} -$$

$$11\,500\,969\,910\,947\,008\,432\,330\,\sqrt{6+\sqrt{37}} -$$

$$1\,890\,747\,806\,326\,604\,606\,154\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{15}+$$

$$\Big(-6\,381\,418\,926\,461\,010\,175\,210-1\,049\,098\,809\,083\,469\,039\,292\,\sqrt{37}-$$

$$5\,158\,150\,836\,524\,518\,787\,315\,i\,\sqrt{6+\sqrt{37}}-$$

$$847\,994\,773\,820\,029\,730\,009\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{16}+$$

$$\Big(1\,562\,656\,324\,994\,702\,883\,503\,\sqrt{6+\sqrt{37}}+$$

$$256\,899\,117\,724\,999\,301\,339\,\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$2\,i\,\left(2\,153\,840\,186\,969\,829\,907\,747+354\,089\,145\,536\,698\,918\,969\,\sqrt{37}\right)\Big)\,\omega_0^{17}+$$

$$\Big(1\,606\,481\,492\,682\,118\,247\,919+264\,103\,930\,245\,863\,658\,633\,\sqrt{37}+$$

$$258\,558\,853\,582\,173\,328\,430\,i\,\sqrt{6+\sqrt{37}}+$$

$$42\,506\,815\,468\,294\,821\,944\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{18}+$$

$$\Big(21\,720\,251\,186\,435\,823\,302\,\sqrt{6+\sqrt{37}}+3\,570\,783\,712\,244\,358\,242\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$9\,i\,\left(35\,691\,672\,085\,555\,838\,821+5\,867\,674\,635\,083\,266\,921\,\sqrt{37}\right)\Big)\,\omega_0^{19}+$$

$$\Big(10\,998\,554\,188\,078\,457\,742+1\,808\,153\,266\,793\,804\,328\,\sqrt{37}+$$

$$28\,753\,919\,811\,220\,165\,619\,i\,\sqrt{6+\sqrt{37}}+$$

$$4\,727\,110\,830\,515\,839\,013\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{20}+$$

$$\Big(31\,431\,831\,834\,897\,537\,374\,i+5\,167\,364\,091\,063\,904\,598\,i\,\sqrt{37}-$$

$$9\,218\,976\,687\,054\,499\,443\,\sqrt{6+\sqrt{37}}-$$

$$1\,515\,586\,662\,277\,672\,011\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{21}+$$

$$\Big(-1\,460\,711\,573\,800\,118\,926\,i\,\sqrt{6+\sqrt{37}}-240\,137\,411\,923\,774\,732\,i\,\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$3\,\left(3\,596\,565\,920\,971\,674\,551+591\,272\,586\,628\,192\,289\,\sqrt{37}\right)\Big)\,\omega_0^{22}-$$

$$\left(1\,606\,663\,387\,317\,385\,597\,i + 264\,135\,102\,257\,998\,477\,i\sqrt{37} + 5\,860\,012\,099\,557\,502\sqrt{6+\sqrt{37}} + 963\,962\,606\,225\,302\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{23} +$$

$$\left(-64\,309\,562\,954\,938\,133\,i\sqrt{6+\sqrt{37}} - 10\,572\,390\,759\,454\,919\,i\sqrt{37\left(6+\sqrt{37}\right)} - 6\left(19\,200\,995\,383\,648\,525 + 3\,156\,571\,463\,466\,678\sqrt{37}\right)\right)\omega_0^{24} +$$

$$3\left(5\,707\,033\,832\,275\,627\sqrt{6+\sqrt{37}} + 938\,185\,597\,239\,487\sqrt{37\left(6+\sqrt{37}\right)} - 2\,i\left(18\,475\,592\,254\,083\,391 + 3\,037\,377\,448\,778\,401\sqrt{37}\right)\right)\omega_0^{25} +$$

$$\left(22\,867\,756\,129\,140\,383 + 3\,759\,528\,994\,939\,421\sqrt{37} + 2\,370\,848\,414\,306\,058\,i\sqrt{6+\sqrt{37}} + 389\,691\,320\,358\,072\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{26} +$$

$$\left(1\,026\,163\,411\,650\,611\,i + 168\,743\,892\,600\,263\,i\sqrt{37} - 98\,040\,224\,892\,818\sqrt{6+\sqrt{37}} - 16\,104\,913\,220\,246\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{27} +$$

$$\left(546\,185\,060\,407\,722 + 89\,787\,907\,357\,872\sqrt{37} + 48\,723\,047\,049\,687\,i\sqrt{6+\sqrt{37}} + 8\,005\,589\,679\,801\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{28} +$$

$$\left(-16\,147\,377\,354\,031\sqrt{6+\sqrt{37}} - 2\,652\,146\,325\,343\sqrt{37\left(6+\sqrt{37}\right)} + 2\,i\left(66\,645\,744\,404\,719 + 10\,958\,543\,204\,947\sqrt{37}\right)\right)\omega_0^{29} +$$

$$\left(-8\,628\,457\,887\,905 - 1\,420\,123\,809\,263\sqrt{37} - 2\,399\,512\,752\,518\,i\sqrt{6+\sqrt{37}} - 394\,191\,145\,940\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{30} +$$

$$\left(1\,711\,665\,961\,543\,i + 281\,387\,075\,935\,i\sqrt{37} + 105\,794\,164\,898\sqrt{6+\sqrt{37}} + 17\,478\,994\,994\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{31} +$$

$$\left(-381\,468\,782\,658-62\,822\,921\,328\sqrt{37}-28\,221\,915\,057\,i\sqrt{6+\sqrt{37}}-\right.$$
$$\left.4\,611\,769\,251\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{32}+$$
$$\left(7\,412\,119\,165\sqrt{6+\sqrt{37}}+1\,217\,399\,905\sqrt{37\left(6+\sqrt{37}\right)}-\right.$$
$$\left.2\,i\left(6\,295\,673\,533+1\,040\,655\,391\sqrt{37}\right)\right)\omega_0^{33}+$$
$$2\,i\left(435\,985\,319\sqrt{6+\sqrt{37}}+71\,939\,786\sqrt{37\left(6+\sqrt{37}\right)}+\right.$$
$$\left.9\,i\left(176\,911\,395+29\,280\,208\sqrt{37}\right)\right)\omega_0^{34}-$$
$$\left(389\,667\,773\,i+64\,364\,777\,i\sqrt{37}+26\,258\,012\sqrt{6+\sqrt{37}}+\right.$$
$$\left.4\,421\,696\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{35}+$$
$$2\left(19\,177\,247+3\,113\,744\sqrt{37}+3\,109\,924\,i\sqrt{6+\sqrt{37}}+508\,780\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)$$
$$\omega_0^{36}-4\left(1\,438\,307\,i+236\,204\,i\sqrt{37}+288\,896\sqrt{6+\sqrt{37}}+47\,840\sqrt{37\left(6+\sqrt{37}\right)}\right)$$
$$\omega_0^{37}+16\left(89\,990+14\,705\sqrt{37}-6216\,i\sqrt{6+\sqrt{37}}-1032\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{38}-$$
$$\left.\left.16\,i\left(5983+979\sqrt{37}\right)\omega_0^{39}+192\left(43+7\sqrt{37}\right)\omega_0^{40}\right)\right)\Big/$$
$$\left(\left(7+\sqrt{37}\right)^{5/2}\left(-2\,i+\omega_0\right)\left(2\,i+\omega_0\right)^3\left(\sqrt{37\left(6+\sqrt{37}\right)}+8\,\omega_0+2\,i\,\omega_0^2+\omega_0^3\right)\right.$$
$$\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)^2$$
$$\left(-73-12\sqrt{37}-2\,i\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,i\,\omega_0^3+\omega_0^4+2\,i\,\omega_0^5\right)^4$$
$$\left(-73-12\sqrt{37}-3\,i\left(97+16\sqrt{37}\right)\omega_0+\right.$$
$$\left.32\left(7+\sqrt{37}\right)\omega_0^2+108\,i\,\omega_0^3+\omega_0^4+3\,i\,\omega_0^5\right)$$
$$\left(-\sqrt{6+\sqrt{37}}\left(518+85\sqrt{37}\right)+\right.$$
$$\left(-534-88\sqrt{37}-222\,i\sqrt{6+\sqrt{37}}-44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$
$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$

$$2\left(102+9\sqrt{37}-\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$
$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,\mathrm{i}\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-\mathrm{i}\,\omega_0^6\Big)\Big)\Big\}$$

(* Cálculo do vetor complexo h22 *)

```
h22 = Simplify[
      simplifica
  -AI.(bb[h11, h11] + 2 bb[q, h21b] + 2 bb[qb, h21] + bb[h20b, h20] - 4 h11 l1)]
```

$$\Big\{-\Big(\Big(64\,\omega_0^4\Big(11\,244\,966$$
$$\left(802\,360\,677\,406\,663\,688\,461\,023\,733+131\,907\,282\,819\,295\,978\,599\,029\,723\sqrt{37}\right)+$$
$$607\,836\Big(8\,979\,139\,465\,240\,155\,690\,297\,127\,711\sqrt{6+\sqrt{37}}+$$
$$1\,476\,161\,434\,959\,048\,990\,894\,982\,793\sqrt{37\left(6+\sqrt{37}\right)}+$$
$$74\,\mathrm{i}\Big(628\,883\,870\,824\,116\,081\,971\,126\,335+$$
$$103\,387\,871\,496\,157\,812\,869\,147\,603\sqrt{37}\Big)\Big)\omega_0+151\,959$$
$$\Big(443\,894\,413\,671\,475\,770\,684\,622\,145\,037+72\,975\,792\,078\,095\,962\,208\,570\,181\,669$$
$$\sqrt{37}+117\,070\,056\,057\,324\,803\,135\,284\,461\,542\,\mathrm{i}\sqrt{6+\sqrt{37}}+$$
$$19\,246\,198\,659\,605\,606\,493\,500\,698\,570\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^2+$$
$$2738\Big(11\,710\,731\,236\,629\,359\,967\,357\,853\,374\,175\sqrt{6+\sqrt{37}}+$$
$$1\,925\,232\,355\,907\,081\,214\,450\,618\,081\,725\sqrt{37\left(6+\sqrt{37}\right)}+$$
$$74\,\mathrm{i}\Big(1\,359\,740\,169\,936\,446\,287\,663\,303\,455\,023+$$
$$223\,539\,906\,935\,966\,195\,960\,939\,684\,579\sqrt{37}\Big)\Big)\omega_0^3+$$
$$1369\Big(26\,012\,426\,297\,280\,485\,103\,820\,374\,837\,717+$$
$$4\,276\,416\,540\,628\,156\,577\,574\,160\,275\,093\sqrt{37}+$$
$$108\,191\,148\,381\,115\,831\,967\,644\,618\,547\,149\,\mathrm{i}\sqrt{6+\sqrt{37}}+$$
$$17\,786\,515\,229\,259\,088\,456\,899\,997\,650\,349\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^4-$$
$$37\Big(976\,044\,420\,877\,751\,006\,788\,341\,937\,404\,265\sqrt{6+\sqrt{37}}+$$

$$160\,460\,714\,357\,346\,194\,551\,511\,940\,519\,601 \sqrt{37\left(6+\sqrt{37}\right)} -$$
$$222\,\mathbb{i}\left(93\,201\,623\,787\,509\,332\,405\,827\,576\,200\,311 + 15\,322\,252\,565\,881\,432\,660\,639\,441\,634\,777\sqrt{37}\right)\Big)\,\omega_0^5 +$$

$$37\,\mathbb{i}\Big(7\,025\,757\,666\,892\,720\,364\,885\,796\,779\,966\,099\sqrt{6+\sqrt{37}} +$$
$$1\,155\,027\,445\,490\,012\,979\,235\,220\,179\,169\,195\sqrt{37\left(6+\sqrt{37}\right)} +$$
$$3\,\mathbb{i}\left(5\,151\,935\,999\,811\,193\,432\,699\,275\,532\,840\,531 + 846\,973\,061\,030\,973\,601\,043\,654\,091\,486\,919\sqrt{37}\right)\Big)\,\omega_0^6 +$$

$$\Big(-311\,337\,582\,453\,024\,711\,455\,122\,628\,103\,395\,689\sqrt{6+\sqrt{37}} -$$
$$51\,183\,583\,265\,375\,437\,757\,198\,277\,810\,986\,089\sqrt{37\left(6+\sqrt{37}\right)} +$$
$$111\,\mathbb{i}\left(2\,035\,983\,745\,594\,614\,765\,419\,941\,133\,658\,253 + 334\,713\,665\,945\,922\,856\,872\,994\,483\,956\,873\sqrt{37}\right)\Big)\,\omega_0^7 +$$

$$\Big(-829\,714\,260\,210\,421\,641\,587\,106\,453\,195\,750\,302 -$$
$$136\,404\,184\,131\,407\,021\,772\,586\,375\,537\,154\,330\sqrt{37} -$$
$$248\,892\,829\,138\,049\,975\,698\,675\,537\,053\,981\,289\,\mathbb{i}\sqrt{6+\sqrt{37}} -$$
$$40\,917\,729\,057\,860\,738\,690\,462\,565\,949\,565\,653\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^8 +$$

$$\Big(6\,016\,832\,557\,760\,253\,426\,290\,432\,871\,250\,413\sqrt{6+\sqrt{37}} +$$
$$989\,161\,179\,281\,688\,329\,699\,513\,498\,278\,857\sqrt{37\left(6+\sqrt{37}\right)} -$$
$$\mathbb{i}\left(1\,327\,747\,377\,288\,231\,741\,923\,500\,308\,561\,479\,197 + 218\,280\,324\,223\,529\,444\,855\,511\,910\,653\,857\,761\sqrt{37}\right)\Big)\,\omega_0^9 +$$

$$6\Big(112\,381\,611\,155\,960\,172\,090\,303\,214\,550\,867\,017 +$$
$$18\,475\,423\,065\,784\,295\,407\,877\,352\,913\,318\,867\sqrt{37} -$$
$$53\,327\,396\,296\,257\,917\,643\,770\,475\,236\,632\,881\,\mathbb{i}\sqrt{6+\sqrt{37}} -$$
$$8\,766\,969\,946\,736\,262\,708\,053\,672\,550\,593\,411\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{10} +$$

$$\left(261\,499\,569\,273\,692\,642\,156\,761\,545\,249\,442\,202\,\sqrt{6+\sqrt{37}}\,+\right.$$

$$42\,990\,264\,369\,382\,195\,639\,735\,122\,147\,756\,134\,\sqrt{37\left(6+\sqrt{37}\right)}\,-$$

$$3\,i\left(200\,435\,076\,934\,270\,636\,714\,376\,770\,439\,323\,761\,+\right.$$

$$\left.\left.32\,951\,323\,668\,465\,469\,951\,173\,494\,754\,496\,777\,\sqrt{37}\right)\right)\omega_0^{11}\,+$$

$$6\left(129\,844\,810\,109\,574\,598\,914\,332\,036\,952\,907\,421\,+\right.$$

$$21\,346\,355\,288\,870\,482\,254\,188\,594\,271\,406\,043\,\sqrt{37}\,-$$

$$4\,387\,515\,298\,497\,318\,394\,797\,283\,892\,065\,397\,i\,\sqrt{6+\sqrt{37}}\,-$$

$$\left.721\,303\,071\,859\,721\,542\,841\,943\,040\,136\,775\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{12}\,+$$

$$2\left(88\,111\,424\,447\,601\,336\,113\,266\,314\,246\,229\,403\,i\,+\right.$$

$$14\,485\,428\,949\,218\,159\,926\,589\,054\,537\,567\,341\,i\,\sqrt{37}\,+$$

$$63\,871\,529\,724\,470\,428\,043\,397\,844\,255\,429\,016\,\sqrt{6+\sqrt{37}}\,+$$

$$\left.10\,500\,414\,804\,346\,964\,635\,832\,629\,164\,778\,862\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{13}\,+$$

$$4\left(40\,431\,151\,952\,039\,859\,240\,121\,226\,409\,550\,035\,+\right.$$

$$6\,646\,840\,436\,504\,372\,389\,549\,188\,402\,158\,310\,\sqrt{37}\,+$$

$$14\,223\,654\,109\,429\,439\,306\,138\,539\,255\,264\,367\,i\,\sqrt{6+\sqrt{37}}\,+$$

$$\left.2\,338\,354\,331\,371\,883\,426\,898\,427\,966\,909\,658\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{14}\,+$$

$$4\left(1\,505\,270\,473\,681\,744\,452\,743\,910\,509\,158\,067\,\sqrt{6+\sqrt{37}}\,+\right.$$

$$247\,464\,941\,493\,934\,259\,316\,422\,695\,518\,272\,\sqrt{37\left(6+\sqrt{37}\right)}\,+$$

$$91\,i\left(353\,233\,427\,889\,515\,363\,592\,296\,958\,176\,266\,+\right.$$

$$\left.\left.58\,071\,217\,827\,436\,275\,581\,839\,296\,015\,631\,\sqrt{37}\right)\right)\omega_0^{15}\,+$$

$$4\,i\left(4\,247\,012\,153\,376\,299\,829\,484\,370\,892\,230\,544\,\sqrt{6+\sqrt{37}}\,+\right.$$

$$698\,204\,497\,088\,608\,438\,821\,398\,577\,125\,715\,\sqrt{37\left(6+\sqrt{37}\right)}\,+$$

$$4\,i\left(1\,684\,307\,208\,294\,979\,682\,400\,938\,711\,674\,883\,+\right.$$

$$276\,898\,399\,354\,808\,139\,758\,307\,106\,093\,580\,\sqrt{37}\Big)\Big)\,\omega_0^{16} -$$

$$12\,\Big(654\,528\,792\,367\,416\,002\,737\,630\,234\,249\,978\,\sqrt{6+\sqrt{37}} +$$

$$107\,603\,870\,627\,401\,660\,196\,849\,187\,712\,565\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$i\,\Big(1\,233\,503\,318\,992\,754\,622\,831\,877\,512\,942\,114 +$$

$$202\,786\,696\,480\,205\,949\,659\,894\,654\,541\,821\,\sqrt{37}\Big)\Big)\,\omega_0^{17} +$$

$$\Big(-13\,707\,843\,200\,899\,298\,645\,728\,683\,150\,377\,137 -$$

$$2\,253\,555\,540\,368\,472\,022\,741\,812\,733\,342\,573\,\sqrt{37} -$$

$$380\,961\,543\,402\,477\,691\,658\,151\,312\,438\,648\,i\,\sqrt{6+\sqrt{37}} -$$

$$62\,629\,691\,937\,653\,249\,843\,678\,808\,134\,208\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{18} -$$

$$4\,\Big(820\,306\,037\,963\,198\,717\,064\,898\,130\,141\,911\,i +$$

$$134\,857\,481\,921\,619\,840\,198\,916\,170\,676\,078\,i\,\sqrt{37} +$$

$$362\,914\,246\,687\,764\,595\,910\,986\,632\,896\,815\,\sqrt{6+\sqrt{37}} +$$

$$59\,662\,734\,634\,155\,115\,312\,140\,642\,829\,351\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{19} +$$

$$\Big(-1\,429\,848\,385\,305\,440\,246\,327\,645\,627\,391\,451 -$$

$$235\,065\,626\,544\,414\,688\,301\,882\,195\,290\,855\,\sqrt{37} -$$

$$645\,812\,820\,206\,515\,798\,608\,949\,583\,328\,655\,i\,\sqrt{6+\sqrt{37}} -$$

$$106\,170\,973\,630\,767\,981\,798\,342\,436\,869\,331\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{20} -$$

$$\Big(25\,158\,728\,702\,923\,764\,681\,242\,510\,585\,417\,\sqrt{6+\sqrt{37}} +$$

$$4\,136\,069\,520\,650\,883\,116\,994\,802\,050\,941\,\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$4\,i\,\Big(251\,891\,372\,220\,715\,306\,586\,011\,345\,033\,565 +$$

$$41\,410\,686\,504\,042\,402\,859\,984\,346\,956\,511\,\sqrt{37}\Big)\Big)$$

$$\omega_0^{21} + \Big(60\,020\,077\,678\,560\,823\,386\,937\,926\,083\,383 +$$

$$9\,867\,239\,988\,344\,269\,379\,627\,613\,209\,983\,\sqrt{37} -$$

$$81\,252\,383\,281\,570\,193\,037\,246\,987\,369\,005\,i\,\sqrt{6+\sqrt{37}} -$$

$$13\,357\,809\,527\,636\,872\,220\,194\,783\,113\,185\, i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{22} +$$

$$\Bigg(41\,773\,228\,406\,857\,532\,056\,669\,323\,555 \sqrt{6+\sqrt{37}} +$$

$$6\,867\,476\,446\,533\,633\,522\,075\,757\,167 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$i \Big(87\,605\,631\,028\,119\,520\,424\,156\,530\,257\,605 +$$

$$14\,402\,277\,023\,269\,630\,943\,347\,194\,811\,421 \sqrt{37} \Big) \Bigg) \omega_0^{23} +$$

$$\Bigg(1\,464\,190\,231\,079\,992\,251\,876\,191\,709\,296 + 240\,711\,391\,211\,878\,168\,161\,645\,551\,528$$

$$\sqrt{37} - 10\,823\,463\,889\,783\,459\,055\,982\,462\,554\,329\, i \sqrt{6+\sqrt{37}} -$$

$$1\,779\,366\,502\,616\,363\,754\,109\,761\,948\,625\, i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{24} -$$

$$\Bigg(12\,761\,740\,375\,427\,571\,123\,822\,757\,643\,045\, i +$$

$$2\,098\,017\,193\,974\,294\,252\,627\,156\,693\,293\, i \sqrt{37} +$$

$$880\,915\,507\,198\,905\,714\,461\,337\,485\,111 \sqrt{6+\sqrt{37}} +$$

$$144\,821\,617\,286\,027\,634\,249\,916\,382\,015 \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{25} +$$

$$\Bigg(-2\,653\,324\,453\,027\,815\,547\,392\,782\,968\,774 -$$

$$436\,203\,853\,070\,056\,871\,062\,207\,052\,998 \sqrt{37} -$$

$$2\,362\,192\,434\,637\,976\,108\,597\,604\,786\,260\, i \sqrt{6+\sqrt{37}} -$$

$$388\,342\,044\,075\,389\,123\,310\,254\,425\,724\, i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{26} +$$

$$3 \Bigg(63\,123\,958\,877\,129\,621\,473\,018\,952\,300 \sqrt{6+\sqrt{37}} +$$

$$10\,377\,514\,913\,295\,361\,384\,722\,382\,532 \sqrt{37 \left(6+\sqrt{37}\right)} -$$

$$5\, i \Big(288\,034\,511\,000\,371\,822\,089\,688\,291\,045 +$$

$$47\,352\,581\,917\,460\,817\,581\,350\,927\,881 \sqrt{37} \Big) \Bigg) \omega_0^{27} +$$

$$2 \Bigg(331\,726\,053\,969\,527\,593\,257\,560\,957\,587 + 54\,535\,427\,336\,439\,902\,821\,556\,487\,427$$

$$\sqrt{37} - 179\,318\,350\,665\,683\,872\,740\,686\,889\,649\, i \sqrt{6+\sqrt{37}} -$$

$$29\,479\,755\,256\,365\,646\,370\,077\,146\,085\, i \sqrt{37 \left(6+\sqrt{37}\right)} \Bigg) \omega_0^{28} +$$

$$4\left(22\,232\,430\,921\,687\,782\,822\,741\,159\,569\sqrt{6+\sqrt{37}}+3\,654\,989\,129\,877\,983\,322\,814\,018\,951\sqrt{37\left(6+\sqrt{37}\right)}-3\,i\left(36\,473\,742\,618\,012\,068\,454\,839\,325\,061+5\,996\,246\,349\,481\,869\,837\,971\,404\,451\sqrt{37}\right)\right)\omega_0^{29}+$$

$$8\left(23\,201\,917\,899\,485\,404\,043\,445\,419\,284+3\,814\,371\,806\,566\,847\,810\,412\,641\,491\sqrt{37}-3\,649\,624\,789\,617\,605\,592\,302\,988\,233\,i\sqrt{6+\sqrt{37}}-599\,994\,619\,250\,197\,988\,140\,575\,650\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{30}+$$

$$8\left(560\,099\,627\,701\,673\,865\,430\,714\,143\,i+92\,079\,812\,068\,229\,000\,312\,743\,350\,i\sqrt{37}+1\,842\,722\,504\,549\,042\,243\,295\,591\,392\sqrt{6+\sqrt{37}}+302\,941\,715\,745\,161\,884\,959\,340\,289\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{31}+$$

$$2\left(5\,224\,485\,426\,954\,290\,052\,460\,397\,423+858\,900\,112\,228\,103\,461\,823\,425\,745\sqrt{37}+211\,159\,450\,777\,368\,055\,066\,446\,428\,i\sqrt{6+\sqrt{37}}+34\,714\,410\,297\,672\,218\,817\,789\,992\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{32}+$$

$$4\left(230\,951\,450\,342\,520\,499\,868\,952\,964\,i+37\,968\,183\,987\,093\,648\,332\,648\,542\,i\sqrt{37}+233\,732\,791\,748\,103\,278\,575\,866\,447\sqrt{6+\sqrt{37}}+38\,425\,434\,308\,956\,329\,947\,024\,307\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{33}+$$

$$\left(234\,833\,650\,678\,014\,043\,466\,916\,233+38\,606\,414\,059\,437\,773\,291\,725\,433\sqrt{37}+156\,495\,697\,087\,929\,971\,980\,815\,642\,i\sqrt{6+\sqrt{37}}+25\,727\,738\,885\,677\,376\,029\,972\,038\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{34}+$$

$$2\left(7\,442\,770\,852\,085\,429\,726\,659\,803\sqrt{6+\sqrt{37}}+1\,223\,582\,594\,861\,301\,952\,532\,553\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\right.$$

$$\left(41\,125\,616\,729\,723\,984\,074\,183\,057 + 6\,761\,010\,008\,496\,661\,566\,091\,555\,\sqrt{37}\right)\Big)$$

$$\omega_0^{35} + \Big(99\,825\,374\,255\,753\,277\,202\,274\,423 + 16\,411\,190\,886\,288\,388\,436\,159\,327\,\sqrt{37} - 9\,314\,839\,084\,299\,716\,196\,398\,105\,\mathrm{i}\,\sqrt{6+\sqrt{37}} - 1\,531\,352\,934\,482\,647\,161\,603\,137\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{36} +$$

$$3\,\Big(2\,239\,571\,026\,623\,571\,865\,433\,254\,\mathrm{i} + 368\,183\,351\,843\,735\,204\,063\,450\,\mathrm{i}\,\sqrt{37} + 2\,357\,128\,638\,408\,526\,759\,225\,435\,\sqrt{6+\sqrt{37}} + 387\,509\,378\,906\,799\,644\,453\,355\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{37} +$$

$$\Big(11\,727\,993\,763\,051\,102\,125\,363\,297 + 1\,928\,070\,390\,741\,199\,103\,854\,053\,\sqrt{37} + 674\,486\,271\,097\,066\,534\,044\,085\,\mathrm{i}\,\sqrt{6+\sqrt{37}} + 110\,883\,614\,603\,550\,921\,655\,621\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{38} +$$

$$\Big(4\,699\,903\,682\,997\,257\,316\,133\,449\,\mathrm{i} + 772\,659\,659\,558\,056\,232\,010\,525\,\mathrm{i}\,\sqrt{37} + 919\,743\,181\,439\,075\,117\,665\,709\,\sqrt{6+\sqrt{37}} + 151\,204\,883\,546\,628\,492\,467\,525\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{39} +$$

$$\Big(-75\,257\,163\,978\,113\,396\,043\,758 - 12\,372\,212\,774\,311\,731\,493\,274\,\sqrt{37} + 528\,994\,681\,696\,056\,747\,357\,973\,\mathrm{i}\,\sqrt{6+\sqrt{37}} + 86\,966\,079\,702\,052\,777\,418\,233\,\mathrm{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{40} +$$

$$\Big(503\,265\,335\,813\,989\,837\,418\,089\,\mathrm{i} + 82\,736\,340\,239\,102\,636\,721\,877\,\mathrm{i}\,\sqrt{37} - 49\,127\,970\,534\,314\,894\,784\,657\,\sqrt{6+\sqrt{37}} - 8\,076\,563\,044\,424\,049\,892\,677\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{41} +$$

$$2\,\mathrm{i}\,\Big(65\,139\,343\,796\,601\,292\,840\,607\,\mathrm{i} + 10\,708\,844\,684\,624\,790\,875\,201\,\mathrm{i}\,\sqrt{37} + 23\,334\,085\,390\,189\,163\,319\,569\,\sqrt{6+\sqrt{37}} + 3\,836\,116\,621\,100\,386\,202\,243\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{42} +$$

$$\left(23\,601\,398\,503\,944\,466\,361\,863\,\mathbb{i} + 3\,880\,039\,083\,529\,504\,899\,847\,\mathbb{i}\,\sqrt{37} - 16\,066\,946\,176\,094\,476\,436\,750\,\sqrt{6+\sqrt{37}} - 2\,641\,384\,705\,745\,256\,325\,778\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{43} +$$

$$2\,\mathbb{i}\left(199\,032\,699\,640\,597\,921\,427\,\sqrt{6+\sqrt{37}} + 32\,726\,746\,999\,573\,365\,269\,\sqrt{37\left(6+\sqrt{37}\right)} + 3\,\mathbb{i}\left(2\,564\,996\,562\,890\,173\,071\,637 + 421\,682\,948\,853\,373\,943\,703\,\sqrt{37}\right)\right)\omega_0^{44} -$$

$$2\left(173\,541\,557\,432\,327\,268\,137\,\mathbb{i} + 28\,531\,474\,592\,194\,314\,095\,\mathbb{i}\,\sqrt{37} + 649\,435\,738\,988\,044\,388\,680\,\sqrt{6+\sqrt{37}} + 106\,766\,220\,490\,025\,078\,170\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{45} -$$

$$4\,\mathbb{i}\left(39\,545\,058\,060\,881\,910\,615\,\sqrt{6+\sqrt{37}} + 6\,500\,659\,050\,711\,947\,472\,\sqrt{37\left(6+\sqrt{37}\right)} - \mathbb{i}\left(257\,759\,780\,647\,926\,625\,999 + 42\,375\,464\,538\,094\,778\,536\,\sqrt{37}\right)\right)\omega_0^{46} -$$

$$4\left(32\,619\,431\,278\,442\,239\,016\,\mathbb{i} + 5\,362\,724\,290\,443\,342\,989\,\mathbb{i}\,\sqrt{37} + 13\,295\,309\,401\,854\,589\,647\,\sqrt{6+\sqrt{37}} + 2\,185\,714\,698\,022\,740\,558\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{47} - 4\,\mathbb{i}$$

$$\left(2\,584\,849\,538\,738\,436\,590\,\sqrt{6+\sqrt{37}} + 424\,886\,169\,001\,102\,919\,\sqrt{37\left(6+\sqrt{37}\right)} - \mathbb{i}\left(11\,163\,484\,742\,285\,738\,675 + 1\,835\,267\,927\,218\,448\,441\,\sqrt{37}\right)\right)\omega_0^{48} -$$

$$4\left(2\,180\,350\,797\,302\,192\,612\,\mathbb{i} + 358\,461\,472\,448\,444\,567\,\mathbb{i}\,\sqrt{37} + 284\,501\,384\,917\,687\,594\,\sqrt{6+\sqrt{37}} + 46\,769\,820\,259\,517\,683\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{49} +$$

$$\left(-271\,671\,796\,437\,081\,260\,i\sqrt{6+\sqrt{37}} - 44\,640\,581\,096\,017\,484\,i\sqrt{37\left(6+\sqrt{37}\right)} - 3\left(416\,378\,326\,362\,451\,841 + 68\,452\,561\,608\,055\,605\sqrt{37}\right)\right)\omega_0^{50} -$$

$$12\left(648\,256\,059\,840\,158\sqrt{6+\sqrt{37}} + 106\,520\,854\,714\,688\sqrt{37\left(6+\sqrt{37}\right)} + 5\,i\left(4\,657\,589\,796\,509\,717 + 765\,782\,394\,118\,042\sqrt{37}\right)\right)\omega_0^{51} +$$

$$\left(-19\,804\,501\,044\,057\,129 - 3\,255\,953\,491\,704\,069\sqrt{37} - 1\,975\,549\,884\,940\,573\,i\sqrt{6+\sqrt{37}} - 323\,259\,524\,175\,697\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{52} +$$

$$\left(189\,790\,222\,608\,485\sqrt{6+\sqrt{37}} + 31\,247\,378\,184\,209\sqrt{37\left(6+\sqrt{37}\right)} - 24\,i\left(143\,201\,097\,657\,193 + 23\,555\,104\,897\,169\sqrt{37}\right)\right)\omega_0^{53} +$$

$$\left(-16\,562\,468\,522\,267 - 2\,731\,927\,950\,299\sqrt{37} + 49\,961\,762\,014\,841\,i\sqrt{6+\sqrt{37}} + 8\,290\,186\,712\,549\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{54} +$$

$$\left(44\,514\,958\,651\,673\,i + 7\,302\,809\,344\,793\,i\sqrt{37} + 5\,078\,009\,895\,641\sqrt{6+\sqrt{37}} + 837\,599\,160\,629\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{55} +$$

$$\left(7\,373\,695\,266\,108 + 1\,211\,659\,283\,436\sqrt{37} + 1\,062\,782\,716\,141\,i\sqrt{6+\sqrt{37}} + 177\,394\,220\,077\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{56} +$$

$$\left(53\,533\,660\,123\sqrt{6+\sqrt{37}} + 8\,918\,592\,763\sqrt{37\left(6+\sqrt{37}\right)} + 3\,i\left(629\,489\,639\,027 + 103\,305\,656\,659\sqrt{37}\right)\right)\omega_0^{57} +$$

$$6\left(27\,892\,700\,941 + 4\,581\,321\,305\sqrt{37} + 1\,610\,598\,900\,i\sqrt{6+\sqrt{37}} + 274\,930\,972\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{58} +$$

$$\Big(19\,463\,991\,031\,i + 3\,186\,985\,243\,i\sqrt{37} + 687\,726\,992\sqrt{6+\sqrt{37}} +$$

$$116\,219\,264\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{59} +$$

$$2\Big(736\,750\,485 + 120\,760\,593\sqrt{37} + 109\,678\,397\,i\sqrt{6+\sqrt{37}} +$$

$$18\,439\,109\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{60} +$$

$$4\Big(4\,387\,719\sqrt{6+\sqrt{37}} + 733\,407\sqrt{37\left(6+\sqrt{37}\right)} +$$

$$2\,i\left(8\,383\,631 + 1\,355\,633\sqrt{37}\right)\Big)\,\omega_0^{61} +$$

$$24\left(174\,878 + 28\,258\sqrt{37} + 120\,583\,i\sqrt{6+\sqrt{37}} + 20\,031\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{62} +$$

$$16\left(12\,691\sqrt{6+\sqrt{37}} + 2107\sqrt{37\left(6+\sqrt{37}\right)} + 81\,i\left(267 + 43\sqrt{37}\right)\right)\omega_0^{63} +$$

$$96\left(267 + 43\sqrt{37}\right)\omega_0^{64}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^2\left(-2\,i+\omega_0\right)^2\left(2\,i+\omega_0\right)^2\left(\sqrt{37\left(6+\sqrt{37}\right)} + 8\,\omega_0 + 2\,i\,\omega_0^2 + \omega_0^3\right)$$

$$\left(73 + 12\sqrt{37} + 4\left(7+\sqrt{37}\right)\omega_0^2 - \omega_0^4\right)^4$$

$$\Big(-73 - 12\sqrt{37} + 2\,i\left(97 + 16\sqrt{37}\right)\omega_0 +$$

$$12\left(7+\sqrt{37}\right)\omega_0^2 - 32\,i\,\omega_0^3 + \omega_0^4 - 2\,i\,\omega_0^5\Big)$$

$$\left(-73 - 12\sqrt{37} - 2\,i\left(97 + 16\sqrt{37}\right)\omega_0 + 12\left(7+\sqrt{37}\right)\omega_0^2 + 32\,i\,\omega_0^3 + \omega_0^4 + 2\,i\,\omega_0^5\right)^2$$

$$\Big(-\sqrt{6+\sqrt{37}}\left(518 + 85\sqrt{37}\right) +$$

$$\left(-534 - 88\sqrt{37} - 222\,i\sqrt{6+\sqrt{37}} - 44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0 +$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)} - i\left(583 + 86\sqrt{37}\right)\right)\omega_0^2 +$$

$$2\left(102 + 9\sqrt{37} - i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3 +$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)} - 2\,i\left(15 + 2\sqrt{37}\right)\right)\omega_0^4 + 12\,\omega_0^5 - i\,\omega_0^6\Big)$$

$$\Big(37\left(6+\sqrt{37}\right) + 16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0 + 64\,\omega_0^2 +$$

$$2\sqrt{37\left(6+\sqrt{37}\right)}\ \omega_0^3+20\ \omega_0^4+\omega_0^6\Big)$$

$$\left(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-2\left(89+28\sqrt{37}\right)\omega_0^4-\right.$$

$$\left.104\left(-4+\sqrt{37}\right)\omega_0^6+129\ \omega_0^8+4\ \omega_0^{10}\right)$$

$$\Big(37\left(174\,922+28\,757\sqrt{37}\right)+4\sqrt{6+\sqrt{37}}\left(276\,686+45\,487\sqrt{37}\right)\omega_0+$$

$$4\left(314\,823+51\,764\sqrt{37}\right)\omega_0^2+$$

$$56\sqrt{6+\sqrt{37}}\left(2664+455\sqrt{37}\right)\omega_0^3+$$

$$\left(296\,509+48\,182\sqrt{37}\right)\omega_0^4+$$

$$8\sqrt{6+\sqrt{37}}\left(4181+965\sqrt{37}\right)\omega_0^5+8\left(13\,319+1993\sqrt{37}\right)\omega_0^6+$$

$$8\sqrt{6+\sqrt{37}}\left(296+107\sqrt{37}\right)\omega_0^7+\left(7776+881\sqrt{37}\right)\omega_0^8+$$

$$28\sqrt{37\left(6+\sqrt{37}\right)}\ \omega_0^9+4\left(51+2\sqrt{37}\right)\omega_0^{10}+\omega_0^{12}\Big)\Big)\Big),$$

$$-\Big(\Big(64\ \omega_0^4\Big(67\,469\,796\Big(6\,393\,033\,890\,621\,681\,382\,129\,205+$$

$$1\,051\,008\,297\,427\,032\,716\,092\,967\sqrt{37}\Big)+$$

$$1\,215\,672\Big(749\,405\,998\,021\,908\,776\,724\,835\,393\sqrt{6+\sqrt{37}}+$$

$$123\,201\,587\,155\,362\,390\,535\,258\,283\sqrt{37\left(6+\sqrt{37}\right)}+222\,\mathbb{i}$$

$$\left(5\,010\,808\,745\,561\,452\,494\,643\,321+823\,771\,883\,351\,130\,510\,068\,369\sqrt{37}\right)\Big)$$

$$\omega_0+151\,959\Big(101\,673\,886\,598\,698\,810\,503\,962\,107\,603+$$

$$16\,715\,083\,992\,225\,822\,327\,835\,790\,479\sqrt{37}+$$

$$19\,009\,595\,593\,091\,053\,976\,487\,635\,908\,\mathbb{i}\sqrt{6+\sqrt{37}}+$$

$$3\,125\,158\,264\,588\,552\,045\,734\,266\,892\,\mathbb{i}\sqrt{37\left(6+\sqrt{37}\right)}\Big)\omega_0^2+$$

$$2738\Big(4\,176\,778\,829\,199\,445\,276\,935\,283\,442\,665\sqrt{6+\sqrt{37}}+$$

$$686\,658\,209\,718\,844\,684\,075\,452\,411\,121\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$148\,\mathbb{i}\Big(128\,547\,620\,135\,979\,688\,454\,777\,650\,907+$$

$$21\,133\,098\,570\,868\,815\,812\,767\,473\,116\sqrt{37}\Big)\Big)\omega_0^3+$$

$$1369\Big(64\,819\,352\,958\,880\,157\,187\,936\,873\,085\,985+$$

$$10\,656\,235\,984\,228\,412\,383\,347\,313\,050\,161\sqrt{37}+$$

$$31\,631\,804\,224\,821\,966\,945\,293\,540\,881\,635\,i\sqrt{6+\sqrt{37}}\;+$$

$$5\,200\,236\,581\,202\,053\,602\,004\,200\,418\,775\,i\sqrt{37\left(6+\sqrt{37}\right)}\;)\;\omega_0^4\;+$$

$$111\;(304\,108\,922\,999\,556\,199\,144\,047\,994\,338\,939\sqrt{6+\sqrt{37}}\;+$$

$$49\,995\,198\,971\,649\,916\,843\,826\,750\,958\,967\sqrt{37\left(6+\sqrt{37}\right)}\;+$$

$$74\,i\;(49\,233\,697\,857\,493\,575\,631\,075\,142\,210\,639\;+$$

$$8\,093\,970\,069\,069\,882\,692\,643\,003\,603\,275\sqrt{37}\;))\;\omega_0^5\;+$$

$$37\;(951\,445\,600\,516\,514\,392\,537\,248\,305\,498\,541\;+$$

$$156\,416\,693\,201\,052\,492\,005\,876\,712\,432\,045\sqrt{37}\;+$$

$$5\,780\,202\,746\,606\,641\,074\,110\,600\,677\,590\,667\,i\sqrt{6+\sqrt{37}}\;+$$

$$950\,259\,477\,961\,776\,520\,005\,596\,433\,976\,939\,i\sqrt{37\left(6+\sqrt{37}\right)}\;)\;\omega_0^6\;+$$

$$(-51\,970\,384\,524\,520\,640\,672\,650\,333\,524\,329\,841\sqrt{6+\sqrt{37}}\;-$$

$$8\,543\,878\,585\,701\,205\,708\,251\,026\,837\,594\,889\sqrt{37\left(6+\sqrt{37}\right)}\;+$$

$$37\,i\;(29\,870\,371\,490\,873\,349\,591\,373\,384\,560\,027\,221\;+$$

$$4\,910\,658\,823\,534\,394\,074\,765\,138\,372\,471\,013\sqrt{37}\;))\;\omega_0^7\;+$$

$$(-589\,363\,057\,396\,352\,665\,416\,389\,850\,375\,188\,586\;-$$

$$96\,890\,689\,791\,149\,541\,068\,786\,118\,300\,978\,462\sqrt{37}\;+$$

$$383\,045\,856\,909\,796\,593\,839\,991\,622\,105\,264\,849\,i\sqrt{6+\sqrt{37}}\;+$$

$$62\,972\,350\,967\,483\,354\,790\,588\,836\,409\,645\,913\,i\sqrt{37\left(6+\sqrt{37}\right)}\;)\;\omega_0^8\;+$$

$$(1\,097\,755\,450\,219\,718\,270\,056\,593\,297\,926\,505\,023\,i\;+$$

$$180\,469\,884\,325\,058\,239\,168\,078\,363\,099\,729\,811\,i\sqrt{37}\;-$$

$$293\,516\,254\,176\,344\,636\,563\,189\,712\,209\,329\,367\sqrt{6+\sqrt{37}}\;-$$

$$48\,253\,774\,944\,252\,905\,234\,861\,084\,546\,366\,335\sqrt{37\left(6+\sqrt{37}\right)}\;)\;\omega_0^9\;+$$

$$2\,i\;(558\,136\,577\,556\,499\,807\,531\,487\,331\,934\,878\,787\,i\;+$$

$$91\,757\,088\,128\,366\,575\,777\,918\,660\,975\,445\,563\,i\sqrt{37}\;+$$

$$83\,268\,476\,694\,788\,844\,054\,531\,894\,128\,391\,473\,\sqrt{6+\sqrt{37}}\,+$$

$$13\,689\,253\,243\,083\,030\,435\,143\,709\,915\,342\,923\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{10}\,+$$

$$\Bigg(-257\,357\,867\,356\,918\,955\,495\,238\,347\,567\,821\,852\,\sqrt{6+\sqrt{37}}\,-$$

$$42\,309\,372\,768\,543\,944\,375\,620\,966\,889\,938\,648\,\sqrt{37\left(6+\sqrt{37}\right)}\,+$$

$$3\,i\,\Big(40\,581\,870\,352\,740\,587\,771\,919\,208\,787\,720\,363\,+$$

$$6\,671\,618\,388\,947\,855\,212\,562\,869\,023\,350\,687\,\sqrt{37}\Big)\Bigg)\,\omega_0^{11}\,+$$

$$\Bigg(-561\,384\,821\,479\,190\,649\,428\,203\,806\,109\,707\,340\,-$$

$$92\,291\,096\,139\,777\,731\,457\,808\,419\,797\,122\,000\,\sqrt{37}\,-$$

$$41\,359\,084\,436\,756\,044\,548\,225\,187\,035\,001\,466\,i\,\sqrt{6+\sqrt{37}}\,-$$

$$6\,799\,391\,597\,279\,457\,184\,513\,405\,787\,610\,530\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{12}\,-$$

$$4\,\Bigg(51\,215\,149\,173\,213\,835\,654\,704\,931\,174\,159\,024\,i\,+$$

$$8\,419\,718\,658\,769\,161\,217\,743\,930\,792\,736\,495\,i\,\sqrt{37}\,+$$

$$18\,107\,094\,937\,941\,679\,625\,265\,807\,106\,244\,024\,\sqrt{6+\sqrt{37}}\,+$$

$$2\,976\,788\,070\,839\,573\,415\,430\,054\,517\,639\,540\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{13}\,+$$

$$\Bigg(-41\,019\,222\,901\,387\,606\,085\,349\,623\,303\,581\,434\,i\,\sqrt{6+\sqrt{37}}\,-$$

$$6\,743\,518\,705\,040\,842\,032\,406\,086\,758\,125\,086\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\,-$$

$$130\,\Big(502\,388\,882\,353\,955\,466\,380\,476\,361\,079\,893\,+$$

$$82\,592\,223\,492\,460\,561\,201\,090\,050\,527\,297\,\sqrt{37}\Big)\Bigg)\,\omega_0^{14}\,+$$

$$4\,\Bigg(125\,606\,315\,386\,326\,534\,458\,216\,970\,864\,898\,\sqrt{6+\sqrt{37}}\,+$$

$$20\,649\,551\,048\,669\,399\,850\,221\,005\,468\,669\,\sqrt{37\left(6+\sqrt{37}\right)}\,-$$

$$i\,\Big(16\,585\,403\,500\,196\,451\,911\,608\,144\,454\,822\,773\,+$$

$$2\,726\,623\,539\,483\,024\,843\,260\,534\,071\,199\,167\,\sqrt{37}\Big)\Bigg)$$

$$\omega_0^{15}\,+\,2\,\Bigg(6\,257\,389\,100\,813\,909\,288\,178\,649\,090\,181\,628\,+$$

$$1\,028\,708\,431\,349\,386\,933\,852\,613\,952\,322\,386\,\sqrt{37} -$$
$$2\,944\,892\,775\,030\,545\,803\,517\,660\,778\,187\,157\,i\,\sqrt{6+\sqrt{37}} -$$
$$484\,137\,389\,937\,885\,098\,016\,055\,139\,328\,401\,i\,\sqrt{37\,(6+\sqrt{37})}\,)\,\omega_0^{16} +$$
$$4\,(450\,880\,478\,560\,140\,347\,544\,630\,899\,755\,200\,\sqrt{6+\sqrt{37}} +$$
$$74\,124\,294\,071\,041\,932\,261\,864\,179\,274\,744\,\sqrt{37\,(6+\sqrt{37})} -$$
$$i\,(368\,254\,716\,319\,377\,546\,857\,079\,724\,002\,098 +$$
$$60\,540\,702\,433\,327\,298\,970\,783\,772\,500\,323\,\sqrt{37}\,)\,)$$
$$\omega_0^{17} + (-74\,349\,458\,001\,200\,566\,369\,223\,918\,032\,453 -$$
$$12\,222\,975\,602\,099\,566\,824\,715\,486\,873\,561\,\sqrt{37} +$$
$$593\,724\,147\,073\,078\,573\,754\,097\,572\,490\,524\,i\,\sqrt{6+\sqrt{37}} +$$
$$97\,607\,648\,517\,551\,128\,386\,281\,063\,409\,056\,i\,\sqrt{37\,(6+\sqrt{37})}\,)\,\omega_0^{18} +$$
$$(-526\,740\,717\,987\,098\,019\,501\,926\,918\,697\,124\,\sqrt{6+\sqrt{37}} -$$
$$86\,595\,640\,609\,575\,711\,105\,158\,718\,632\,392\,\sqrt{37\,(6+\sqrt{37})} +$$
$$20\,i\,(33\,408\,848\,816\,171\,622\,253\,138\,061\,487\,761 +$$
$$5\,492\,380\,912\,416\,399\,417\,999\,985\,802\,729\,\sqrt{37}\,)\,)$$
$$\omega_0^{19} + (-1\,269\,672\,384\,143\,103\,668\,721\,489\,800\,198\,441 -$$
$$208\,732\,854\,162\,704\,823\,299\,064\,980\,652\,509\,\sqrt{37} -$$
$$85\,017\,329\,482\,855\,813\,014\,790\,037\,580\,919\,i\,\sqrt{6+\sqrt{37}} -$$
$$13\,976\,762\,870\,387\,389\,889\,045\,801\,248\,459\,i\,\sqrt{37\,(6+\sqrt{37})}\,)\,\omega_0^{20} -$$
$$(164\,666\,967\,458\,487\,134\,163\,248\,733\,963\,419\,\sqrt{6+\sqrt{37}} +$$
$$27\,071\,082\,692\,819\,512\,518\,035\,475\,277\,463\,\sqrt{37\,(6+\sqrt{37})} +$$
$$12\,i\,(37\,580\,999\,581\,045\,799\,423\,755\,095\,625\,559 +$$
$$6\,178\,278\,273\,046\,986\,048\,629\,249\,970\,214\,\sqrt{37}\,)\,)$$

$$\omega_0^{21} + \left(-165\,571\,540\,853\,413\,444\,687\,251\,415\,159\,297 - 27\,219\,793\,642\,888\,757\,888\,601\,509\,675\,785\,\sqrt{37} - 93\,193\,145\,630\,735\,888\,623\,705\,914\,471\,967\,i\,\sqrt{6+\sqrt{37}} - 15\,320\,858\,765\,493\,667\,660\,174\,440\,133\,675\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{22} -$$

$$3\left(50\,234\,123\,563\,250\,314\,298\,376\,853\,445\,953\,i + 8\,258\,439\,041\,970\,538\,629\,486\,558\,840\,809\,i\,\sqrt{37} + 44\,401\,207\,896\,928\,898\,362\,059\,835\,587\,\sqrt{6+\sqrt{37}} + 7\,299\,513\,613\,385\,134\,664\,208\,902\,143\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{23} +$$

$$\left(18\,918\,221\,210\,172\,862\,140\,023\,646\,399\,322 + 3\,110\,136\,408\,571\,182\,995\,825\,564\,916\,466\,\sqrt{37} - 13\,494\,391\,026\,076\,661\,358\,814\,169\,565\,725\,i\,\sqrt{6+\sqrt{37}} - 2\,218\,464\,218\,989\,571\,558\,897\,176\,813\,517\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{24} +$$

$$\left(2\,467\,093\,778\,997\,942\,890\,927\,442\,208\,729\,\sqrt{6+\sqrt{37}} + 405\,587\,718\,855\,059\,959\,344\,075\,801\,073\,\sqrt{37\left(6+\sqrt{37}\right)} - i\left(12\,438\,734\,781\,784\,815\,513\,152\,966\,580\,089 + 2\,044\,915\,401\,485\,294\,632\,519\,021\,842\,449\,\sqrt{37}\right)\right)$$

$$\omega_0^{25} + 2\left(1\,729\,379\,733\,167\,199\,363\,622\,028\,886\,362 + 284\,308\,276\,799\,035\,801\,501\,650\,441\,638\,\sqrt{37} - 437\,581\,070\,771\,936\,054\,258\,636\,059\,675\,i\,\sqrt{6+\sqrt{37}} - 71\,937\,884\,898\,773\,283\,905\,050\,897\,999\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{26} +$$

$$3\left(50\,805\,350\,844\,918\,999\,221\,290\,749\,398\,\sqrt{6+\sqrt{37}} + 8\,352\,348\,228\,986\,717\,615\,181\,486\,638\,\sqrt{37\left(6+\sqrt{37}\right)} - i\left(121\,617\,994\,495\,777\,504\,431\,369\,319\,457 + 19\,993\,875\,133\,200\,394\,793\,009\,205\,349\,\sqrt{37}\right)\right)\omega_0^{27} -$$

$$2\,i\left(50\,463\,828\,154\,369\,611\,311\,817\,092\,412\sqrt{6+\sqrt{37}}+8\,296\,202\,243\,344\,404\,896\,861\,764\,212\sqrt{37\left(6+\sqrt{37}\right)}-i\left(112\,252\,558\,873\,789\,148\,243\,312\,296\,081+18\,454\,207\,001\,782\,484\,260\,264\,031\,305\sqrt{37}\right)\right)\omega_0^{28}-$$

$$2\left(143\,768\,673\,370\,596\,897\,729\,394\,262\,021\,i+23\,635\,424\,308\,797\,386\,202\,977\,755\,901\,i\sqrt{37}+7\,060\,843\,987\,763\,483\,544\,238\,682\,832\sqrt{6+\sqrt{37}}+1\,160\,795\,602\,025\,466\,400\,614\,983\,184\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{29}+$$

$$4\left(248\,028\,016\,400\,601\,807\,874\,056\,254+40\,775\,554\,371\,701\,350\,495\,311\,518\sqrt{37}-8\,202\,050\,172\,529\,825\,587\,122\,942\,855\,i\sqrt{6+\sqrt{37}}-1\,348\,408\,742\,353\,976\,826\,609\,966\,833\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{30}+$$

$$8\left(286\,835\,914\,376\,086\,885\,593\,958\,321\sqrt{6+\sqrt{37}}+47\,155\,532\,821\,675\,208\,649\,440\,872\sqrt{37\left(6+\sqrt{37}\right)}-i\left(6\,707\,912\,668\,021\,574\,521\,886\,351\,543+1\,102\,774\,049\,793\,801\,005\,412\,390\,485\sqrt{37}\right)\right)\omega_0^{31}+$$

$$4\left(2\,997\,001\,187\,539\,364\,833\,128\,324\,321+492\,703\,960\,452\,327\,794\,510\,294\,361\sqrt{37}-1\,302\,358\,211\,209\,876\,280\,358\,701\,071\,i\sqrt{6+\sqrt{37}}-214\,106\,373\,549\,634\,143\,443\,185\,723\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{32}+$$

$$8\left(164\,333\,021\,495\,926\,119\,241\,941\,460\sqrt{6+\sqrt{37}}+27\,016\,182\,287\,480\,421\,107\,080\,930\sqrt{37\left(6+\sqrt{37}\right)}-i\left(448\,075\,795\,284\,723\,883\,095\,127\,201+73\,663\,206\,851\,622\,620\,055\,643\,882\sqrt{37}\right)\right)\omega_0^{33}+$$

$$\left(1\,426\,903\,162\,221\,451\,788\,132\,301\,083+234\,581\,434\,999\,592\,681\,426\,084\,631\sqrt{37}-\right.$$

$$288\,260\,438\,130\,063\,634\,308\,962\,876\,\mathrm{i}\sqrt{6+\sqrt{37}} -$$
$$47\,389\,726\,608\,028\,029\,184\,942\,460\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{34} +$$
$$2\Bigg(70\,184\,787\,443\,507\,058\,780\,671\,829\sqrt{6+\sqrt{37}} +$$
$$5\Bigg(2\,307\,661\,743\,533\,396\,141\,357\,253\sqrt{37\left(6+\sqrt{37}\right)} -$$
$$4\,\mathrm{i}\Big(5\,682\,798\,529\,953\,741\,904\,260\,295 +$$
$$934\,246\,310\,604\,234\,980\,100\,202\sqrt{37}\Big)\Bigg)\Bigg)\omega_0^{35} +$$
$$\Bigg(105\,978\,609\,283\,172\,784\,224\,481\,607 + 17\,422\,775\,805\,782\,953\,104\,186\,991\sqrt{37} -$$
$$9\,490\,522\,063\,409\,416\,406\,208\,659\,\mathrm{i}\sqrt{6+\sqrt{37}} -$$
$$1\,560\,230\,760\,367\,641\,493\,586\,279\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{36} +$$
$$\Bigg(7\,016\,178\,817\,978\,996\,829\,718\,863\sqrt{6+\sqrt{37}} +$$
$$1\,153\,453\,006\,774\,384\,638\,365\,723\sqrt{37\left(6+\sqrt{37}\right)} - 2\,\mathrm{i}$$
$$\Big(16\,310\,048\,443\,683\,377\,165\,948\,327 + 2\,681\,355\,562\,073\,414\,217\,533\,255\sqrt{37}\Big)\Bigg)$$
$$\omega_0^{37} + \Bigg(12\,534\,771\,083\,988\,159\,265\,104\,655 + 2\,060\,703\,621\,281\,618\,526\,141\,631\sqrt{37} -$$
$$3\,160\,407\,510\,567\,057\,010\,336\,503\,\mathrm{i}\sqrt{6+\sqrt{37}} -$$
$$519\,567\,110\,603\,220\,722\,759\,151\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{38} +$$
$$\Bigg(998\,777\,930\,424\,451\,957\,477\,869\sqrt{6+\sqrt{37}} +$$
$$164\,198\,053\,897\,913\,354\,715\,309\sqrt{37\left(6+\sqrt{37}\right)} -$$
$$\mathrm{i}\Big(3\,602\,222\,019\,176\,644\,134\,954\,829 + 592\,201\,788\,640\,019\,068\,925\,101\sqrt{37}\Big)\Bigg)$$
$$\omega_0^{39} + \Bigg(1\,719\,269\,803\,273\,187\,412\,165\,994 + 282\,646\,221\,969\,385\,836\,575\,398\sqrt{37} -$$
$$382\,237\,010\,218\,382\,592\,321\,337\,\mathrm{i}\sqrt{6+\sqrt{37}} -$$
$$62\,839\,320\,551\,162\,269\,367\,377\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\omega_0^{40} +$$

$$3\left(56\,319\,996\,207\,407\,453\,764\,313\sqrt{6+\sqrt{37}}+ 9\,258\,944\,606\,549\,840\,936\,177\sqrt{37\left(6+\sqrt{37}\right)}- i\left(76\,027\,624\,499\,929\,313\,354\,689+12\,498\,869\,124\,767\,999\,069\,493\sqrt{37}\right)\right)\omega_0^{41}+$$

$$2\left(90\,895\,614\,965\,983\,356\,889\,781+14\,943\,148\,676\,503\,613\,150\,981\sqrt{37}- 4\,711\,266\,655\,455\,299\,710\,933\,i\sqrt{6+\sqrt{37}}- 774\,538\,787\,343\,380\,079\,427\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{42}+$$

$$\left(14\,869\,923\,716\,096\,736\,740\,568\sqrt{6+\sqrt{37}}+ 2\,444\,597\,692\,538\,190\,016\,044\sqrt{37\left(6+\sqrt{37}\right)}- i\left(1\,587\,527\,509\,617\,887\,194\,517+260\,983\,112\,716\,313\,516\,561\sqrt{37}\right)\right)\omega_0^{43}+$$

$$2\left(6\,897\,026\,909\,186\,934\,865\,292+1\,133\,864\,408\,015\,972\,497\,946\sqrt{37}+ 602\,958\,645\,729\,206\,481\,671\,i\sqrt{6+\sqrt{37}}+ 99\,122\,158\,368\,568\,817\,075\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{44}+$$

$$4\left(188\,725\,472\,868\,804\,389\,698\sqrt{6+\sqrt{37}}+ 31\,026\,213\,358\,083\,565\,822\sqrt{37\left(6+\sqrt{37}\right)}+ 15\,i\left(20\,085\,922\,652\,496\,489\,641+3\,302\,133\,754\,925\,335\,104\sqrt{37}\right)\right)\omega_0^{45}+$$

$$2\left(365\,566\,764\,617\,958\,609\,847+60\,098\,811\,838\,053\,750\,571\sqrt{37}+ 57\,043\,299\,918\,533\,701\,187\,i\sqrt{6+\sqrt{37}}+ 9\,377\,364\,784\,549\,046\,153\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{46}+$$

$$4\left(28\,931\,029\,006\,543\,033\,553\,i+4\,756\,292\,065\,156\,492\,031\,i\sqrt{37}+ 5\,980\,611\,644\,644\,193\,068\sqrt{6+\sqrt{37}}+ 983\,201\,467\,233\,849\,523\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{47}+$$

$$\left(26\,416\,139\,669\,314\,554\,272 + 4\,342\,788\,090\,322\,166\,156\sqrt{37} + 4\,633\,160\,822\,729\,282\,182\,i\sqrt{6+\sqrt{37}} + 761\,613\,287\,534\,633\,518\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{48} +$$

$$\left(475\,525\,361\,455\,244\,456\sqrt{6+\sqrt{37}} + 78\,174\,400\,376\,514\,728\sqrt{37\left(6+\sqrt{37}\right)} + 12\,i\left(441\,874\,004\,161\,048\,288 + 72\,645\,031\,723\,128\,273\sqrt{37}\right)\right)\omega_0^{49} +$$

$$\left(613\,979\,682\,597\,494\,613 + 100\,938\,016\,291\,691\,529\sqrt{37} + 109\,058\,444\,263\,627\,460\,i\sqrt{6+\sqrt{37}} + 17\,926\,653\,749\,927\,624\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{50} +$$

$$12\left(10\,541\,971\,257\,964\,801\,i + 1\,733\,105\,226\,582\,885\,i\sqrt{37} + 403\,492\,886\,532\,610\sqrt{6+\sqrt{37}} + 66\,328\,259\,676\,915\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{51} +$$

$$\left(7\,049\,771\,902\,160\,585 + 1\,159\,007\,036\,491\,397\sqrt{37} + 1\,422\,388\,508\,717\,887\,i\sqrt{6+\sqrt{37}} + 233\,900\,039\,556\,499\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{52} +$$

$$\left(-30\,984\,649\,122\,953\sqrt{6+\sqrt{37}} - 5\,092\,956\,804\,125\sqrt{37\left(6+\sqrt{37}\right)} + 108\,i\left(8\,842\,755\,838\,090 + 1\,453\,093\,327\,587\sqrt{37}\right)\right)\omega_0^{53} +$$

$$\left(-59\,034\,570\,811\,631 - 9\,705\,254\,824\,343\sqrt{37} - 5\,580\,311\,385\,569\,i\sqrt{6+\sqrt{37}} - 906\,954\,538\,493\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{54} -$$

$$3\left(9\,475\,299\,948\,371\,i + 1\,559\,977\,617\,787\,i\sqrt{37} + 801\,897\,102\,161\sqrt{6+\sqrt{37}} + 131\,726\,457\,613\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{55} +$$

$$\left(-3\,938\,144\,708\,826 - 647\,606\,422\,842\sqrt{37} - 793\,746\,038\,939\,i\sqrt{6+\sqrt{37}} - 129\,981\,951\,323\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{56} -$$

$$\Big(878\,258\,078\,531\,i + 144\,703\,879\,523\,i\sqrt{37} + 58\,911\,466\,637\sqrt{6+\sqrt{37}} +$$

$$9\,666\,368\,453\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{57} -$$

$$2\,i\Big(7\,859\,889\,761\sqrt{6+\sqrt{37}} + 1\,285\,554\,617\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$4\,i\left(9\,057\,123\,431 + 1\,490\,666\,864\sqrt{37}\right)\Big)\,\omega_0^{58} -$$

$$\Big(9\,378\,801\,865\,i + 1\,550\,561\,629\,i\sqrt{37} + 868\,729\,438\sqrt{6+\sqrt{37}} +$$

$$142\,270\,654\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{59} -$$

$$2\,i\Big(38\,842\,526\sqrt{6+\sqrt{37}} + 6\,293\,630\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$3\,i\left(108\,953\,303 + 17\,991\,123\sqrt{37}\right)\Big)\,\omega_0^{60} -$$

$$2\left(9\,652\,537\,i + 1\,651\,861\,i\sqrt{37} + 2\,185\,220\sqrt{6+\sqrt{37}} + 355\,076\sqrt{37\left(6+\sqrt{37}\right)}\right)$$

$$\omega_0^{61} +$$

$$4\,i\left(331\,991\,i + 56\,699\,i\sqrt{37} + 143\,708\sqrt{6+\sqrt{37}} + 23\,900\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{62} +$$

$$8\left(4144\sqrt{6+\sqrt{37}} + 688\sqrt{37\left(6+\sqrt{37}\right)} + 27\,i\left(1331 + 215\sqrt{37}\right)\right)\omega_0^{63} +$$

$$16\left(1331 + 215\sqrt{37}\right)\omega_0^{64}\Big)\Big)\Big/$$

$$\Big(\left(7+\sqrt{37}\right)^{5/2}\left(-2\,i+\omega_0\right)^2\left(2\,i+\omega_0\right)^2$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)} + 8\,\omega_0 + 2\,i\,\omega_0^2 + \omega_0^3\right)$$

$$\left(73 + 12\sqrt{37} + 4\left(7+\sqrt{37}\right)\omega_0^2 - \omega_0^4\right)^4$$

$$\Big(-73 - 12\sqrt{37} + 2\,i\left(97 + 16\sqrt{37}\right)\omega_0 +$$

$$12\left(7+\sqrt{37}\right)\omega_0^2 - 32\,i\,\omega_0^3 + \omega_0^4 - 2\,i\,\omega_0^5\Big)$$

$$\left(-73 - 12\sqrt{37} - 2\,i\left(97 + 16\sqrt{37}\right)\omega_0 + 12\left(7+\sqrt{37}\right)\omega_0^2 + 32\,i\,\omega_0^3 + \omega_0^4 + 2\,i\,\omega_0^5\right)^2$$

$$\Big(-\sqrt{6+\sqrt{37}}\left(518 + 85\sqrt{37}\right) +$$

$$\left(-534 - 88\sqrt{37} - 222\,i\sqrt{6+\sqrt{37}} - 44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0 +$$

$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$
$$2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$
$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\omega_0^5-i\omega_0^6\Bigg)$$
$$\Bigg(37\left(6+\sqrt{37}\right)+16\sqrt{37\left(6+\sqrt{37}\right)}\omega_0+64\omega_0^2+$$
$$2\sqrt{37\left(6+\sqrt{37}\right)}\omega_0^3+20\omega_0^4+\omega_0^6\Bigg)$$
$$\Big(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-2\left(89+28\sqrt{37}\right)\omega_0^4-$$
$$104\left(-4+\sqrt{37}\right)\omega_0^6+129\omega_0^8+4\omega_0^{10}\Big)$$
$$\Bigg(37\left(174\,922+28\,757\sqrt{37}\right)+4\sqrt{6+\sqrt{37}}\left(276\,686+45\,487\sqrt{37}\right)\omega_0+$$
$$4\left(314\,823+51\,764\sqrt{37}\right)\omega_0^2+$$
$$56\sqrt{6+\sqrt{37}}\left(2664+455\sqrt{37}\right)\omega_0^3+$$
$$\left(296\,509+48\,182\sqrt{37}\right)\omega_0^4+$$
$$8\sqrt{6+\sqrt{37}}\left(4181+965\sqrt{37}\right)\omega_0^5+$$
$$8\left(13\,319+1993\sqrt{37}\right)\omega_0^6+$$
$$8\sqrt{6+\sqrt{37}}\left(296+107\sqrt{37}\right)\omega_0^7+$$
$$\left(7776+881\sqrt{37}\right)\omega_0^8+28\sqrt{37\left(6+\sqrt{37}\right)}\omega_0^9+$$
$$4\left(51+2\sqrt{37}\right)\omega_0^{10}+\omega_0^{12}\Bigg)\Bigg)\Bigg),$$

$$-\Bigg(\Bigg(64\left(6+\sqrt{37}\right)\omega_0^2\Bigg(67\,469\,796\Big(6\,393\,033\,890\,621\,681\,382\,129\,205+$$
$$1\,051\,008\,297\,427\,032\,716\,092\,967\sqrt{37}\Big)+$$
$$1\,215\,672\Big(749\,405\,998\,021\,908\,776\,724\,835\,393\sqrt{6+\sqrt{37}}+$$
$$123\,201\,587\,155\,362\,390\,535\,258\,283\sqrt{37\left(6+\sqrt{37}\right)}+222\,i$$
$$\left(5\,010\,808\,745\,561\,452\,494\,643\,321+823\,771\,883\,351\,130\,510\,068\,369\sqrt{37}\right)\Big)$$
$$\omega_0+151\,959\Big(101\,673\,886\,598\,698\,810\,503\,962\,107\,603+$$
$$16\,715\,083\,992\,225\,822\,327\,835\,790\,479\sqrt{37}+$$
$$19\,009\,595\,593\,091\,053\,976\,487\,635\,908\,i\sqrt{6+\sqrt{37}}+$$

$3\,125\,158\,264\,588\,552\,045\,734\,266\,892\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^2 +$

$2738\,\Big(4\,176\,778\,829\,199\,445\,276\,935\,283\,442\,665\,\sqrt{6+\sqrt{37}} +$

$686\,658\,209\,718\,844\,684\,075\,452\,411\,121\,\sqrt{37\left(6+\sqrt{37}\right)} +$

$148\,\mathbb{i}\,\big(128\,547\,620\,135\,979\,688\,454\,777\,650\,907 +$

$21\,133\,098\,570\,868\,815\,812\,767\,473\,116\,\sqrt{37}\big)\Big)\,\omega_0^3 +$

$1369\,\Big(64\,819\,352\,958\,880\,157\,187\,936\,873\,085\,985 +$

$10\,656\,235\,984\,228\,412\,383\,347\,313\,050\,161\,\sqrt{37} +$

$31\,631\,804\,224\,821\,966\,945\,293\,540\,881\,635\,\mathbb{i}\,\sqrt{6+\sqrt{37}} +$

$5\,200\,236\,581\,202\,053\,602\,004\,200\,418\,775\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^4 +$

$111\,\Big(304\,108\,922\,999\,556\,199\,144\,047\,994\,338\,939\,\sqrt{6+\sqrt{37}} +$

$49\,995\,198\,971\,649\,916\,843\,826\,750\,958\,967\,\sqrt{37\left(6+\sqrt{37}\right)} +$

$74\,\mathbb{i}\,\big(49\,233\,697\,857\,493\,575\,631\,075\,142\,210\,639 +$

$8\,093\,970\,069\,069\,882\,692\,643\,003\,603\,275\,\sqrt{37}\big)\Big)\,\omega_0^5 +$

$37\,\Big(951\,445\,600\,516\,514\,392\,537\,248\,305\,498\,541 +$

$156\,416\,693\,201\,052\,492\,005\,876\,712\,432\,045\,\sqrt{37} +$

$5\,780\,202\,746\,606\,641\,074\,110\,600\,677\,590\,667\,\mathbb{i}\,\sqrt{6+\sqrt{37}} +$

$950\,259\,477\,961\,776\,520\,005\,596\,433\,976\,939\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^6 +$

$\Big(-51\,970\,384\,524\,520\,640\,672\,650\,333\,524\,329\,841\,\sqrt{6+\sqrt{37}} -$

$8\,543\,878\,585\,701\,205\,708\,251\,026\,837\,594\,889\,\sqrt{37\left(6+\sqrt{37}\right)} +$

$37\,\mathbb{i}\,\big(29\,870\,371\,490\,873\,349\,591\,373\,384\,560\,027\,221 +$

$4\,910\,658\,823\,534\,394\,074\,765\,138\,372\,471\,013\,\sqrt{37}\big)\Big)\,\omega_0^7 +$

$\Big(-589\,363\,057\,396\,352\,665\,416\,389\,850\,375\,188\,586 -$

$96\,890\,689\,791\,149\,541\,068\,786\,118\,300\,978\,462\,\sqrt{37} +$

$$383\,045\,856\,909\,796\,593\,839\,991\,622\,105\,264\,849\,i\,\sqrt{6+\sqrt{37}}\,+$$

$$62\,972\,350\,967\,483\,354\,790\,588\,836\,409\,645\,913\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^8\,+$$

$$\Big(1\,097\,755\,450\,219\,718\,270\,056\,593\,297\,926\,505\,023\,i\,+$$

$$180\,469\,884\,325\,058\,239\,168\,078\,363\,099\,729\,811\,i\,\sqrt{37}\,-$$

$$293\,516\,254\,176\,344\,636\,563\,189\,712\,209\,329\,367\,\sqrt{6+\sqrt{37}}\,-$$

$$48\,253\,774\,944\,252\,905\,234\,861\,084\,546\,366\,335\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^9\,+$$

$$2\,i\,\Big(558\,136\,577\,556\,499\,807\,531\,487\,331\,934\,878\,787\,i\,+$$

$$91\,757\,088\,128\,366\,575\,777\,918\,660\,975\,445\,563\,i\,\sqrt{37}\,+$$

$$83\,268\,476\,694\,788\,844\,054\,531\,894\,128\,391\,473\,\sqrt{6+\sqrt{37}}\,+$$

$$13\,689\,253\,243\,083\,030\,435\,143\,709\,915\,342\,923\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{10}\,+$$

$$\Big(-257\,357\,867\,356\,918\,955\,495\,238\,347\,567\,821\,852\,\sqrt{6+\sqrt{37}}\,-$$

$$42\,309\,372\,768\,543\,944\,375\,620\,966\,889\,938\,648\,\sqrt{37\left(6+\sqrt{37}\right)}\,+$$

$$3\,i\,\Big(40\,581\,870\,352\,740\,587\,771\,919\,208\,787\,720\,363\,+$$

$$6\,671\,618\,388\,947\,855\,212\,562\,869\,023\,350\,687\,\sqrt{37}\Big)\Bigg)\,\omega_0^{11}\,+$$

$$\Big(-561\,384\,821\,479\,190\,649\,428\,203\,806\,109\,707\,340\,-$$

$$92\,291\,096\,139\,777\,731\,457\,808\,419\,797\,122\,000\,\sqrt{37}\,-$$

$$41\,359\,084\,436\,756\,044\,548\,225\,187\,035\,001\,466\,i\,\sqrt{6+\sqrt{37}}\,-$$

$$6\,799\,391\,597\,279\,457\,184\,513\,405\,787\,610\,530\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{12}\,-$$

$$4\,\Big(51\,215\,149\,173\,213\,835\,654\,704\,931\,174\,159\,024\,i\,+$$

$$8\,419\,718\,658\,769\,161\,217\,743\,930\,792\,736\,495\,i\,\sqrt{37}\,+$$

$$18\,107\,094\,937\,941\,679\,625\,265\,807\,106\,244\,024\,\sqrt{6+\sqrt{37}}\,+$$

$$2\,976\,788\,070\,839\,573\,415\,430\,054\,517\,639\,540\,\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{13}\,+$$

$$\Big(-41\,019\,222\,901\,387\,606\,085\,349\,623\,303\,581\,434\,i\,\sqrt{6+\sqrt{37}}\,-$$

$$6\,743\,518\,705\,040\,842\,032\,406\,086\,758\,125\,086\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\,-$$

$$130\left(502\,388\,882\,353\,955\,466\,380\,476\,361\,079\,893+82\,592\,223\,492\,460\,561\,201\,090\,050\,527\,297\sqrt{37}\right)\right)\omega_0^{14}+$$

$$4\left(125\,606\,315\,386\,326\,534\,458\,216\,970\,864\,898\sqrt{6+\sqrt{37}}+20\,649\,551\,048\,669\,399\,850\,221\,005\,468\,669\sqrt{37\left(6+\sqrt{37}\right)}-\mathrm{i}\left(16\,585\,403\,500\,196\,451\,911\,608\,144\,454\,822\,773+2\,726\,623\,539\,483\,024\,843\,260\,534\,071\,199\,167\sqrt{37}\right)\right)$$

$$\omega_0^{15}+2\left(6\,257\,389\,100\,813\,909\,288\,178\,649\,090\,181\,628+1\,028\,708\,431\,349\,386\,933\,852\,613\,952\,322\,386\sqrt{37}-2\,944\,892\,775\,030\,545\,803\,517\,660\,778\,187\,157\,\mathrm{i}\sqrt{6+\sqrt{37}}-484\,137\,389\,937\,885\,098\,016\,055\,139\,328\,401\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{16}+$$

$$4\left(450\,880\,478\,560\,140\,347\,544\,630\,899\,755\,200\sqrt{6+\sqrt{37}}+74\,124\,294\,071\,041\,932\,261\,864\,179\,274\,744\sqrt{37\left(6+\sqrt{37}\right)}-\mathrm{i}\left(368\,254\,716\,319\,377\,546\,857\,079\,724\,002\,098+60\,540\,702\,433\,327\,298\,970\,783\,772\,500\,323\sqrt{37}\right)\right)$$

$$\omega_0^{17}+\left(-74\,349\,458\,001\,200\,566\,369\,223\,918\,032\,453-12\,222\,975\,602\,099\,566\,824\,715\,486\,873\,561\sqrt{37}+593\,724\,147\,073\,078\,573\,754\,097\,572\,490\,524\,\mathrm{i}\sqrt{6+\sqrt{37}}+97\,607\,648\,517\,551\,128\,386\,281\,063\,409\,056\,\mathrm{i}\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^{18}+$$

$$\left(-526\,740\,717\,987\,098\,019\,501\,926\,918\,697\,124\sqrt{6+\sqrt{37}}-86\,595\,640\,609\,575\,711\,105\,158\,718\,632\,392\sqrt{37\left(6+\sqrt{37}\right)}+20\,\mathrm{i}\left(33\,408\,848\,816\,171\,622\,253\,138\,061\,487\,761+5\,492\,380\,912\,416\,399\,417\,999\,985\,802\,729\sqrt{37}\right)\right)$$

$$\omega_0^{19}+\left(-1\,269\,672\,384\,143\,103\,668\,721\,489\,800\,198\,441-208\,732\,854\,162\,704\,823\,299\,064\,980\,652\,509\sqrt{37}-\right.$$

$85\,017\,329\,482\,855\,813\,014\,790\,037\,580\,919\,i\sqrt{6+\sqrt{37}}\,-$

$13\,976\,762\,870\,387\,389\,889\,045\,801\,248\,459\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{20}\,-$

$\Bigg(164\,666\,967\,458\,487\,134\,163\,248\,733\,963\,419\sqrt{6+\sqrt{37}}\,+$

$27\,071\,082\,692\,819\,512\,518\,035\,475\,277\,463\sqrt{37\left(6+\sqrt{37}\right)}\,+$

$12\,i\,\Big(37\,580\,999\,581\,045\,799\,423\,755\,095\,625\,559\,+$

$6\,178\,278\,273\,046\,986\,048\,629\,249\,970\,214\sqrt{37}\Big)\Bigg)$

$\omega_0^{21}+\Bigg(-165\,571\,540\,853\,413\,444\,687\,251\,415\,159\,297\,-$

$27\,219\,793\,642\,888\,757\,888\,601\,509\,675\,785\sqrt{37}\,-$

$93\,193\,145\,630\,735\,888\,623\,705\,914\,471\,967\,i\sqrt{6+\sqrt{37}}\,-$

$15\,320\,858\,765\,493\,667\,660\,174\,440\,133\,675\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{22}\,-$

$3\,\Bigg(50\,234\,123\,563\,250\,314\,298\,376\,853\,445\,953\,i\,+$

$8\,258\,439\,041\,970\,538\,629\,486\,558\,840\,809\,i\sqrt{37}\,+$

$44\,401\,207\,896\,928\,898\,362\,059\,835\,587\sqrt{6+\sqrt{37}}\,+$

$7\,299\,513\,613\,385\,134\,664\,208\,902\,143\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{23}\,+$

$\Bigg(18\,918\,221\,210\,172\,862\,140\,023\,646\,399\,322\,+$

$3\,110\,136\,408\,571\,182\,995\,825\,564\,916\,466\sqrt{37}\,-$

$13\,494\,391\,026\,076\,661\,358\,814\,169\,565\,725\,i\sqrt{6+\sqrt{37}}\,-$

$2\,218\,464\,218\,989\,571\,558\,897\,176\,813\,517\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{24}\,+$

$\Bigg(2\,467\,093\,778\,997\,942\,890\,927\,442\,208\,729\sqrt{6+\sqrt{37}}\,+$

$405\,587\,718\,855\,059\,959\,344\,075\,801\,073\sqrt{37\left(6+\sqrt{37}\right)}\,-$

$i\,\Big(12\,438\,734\,781\,784\,815\,513\,152\,966\,580\,089\,+$

$2\,044\,915\,401\,485\,294\,632\,519\,021\,842\,449\sqrt{37}\Big)\Bigg)$

$\omega_0^{25}+2\,\Bigg(1\,729\,379\,733\,167\,199\,363\,622\,028\,886\,362\,+$

$284\,308\,276\,799\,035\,801\,501\,650\,441\,638\sqrt{37}\,-$

$$437\,581\,070\,771\,936\,054\,258\,636\,059\,675\,i\sqrt{6+\sqrt{37}}\;-$$

$$71\,937\,884\,898\,773\,283\,905\,050\,897\,999\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{26}+$$

$$3\Bigg(50\,805\,350\,844\,918\,999\,221\,290\,749\,398\sqrt{6+\sqrt{37}}\;+$$

$$8\,352\,348\,228\,986\,717\,615\,181\,486\,638\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$i\Big(121\,617\,994\,495\,777\,504\,431\,369\,319\,457\;+$$

$$19\,993\,875\,133\,200\,394\,793\,009\,205\,349\sqrt{37}\Big)\Bigg)\,\omega_0^{27}-$$

$$2\,i\Bigg(50\,463\,828\,154\,369\,611\,311\,817\,092\,412\sqrt{6+\sqrt{37}}\;+$$

$$8\,296\,202\,243\,344\,404\,896\,861\,764\,212\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$i\Big(112\,252\,558\,873\,789\,148\,243\,312\,296\,081\;+$$

$$18\,454\,207\,001\,782\,484\,260\,264\,031\,305\sqrt{37}\Big)\Bigg)\,\omega_0^{28}-$$

$$2\Bigg(143\,768\,673\,370\,596\,897\,729\,394\,262\,021\,i+23\,635\,424\,308\,797\,386\,202\,977\,755\,901$$

$$i\sqrt{37}+7\,060\,843\,987\,763\,483\,544\,238\,682\,832\sqrt{6+\sqrt{37}}\;+$$

$$1\,160\,795\,602\,025\,466\,400\,614\,983\,184\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{29}+$$

$$4\Bigg(248\,028\,016\,400\,601\,807\,874\,056\,254+40\,775\,554\,371\,701\,350\,495\,311\,518\sqrt{37}\;-$$

$$8\,202\,050\,172\,529\,825\,587\,122\,942\,855\,i\sqrt{6+\sqrt{37}}\;-$$

$$1\,348\,408\,742\,353\,976\,826\,609\,966\,833\,i\sqrt{37\left(6+\sqrt{37}\right)}\Bigg)\,\omega_0^{30}+$$

$$8\Bigg(286\,835\,914\,376\,086\,885\,593\,958\,321\sqrt{6+\sqrt{37}}\;+$$

$$47\,155\,532\,821\,675\,208\,649\,440\,872\sqrt{37\left(6+\sqrt{37}\right)}\;-$$

$$i\Big(6\,707\,912\,668\,021\,574\,521\,886\,351\,543\;+$$

$$1\,102\,774\,049\,793\,801\,005\,412\,390\,485\sqrt{37}\Big)\Bigg)\,\omega_0^{31}+$$

$$4\Bigg(2\,997\,001\,187\,539\,364\,833\,128\,324\,321+492\,703\,960\,452\,327\,794\,510\,294\,361\sqrt{37}\;-$$

$$1\,302\,358\,211\,209\,876\,280\,358\,701\,071\,i\sqrt{6+\sqrt{37}}\;-$$

$$214\,106\,373\,549\,634\,143\,443\,185\,723\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{32}+$$

$$8\Big(164\,333\,021\,495\,926\,119\,241\,941\,460\sqrt{6+\sqrt{37}}+$$

$$27\,016\,182\,287\,480\,421\,107\,080\,930\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$i\Big(448\,075\,795\,284\,723\,883\,095\,127\,201+$$

$$73\,663\,206\,851\,622\,620\,055\,643\,882\sqrt{37}\Big)\Big)\,\omega_0^{33}+$$

$$\Big(1\,426\,903\,162\,221\,451\,788\,132\,301\,083+234\,581\,434\,999\,592\,681\,426\,084\,631\sqrt{37}-$$

$$288\,260\,438\,130\,063\,634\,308\,962\,876\,i\sqrt{6+\sqrt{37}}-$$

$$47\,389\,726\,608\,028\,029\,184\,942\,460\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{34}+$$

$$2\Big(70\,184\,787\,443\,507\,058\,780\,671\,829\sqrt{6+\sqrt{37}}+$$

$$5\Big(2\,307\,661\,743\,533\,396\,141\,357\,253\sqrt{37\left(6+\sqrt{37}\right)}-$$

$$4\,i\Big(5\,682\,798\,529\,953\,741\,904\,260\,295+$$

$$934\,246\,310\,604\,234\,980\,100\,202\sqrt{37}\Big)\Big)\Big)\,\omega_0^{35}+$$

$$\Big(105\,978\,609\,283\,172\,784\,224\,481\,607+17\,422\,775\,805\,782\,953\,104\,186\,991\sqrt{37}-$$

$$9\,490\,522\,063\,409\,416\,406\,208\,659\,i\sqrt{6+\sqrt{37}}-$$

$$1\,560\,230\,760\,367\,641\,493\,586\,279\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{36}+$$

$$\Big(7\,016\,178\,817\,978\,996\,829\,718\,863\sqrt{6+\sqrt{37}}+$$

$$1\,153\,453\,006\,774\,384\,638\,365\,723\sqrt{37\left(6+\sqrt{37}\right)}-2\,i$$

$$\Big(16\,310\,048\,443\,683\,377\,165\,948\,327+2\,681\,355\,562\,073\,414\,217\,533\,255\sqrt{37}\Big)\Big)$$

$$\omega_0^{37}+\Big(12\,534\,771\,083\,988\,159\,265\,104\,655+2\,060\,703\,621\,281\,618\,526\,141\,631\sqrt{37}-$$

$$3\,160\,407\,510\,567\,057\,010\,336\,503\,i\sqrt{6+\sqrt{37}}-$$

$$519\,567\,110\,603\,220\,722\,759\,151\,i\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{38}+$$

$$\Big(998\,777\,930\,424\,451\,957\,477\,869 \sqrt{6+\sqrt{37}} +$$

$$164\,198\,053\,897\,913\,354\,715\,309 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$\mathbb{i}\left(3\,602\,222\,019\,176\,644\,134\,954\,829 + 592\,201\,788\,640\,019\,068\,925\,101 \sqrt{37}\right)\Big)$$

$$\omega_0^{39} + \Big(1\,719\,269\,803\,273\,187\,412\,165\,994 + 282\,646\,221\,969\,385\,836\,575\,398 \sqrt{37} -$$

$$382\,237\,010\,218\,382\,592\,321\,337\,\mathbb{i} \sqrt{6+\sqrt{37}} -$$

$$62\,839\,320\,551\,162\,269\,367\,377\,\mathbb{i} \sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{40} +$$

$$3\Big(56\,319\,996\,207\,407\,453\,764\,313 \sqrt{6+\sqrt{37}} +$$

$$9\,258\,944\,606\,549\,840\,936\,177 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$\mathbb{i}\left(76\,027\,624\,499\,929\,313\,354\,689 + 12\,498\,869\,124\,767\,999\,069\,493 \sqrt{37}\right)\Big)\,\omega_0^{41} +$$

$$2\Big(90\,895\,614\,965\,983\,356\,889\,781 + 14\,943\,148\,676\,503\,613\,150\,981 \sqrt{37} -$$

$$4\,711\,266\,655\,455\,299\,710\,933\,\mathbb{i} \sqrt{6+\sqrt{37}} -$$

$$774\,538\,787\,343\,380\,079\,427\,\mathbb{i} \sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{42} +$$

$$\Big(14\,869\,923\,716\,096\,736\,740\,568 \sqrt{6+\sqrt{37}} +$$

$$2\,444\,597\,692\,538\,190\,016\,044 \sqrt{37\left(6+\sqrt{37}\right)} -$$

$$\mathbb{i}\left(1\,587\,527\,509\,617\,887\,194\,517 + 260\,983\,112\,716\,313\,516\,561 \sqrt{37}\right)\Big)\,\omega_0^{43} +$$

$$2\Big(6\,897\,026\,909\,186\,934\,865\,292 + 1\,133\,864\,408\,015\,972\,497\,946 \sqrt{37} +$$

$$602\,958\,645\,729\,206\,481\,671\,\mathbb{i} \sqrt{6+\sqrt{37}} +$$

$$99\,122\,158\,368\,568\,817\,075\,\mathbb{i} \sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{44} +$$

$$4\Big(188\,725\,472\,868\,804\,389\,698 \sqrt{6+\sqrt{37}} +$$

$$31\,026\,213\,358\,083\,565\,822 \sqrt{37\left(6+\sqrt{37}\right)} +$$

$$15\,\mathbb{i}\,\left(20\,085\,922\,652\,496\,489\,641+3\,302\,133\,754\,925\,335\,104\,\sqrt{37}\right)\Big)\,\omega_0^{45}+$$

$$2\,\Big(365\,566\,764\,617\,958\,609\,847+60\,098\,811\,838\,053\,750\,571\,\sqrt{37}+$$

$$57\,043\,299\,918\,533\,701\,187\,\mathbb{i}\,\sqrt{6+\sqrt{37}}+$$

$$9\,377\,364\,784\,549\,046\,153\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{46}+$$

$$4\,\Big(28\,931\,029\,006\,543\,033\,553\,\mathbb{i}+4\,756\,292\,065\,156\,492\,031\,\mathbb{i}\,\sqrt{37}+$$

$$5\,980\,611\,644\,644\,193\,068\,\sqrt{6+\sqrt{37}}+$$

$$983\,201\,467\,233\,849\,523\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{47}+$$

$$\Big(26\,416\,139\,669\,314\,554\,272+4\,342\,788\,090\,322\,166\,156\,\sqrt{37}+$$

$$4\,633\,160\,822\,729\,282\,182\,\mathbb{i}\,\sqrt{6+\sqrt{37}}+$$

$$761\,613\,287\,534\,633\,518\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{48}+$$

$$\Big(475\,525\,361\,455\,244\,456\,\sqrt{6+\sqrt{37}}+78\,174\,400\,376\,514\,728\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$12\,\mathbb{i}\,\left(441\,874\,004\,161\,048\,288+72\,645\,031\,723\,128\,273\,\sqrt{37}\right)\Big)\,\omega_0^{49}+$$

$$\Big(613\,979\,682\,597\,494\,613+100\,938\,016\,291\,691\,529\,\sqrt{37}+109\,058\,444\,263\,627\,460$$

$$\mathbb{i}\,\sqrt{6+\sqrt{37}}+17\,926\,653\,749\,927\,624\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{50}+$$

$$12\,\Big(10\,541\,971\,257\,964\,801\,\mathbb{i}+1\,733\,105\,226\,582\,885\,\mathbb{i}\,\sqrt{37}+$$

$$403\,492\,886\,532\,610\,\sqrt{6+\sqrt{37}}+66\,328\,259\,676\,915\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{51}+$$

$$\Big(7\,049\,771\,902\,160\,585+1\,159\,007\,036\,491\,397\,\sqrt{37}+1\,422\,388\,508\,717\,887$$

$$\mathbb{i}\,\sqrt{6+\sqrt{37}}+233\,900\,039\,556\,499\,\mathbb{i}\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{52}+$$

$$\Big(-30\,984\,649\,122\,953\,\sqrt{6+\sqrt{37}}-5\,092\,956\,804\,125\,\sqrt{37\left(6+\sqrt{37}\right)}+$$

$$108\,\mathbb{i}\,\left(8\,842\,755\,838\,090+1\,453\,093\,327\,587\,\sqrt{37}\right)\Big)\,\omega_0^{53}+$$

$$\Big(-59\,034\,570\,811\,631 - 9\,705\,254\,824\,343\,\sqrt{37} - 5\,580\,311\,385\,569\,i\,\sqrt{6+\sqrt{37}} -$$

$$906\,954\,538\,493\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{54} -$$

$$3\,\Big(9\,475\,299\,948\,371\,i + 1\,559\,977\,617\,787\,i\,\sqrt{37} + 801\,897\,102\,161\,\sqrt{6+\sqrt{37}} +$$

$$131\,726\,457\,613\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{55} +$$

$$\Big(-3\,938\,144\,708\,826 - 647\,606\,422\,842\,\sqrt{37} - 793\,746\,038\,939\,i\,\sqrt{6+\sqrt{37}} -$$

$$129\,981\,951\,323\,i\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{56} -$$

$$\Big(878\,258\,078\,531\,i + 144\,703\,879\,523\,i\,\sqrt{37} + 58\,911\,466\,637\,\sqrt{6+\sqrt{37}} +$$

$$9\,666\,368\,453\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{57} -$$

$$2\,i\,\Big(7\,859\,889\,761\,\sqrt{6+\sqrt{37}} + 1\,285\,554\,617\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$4\,i\,\left(9\,057\,123\,431 + 1\,490\,666\,864\,\sqrt{37}\right)\Big)\,\omega_0^{58} -$$

$$\Big(9\,378\,801\,865\,i + 1\,550\,561\,629\,i\,\sqrt{37} + 868\,729\,438\,\sqrt{6+\sqrt{37}} +$$

$$142\,270\,654\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{59} -$$

$$2\,i\,\Big(38\,842\,526\,\sqrt{6+\sqrt{37}} + 6\,293\,630\,\sqrt{37\left(6+\sqrt{37}\right)} -$$

$$3\,i\,\left(108\,953\,303 + 17\,991\,123\,\sqrt{37}\right)\Big)\,\omega_0^{60} -$$

$$2\,\Big(9\,652\,537\,i + 1\,651\,861\,i\,\sqrt{37} + 2\,185\,220\,\sqrt{6+\sqrt{37}} + 355\,076\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)$$

$$\omega_0^{61} +$$

$$4\,i\,\Big(331\,991\,i + 56\,699\,i\,\sqrt{37} + 143\,708\,\sqrt{6+\sqrt{37}} + 23\,900\,\sqrt{37\left(6+\sqrt{37}\right)}\Big)\,\omega_0^{62} +$$

$$8\,\Big(4144\,\sqrt{6+\sqrt{37}} + 688\,\sqrt{37\left(6+\sqrt{37}\right)} + 27\,i\,\left(1331 + 215\,\sqrt{37}\right)\Big)\,\omega_0^{63} +$$

$$16\,\left(1331 + 215\,\sqrt{37}\right)\,\omega_0^{64}\Big)\Big)\Big/$$

$$\Bigg(\left(7+\sqrt{37}\right)^{5/2}\left(-2\,i+\omega_0\right)^2\left(2\,i+\omega_0\right)^2$$

$$\left(\sqrt{37\left(6+\sqrt{37}\right)}+8\,\omega_0+2\,i\,\omega_0^2+\omega_0^3\right)$$

$$\left(73+12\sqrt{37}+4\left(7+\sqrt{37}\right)\omega_0^2-\omega_0^4\right)^4$$

$$\Big(-73-12\sqrt{37}+2\,i\left(97+16\sqrt{37}\right)\omega_0+$$
$$12\left(7+\sqrt{37}\right)\omega_0^2-32\,i\,\omega_0^3+\omega_0^4-2\,i\,\omega_0^5\Big)$$

$$\left(-73-12\sqrt{37}-2\,i\left(97+16\sqrt{37}\right)\omega_0+12\left(7+\sqrt{37}\right)\omega_0^2+32\,i\,\omega_0^3+\omega_0^4+2\,i\,\omega_0^5\right)^2$$

$$\Bigg(-\sqrt{6+\sqrt{37}}\left(518+85\sqrt{37}\right)+$$
$$\left(-534-88\sqrt{37}-222\,i\sqrt{6+\sqrt{37}}-44\,i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0+$$
$$\left(10\sqrt{37\left(6+\sqrt{37}\right)}-i\left(583+86\sqrt{37}\right)\right)\omega_0^2+$$
$$2\left(102+9\sqrt{37}-i\sqrt{37\left(6+\sqrt{37}\right)}\right)\omega_0^3+$$
$$\left(\sqrt{37\left(6+\sqrt{37}\right)}-2\,i\left(15+2\sqrt{37}\right)\right)\omega_0^4+12\,\omega_0^5-i\,\omega_0^6\Bigg)$$

$$\Bigg(37\left(6+\sqrt{37}\right)+16\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0+64\,\omega_0^2+$$
$$2\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^3+20\,\omega_0^4+\omega_0^6\Bigg)$$

$$\Big(10\,657+1752\sqrt{37}+\left(52\,604+8648\sqrt{37}\right)\omega_0^2-2\left(89+28\sqrt{37}\right)\omega_0^4-$$
$$104\left(-4+\sqrt{37}\right)\omega_0^6+129\,\omega_0^8+4\,\omega_0^{10}\Big)$$

$$\Bigg(37\left(174\,922+28\,757\sqrt{37}\right)+4\sqrt{6+\sqrt{37}}\left(276\,686+45\,487\sqrt{37}\right)\omega_0+$$
$$4\left(314\,823+51\,764\sqrt{37}\right)\omega_0^2+$$
$$56\sqrt{6+\sqrt{37}}\left(2664+455\sqrt{37}\right)\omega_0^3+$$
$$\left(296\,509+48\,182\sqrt{37}\right)\omega_0^4+$$
$$8\sqrt{6+\sqrt{37}}\left(4181+965\sqrt{37}\right)\omega_0^5+$$
$$8\left(13\,319+1993\sqrt{37}\right)\omega_0^6+$$
$$8\sqrt{6+\sqrt{37}}\left(296+107\sqrt{37}\right)\omega_0^7+$$
$$\left(7776+881\sqrt{37}\right)\omega_0^8+28\sqrt{37\left(6+\sqrt{37}\right)}\,\omega_0^9+$$
$$4\left(51+2\sqrt{37}\right)\omega_0^{10}+\omega_0^{12}\Bigg)\Bigg)\Bigg)\Bigg\}$$

(* Componentes do número complexo G32 *)

```
G32 = pb.(6 bb[h11, h21] + bb[h20b, h30] + 3 bb[h20, h21b] + 3 bb[q, h22] + 2 bb[qb, h31])

ReG32 = ComplexExpand[Re[G32], ω0 ∈ Reals]
        expande funções … parte real     números r

(* Valor de ω0 *)

ω0 = N[√((-n) * (b + m)), 50]
     valor numérico

3.4760268310670761692396132474965648249422443494620


(* Primeiro coeficiente de Lyapunov em função dos parâmteros *)

l1 = Simplify[l1]
     simplifica

0


(* Segundo coeficiente de Lyapunov em função dos parâmteros *)

l2 = 1/12 * ReG32

0.00025276356395132604656979953754520687758126909 40
```

Modelo Tridimensional
Condição de Transversalidade

(* Parâmtros da bifurcação*)

$$a = \frac{b^2 + 4\,b\,m + m^2 - (b + m)\,n}{2\,m}$$

$$h = \sqrt{\frac{(b + m)\,n\,(-b - m + n)}{m}}$$

$$\sqrt{\frac{(b + m)\,n\,(-b - m + n)}{m}}$$

$$\omega_0 = \sqrt{-(b + m)\,n}$$

$$\sqrt{(-b - m)\,n}$$

(* Matriz Jacobiana J(e*)= A *)

$$A = \left\{\left\{n, -2\sqrt{(-a + b)\,n}, 0\right\}, \{0, -m, m\}, \left\{\sqrt{(-a + b)\,n}, b, -b\right\}\right\}$$

$$\left\{\left\{n, -2\sqrt{(-a + b)\,n}, 0\right\}, \{0, -m, m\}, \left\{\sqrt{(-a + b)\,n}, b, -b\right\}\right\}$$

```
(* Autoveotes q e pb*)
```

$$q = \left\{-\frac{\sqrt{2}\, h\, m}{(m + i\,\omega_0)\,(-n + i\,\omega_0)},\ \frac{m}{m + i\,\omega_0},\ 1\right\}$$

$$\left\{-\frac{\sqrt{2}\ m\sqrt{\frac{(b+m)\ n\ (-b-m+n)}{m}}}{\left(m + i\sqrt{(-b-m)\ n}\right)\left(-n + i\sqrt{(-b-m)\ n}\right)},\ \frac{m}{m + i\sqrt{(-b-m)\ n}},\ 1\right\}$$

$$pb = \left\{\frac{h\,(-i\,m + \omega_0)\,(i\,n + \omega_0)}{\sqrt{2}\,\left(h^2 m - (n - i\,\omega_0)^2\,(b + m + 2\,i\,\omega_0)\right)},\ \frac{b + i\,\omega_0}{m\left(1 + \frac{i\,\omega_0}{m+i\,\omega_0} + \frac{b+\frac{h^2 m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}\right)},\ \frac{1}{1 + \frac{i\,\omega_0}{m+i\,\omega_0} + \frac{b+\frac{h^2 m}{(i\,n+\omega_0)^2}}{m+i\,\omega_0}}\right\}$$

$$\left\{\left(\sqrt{\frac{(b+m)\ n\ (-b-m+n)}{m}}\left(-i\,m + \sqrt{(-b-m)\ n}\right)\left(i\,n + \sqrt{(-b-m)\ n}\right)\right)\Big/\left(\sqrt{2}\left((b+m)\ n\ (-b-m+n) - \left(n - i\sqrt{(-b-m)\ n}\right)^2\left(b + m + 2\,i\sqrt{(-b-m)\ n}\right)\right)\right),\right.$$

$$\left.\frac{b + i\sqrt{(-b-m)\ n}}{m\left(1 + \frac{i\sqrt{(-b-m)\ n}}{m+i\sqrt{(-b-m)\ n}} + \frac{b+\frac{(b+m)\ n\ (-b-m+n)}{\left(i\,n+\sqrt{(-b-m)\ n}\right)^2}}{m+i\sqrt{(-b-m)\ n}}\right)},\ \frac{1}{1 + \frac{i\sqrt{(-b-m)\ n}}{m+i\sqrt{(-b-m)\ n}} + \frac{b+\frac{(b+m)\ n\ (-b-m+n)}{\left(i\,n+\sqrt{(-b-m)\ n}\right)^2}}{m+i\sqrt{(-b-m)\ n}}}\right\}$$

```
(* J'(e*)*)
```

$$A1 = \text{Simplify}\left[D[A, a]\ /.\ a \to \frac{b^2 + 4\,b\,m + m^2 - (b+m)\,n}{2\,m},\ m > 0\ \&\&\ n < 0\ \&\&\ b > 0\right]$$

simplifica derivada

$$\left\{\left\{0,\ \frac{\sqrt{2}\ n}{\sqrt{-\frac{(b+m)\ (b+m-n)\ n}{m}}},\ 0\right\},\ \{0,\ 0,\ 0\},\ \left\{\frac{\sqrt{-\frac{m\,n}{(b+m)\ (b+m-n)}}}{\sqrt{2}},\ 0,\ 0\right\}\right\}$$

```
A2 = Simplify[A1.q, m > 0 && n < 0 && b > 0]
```

simplifica

$$\left\{\frac{\sqrt{2}\ m\,n}{\sqrt{-\frac{(b+m)\ (b+m-n)\ n}{m}}\left(m + i\sqrt{-(b+m)\ n}\right)},\ 0,\ -\frac{i\,m\,n}{-i\,b\,n + (m-n)\sqrt{-(b+m)\ n}}\right\}$$

```
A3 = FullSimplify[pb.A2]
```

simplifica completamente

$$\frac{i\,m\left(i\,n + \sqrt{-(b+m)\ n}\right)}{b^2 + m^2 + 2\,b\,(m-n) - 2\,m\,n + i\,n\sqrt{-(b+m)\ n}}$$

```
(*Condição de Transversalidade*)
```

```
T = FullSimplify[
     simplifica completamente
   ComplexExpand[Re[A3]], m > 0 && n < 0 && b > 0]
   expande funções ⋯ parte real
```

$$-\frac{m\,n}{(b+m)^2-3\,(b+m)\,n+n^2}$$

Apêndice II

Apêndice II

Sistemas acoplados

Lema 3.2.4

```
I0 := {{1, 0}, {0, 0}}
I2 := {{1, 0}, {0, 1}}
I4 := {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
A := {{-1 - c1 - c2 , -b * (1 + c1 + c2)}, {1, 1}}


J := {{-1 - c2, -b * (1 + c1 + c2), -c1, 0},
  {1, 1, 0, 0}, {-c2, 0, -1 - c1, -b * (1 + c1 + c2)}, {0, 0, 1, 1}}

alfa1 := FullSimplify[(c1^2 - c2^2) * (x^2 - x * Tr[A] + Det[A]) / (c1 * (x^2) -
         simplifica completamente                  traço      determinante
      (x * ((c2^2) + (c1 * (c2 + Tr[A])))) + (c1 * Det[A]) + (c2 * (c1 + c2)))]
                                   traço           determinante

alfa2 := FullSimplify[(c2^2 - c1^2) * (x^2 - x * Tr[A] + Det[A]) / (c2 * (x^2) -
         simplifica completamente                  traço      determinante
      (x * ((c1^2) + (c2 * (c1 + Tr[A])))) + (c2 * Det[A]) + (c1 * (c1 + c2)))]
                                   traço           determinante

A1 := A + alfa1 * I0
A2 := A + alfa2 * I0

P1 := FullSimplify[Inverse[x * I2 - A1]]
      simplifica comple··· matriz inversa
P1
```

$$\left\{\left\{\left((-1+x)\left(-c2^2(-1+x)+c1^2(-1+b+x)+c1\left(-1+b+b\,c2+x^2\right)\right)\right)\Big/\left(c1\left(-1+b(1+c1+c2)+x^2\right)\left(b(1+c1+c2)+(-1+x)(1+c1+c2+x)\right)\right),\ -\frac{b(1+c1+c2)\left(\frac{c1-c2}{-1+b(1+c1+c2)+x^2}+\frac{c2}{b(1+c1+c2)+(-1+x)(1+c1+c2+x)}\right)}{c1}\right\},\left\{\frac{\frac{c1-c2}{-1+b(1+c1+c2)+x^2}+\frac{c2}{b(1+c1+c2)+(-1+x)(1+c1+c2+x)}}{c1},\ \frac{\frac{(c1-c2)(1+x)}{-1+b(1+c1+c2)+x^2}+\frac{c2(1+c1+c2+x)}{b(1+c1+c2)+(-1+x)(1+c1+c2+x)}}{c1}\right\}\right\}$$

```
P2 := FullSimplify[Inverse[x * I2 - A2]]
      simplifica comple··· matriz inversa
P2
```

$$\left\{\left\{\left((-1+x)\left(b\,c1\,c2-c1^2(-1+x)+c2\left(b(1+c2)+(-1+x)(1+c2+x)\right)\right)\right)\Big/\left(c2\left(-1+b(1+c1+c2)+x^2\right)\left(b(1+c1+c2)+(-1+x)(1+c1+c2+x)\right)\right),\right.$$

$$-\frac{b\,(1+c1+c2)\left(\frac{-c1+c2}{-1+b\,(1+c1+c2)+x^2}+\frac{c1}{b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x)}\right)}{c2}\Big\},$$

$$\Big\{\frac{\frac{-c1+c2}{-1+b\,(1+c1+c2)+x^2}+\frac{c1}{b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x)}}{c2},$$

$$\left(b\,(1+c1+c2)\,(c2+c1\,(c1+c2)+c2\,x)+c2\,(1+c1+c2+x)\,(-1+x^2)\right)\Big/ \left(c2\,(-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\right)\Big\}\Big\}$$

```
D1 := FullSimplify[c1^2 * P1.{{u1}, {u2}}]
```

simplifica completamente

```
D1
```

$$\Big\{\Big\{-\Big(\big(c1\,(u1+b\,(1+c1+c2)\,u2-u1\,x)\,(-c2^2\,(-1+x)+c1^2\,(-1+b+x)+c1\,(-1+b+b\,c2+x^2))\big)\Big/ \big((-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\big)\Big)\Big\},$$

$$\Big\{c1\left(\frac{(c1-c2)\,(u1+u2+u2\,x)}{-1+b\,(1+c1+c2)+x^2}+\frac{c2\,(u1+u2\,(1+c1+c2+x))}{b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x)}\right)\Big\}\Big\}$$

```
D2 := FullSimplify[c2^2 * P2.{{u1}, {u2}}]
```

simplifica completamente

```
D2
```

$$\Big\{\Big\{-\Big(\big(c2\,(u1+b\,(1+c1+c2)\,u2-u1\,x)\,(b\,c1\,c2-c1^2\,(-1+x)+c2\,(b\,(1+c2)+(-1+x)\,(1+c2+x)))\big)\Big/ \big((-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\big)\Big)\Big\},$$

$$\Big\{\big(c2\,(b\,c1^3\,u2+c1^2\,(u1+b\,(1+2\,c2)\,u2-u1\,x)+c2\,(u1+u2+u2\,x)\,(b\,(1+c2)+(-1+x)\,(1+c2+x))+c1\,c2\,(u2\,(-1+x^2)+b\,(u1+u2\,(2+c2+x))))\big)\Big/ \big((-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\big)\Big\}\Big\}$$

```
U := Simplify[(x * I4 - J) . {{-((c1 (u1 + b (1 + c1 + c2) u2 - u1 x)
```

simplifica

$$(-c2^2\,(-1+x)+c1^2\,(-1+b+x)+c1\,(-1+b+b\,c2+x^2)))\Big/ \big((-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\big)\Big)\Big\},$$

$$\Big\{c1\left(\frac{(c1-c2)\,(u1+u2+u2\,x)}{-1+b\,(1+c1+c2)+x^2}+\frac{c2\,(u1+u2\,(1+c1+c2+x))}{b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x)}\right)\Big\},$$

$$\Big\{-\Big(\big(c2\,(u1+b\,(1+c1+c2)\,u2-u1\,x)\,(b\,c1\,c2-c1^2\,(-1+x)+c2\,(b\,(1+c2)+(-1+x)\,(1+c2+x)))\big)\Big/ \big((-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\big)\Big)\Big\},$$

$$\Big\{\big(c2\,(b\,c1^3\,u2+c1^2\,(u1+b\,(1+2\,c2)\,u2-u1\,x)+c2\,(u1+u2+u2\,x)\,(b\,(1+c2)+(-1+x)\,(1+c2+x))+c1\,c2\,(u2\,(-1+x^2)+b\,(u1+u2\,(2+c2+x))))\big)\Big/ \big((-1+b\,(1+c1+c2)+x^2)\,(b\,(1+c1+c2)+(-1+x)\,(1+c1+c2+x))\big)\Big\}\Big\}\Big]$$

```
U
```

$$\{\{c1^2\,u1\},\{c1^2\,u2\},\{c2^2\,u1\},\{c2^2\,u2\}\}$$

Sistemas Acoplados

Proposição 3.2.2

```
(* Componentes do campo de vetores com c1 e c2 *)

f1[x_, y_, z_, w_] := -a (x + b y + 2 x y + y^2 + x y^2) + c1 (x - z)
f2[x_, y_, z_, w_] := x + y + 2 x y + y^2 + x y^2
f3[x_, y_, z_, w_] := -a (z + b w + 2 z w + w^2 + z w^2) + c2 (z - x)
f4[x_, y_, z_, w_] := z + w + 2 z w + w^2 + z w^2

(* Valor de bifurcação *)
a = c1 + c2 + 1;


(* Definindo b em função de ω0 e de a *)
b = (ω1^2 + 1)/a;

(* Autovetor q e seu conjugado qb*)
q = {-1 + i ω1, 1};
qb = {-1 - i ω1, 1};

(* Autovetor p e seu conjugado pb *)
p = {1 + I ω1, ω1^2 + 1}
        unidade imaginária
pb = {1 - I ω1, ω1^2 + 1}
         unidade imaginária


(* Funções multilineares simétricas B, C *)
                                      constante


(* Função B *)
bb[{x1_, x2_}, {y1_, y2_}] := {-2 a (x1 y2 + x2 y1 + x2 y2), 2 (x1 y2 + x2 y1 + x2 y2)}
(* Função C *)
          constante
cc[{x1_, x2_}, {y1_, y2_}, {u1_, u2_}] :=
 {-2 a (x2 y1 u2 + x2 y2 u1 + x1 y2 u2), 2 (x2 y1 u2 + x2 y2 u1 + x1 y2 u2)}

(*Matrizes*)
A := {{-1 - c1 - c2 , -b * (1 + c1 + c2)}, {1, 1}}
I0 := {{1, 0}, {0, 0}}
I4 := {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
I2 := {{1, 0}, {0, 1}}

(*valores de alpha*)
alpha1 := (c1^2 - c2^2) * (x^2 - x * Tr[A] + Det[A]) /
                                     traço     determinante
```

```
                              traço      determinante
        (c1 * (x^2) - (x * ((c2^2) + (c1 * (c2 + Tr[A])))) + (c1 * Det[A]) + (c2 * (c1 + c2)))
                                                  traço              determinante
alpha2 := (c2^2 - c1^2) * (x^2 - x * Tr[A] + Det[A]) /
                                       traço      determinante
        (c2 * (x^2) - (x * ((c1^2) + (c2 * (c1 + Tr[A])))) + (c2 * Det[A]) + (c1 * (c1 + c2)))
                                                  traço              determinante
(*Calculando alpha para x = 0*)
x := 0
alfa10 = FullSimplify[alfa1]
         simplifica completamente
```

$$\frac{(c1 - c2)\ (c1 + c2)\ (c1 + c2 - \omega_1^2)}{c1^2 - c2^2 - c1\ \omega_1^2}$$

```
alfa20 = FullSimplify[alfa2]
         simplifica completamente
```

$$\frac{(c1 - c2)\ (c1 + c2)\ (c1 + c2 - \omega_1^2)}{c1^2 - c2^2 + c2\ \omega_1^2}$$

```
(*Calculando alpha para x = 2*I*ω1 *)
                             unidade imaginária
x := 2 * I * ω1
         unidade imaginária
alfa11 = FullSimplify[alfa1]
         simplifica completamente
```

$$\frac{(c1 - c2)\ (c1 + c2)\ (i\ (c1 + c2) + 2\ (c1 + c2)\ \omega_1 + 3\ i\ \omega_1^2)}{i\ (c1 - c2)\ (c1 + c2) + 2\ (c1 - c2)\ (c1 + c2)\ \omega_1 + 3\ i\ c1\ \omega_1^2}$$

```
alfa21 = FullSimplify[alfa2]
         simplifica completamente
```

$$\frac{(c1 - c2)\ (c1 + c2)\ (i\ (c1 + c2) + 2\ (c1 + c2)\ \omega_1 + 3\ i\ \omega_1^2)}{i\ (c1 - c2)\ (c1 + c2) + 2\ (c1 - c2)\ (c1 + c2)\ \omega_1 - 3\ i\ c2\ \omega_1^2}$$

```
(*Calculando Z1 e Z2*)

A10 := FullSimplify[ A + alfa10 * I0]
       simplifica completamente
A20 := FullSimplify[A + alfa20 * I0]
       simplifica completamente
D1 = FullSimplify[Inverse[A10].bb[q, qb]]
     simplifica comple··· matriz inversa
```

$$\left\{-2 + \frac{2\ c1 - \frac{2\ c2^2}{c1}}{\omega_1^2},\ -\frac{2\ (c1 - c2)\ (c1 + c2)}{c1\ \omega_1^2}\right\}$$

```
D11 = FullSimplify[bb[q, D1]]
      simplifica completamente
```

$$\left\{\frac{1}{c1\ \omega_1^2} 4\ (1 + c1 + c2)\ (-c1^2 + c2^2 + i\ (c1 - c2)\ (c1 + c2)\ \omega_1 + c1\ \omega_1^2),\right.$$
$$\left.\frac{4\ (c1^2 - c2^2 - i\ (c1 - c2)\ (c1 + c2)\ \omega_1 - c1\ \omega_1^2)}{c1\ \omega_1^2}\right\}$$

```
D2 = FullSimplify[Inverse[A20].bb[q, qb]]
```
simplifica comple··· matriz inversa

$$\left\{-2+\frac{-\frac{2\,c1^2}{c2}+2\,c2}{\omega_1^2},\ \frac{2\,(c1-c2)\,(c1+c2)}{c2\,\omega_1^2}\right\}$$

```
D21 = FullSimplify[bb[q, D2]]
```
simplifica completamente

$$\left\{\frac{1}{c2\,\omega_1^2}4\,(1+c1+c2)\,\left(c1^2-c2^2-i\,(c1-c2)\,(c1+c2)\,\omega_1+c2\,\omega_1^2\right),\ \frac{4\,\left(-c1^2+c2^2+i\,(c1-c2)\,(c1+c2)\,\omega_1-c2\,\omega_1^2\right)}{c2\,\omega_1^2}\right\}$$

```
A11 := A + alfa11 * I0
A21 := A + alfa21 * I0
F1 = FullSimplify[Inverse[2 * I * ω1 * I2 - A11]. bb[q, q]]
```
simplifica comple··· matriz inversa unidade imaginária

$$\left\{-\left(\left(2\,(i+2\,\omega_1)\,\left(-c1-c2+2\,i\,(1+c1+c2)\,\omega_1+\omega_1^2\right)\left(c1^2-c2^2-2\,i\,(c1-c2)\,(c1+c2)\,\omega_1+3\,c1\,\omega_1^2\right)\right)\Big/\left(3\,c1\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)\right)\right),\ \left(2\,(i+2\,\omega_1)\,\left(-(c1-c2)\,(c1+c2)^2+\omega_1\,\left(2\,i\,(c1-c2)\,(c1+c2)\,(1+c1+c2)+\omega_1\,\left(c1^2-c2^2+6\,i\,c1\,\omega_1\right)\right)\right)\right)\Big/\left(3\,c1\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)\right)\right\}$$

```
F11 = FullSimplify[bb[qb, F1]]
```
simplifica completamente

$$\left\{\left(4\,(1+c1+c2)\,(i+2\,\omega_1)\left(-(c1-c2)\,(c1+c2)^2+\omega_1\,\left(i\,(c1-c2)\,(c1+c2)\,(2+3\,c1+3\,c2)+\omega_1\,\left((c1+c2)\left(2\,c1^2-c2\,(3+2\,c2)\right)+i\,\omega_1\,\left(5\,c1^2+c2^2+6\,c1\,(1+c2)+3\,i\,c1\,\omega_1\right)\right)\right)\right)\right)\Big/\left(3\,c1\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)\right),\ \left(4\,(i+2\,\omega_1)\left((c1-c2)\,(c1+c2)^2-i\,(c1-c2)\,(c1+c2)\,(2+3\,c1+3\,c2)\,\omega_1-(c1+c2)\,\left(2\,c1^2-c2\,(3+2\,c2)\right)\omega_1^2-i\,\left(5\,c1^2+c2^2+6\,c1\,(1+c2)\right)\omega_1^3+3\,c1\,\omega_1^4\right)\right)\Big/\left(3\,c1\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)\right)\right\}$$

```
F2  = FullSimplify[Inverse[2 * I * ω1 * I2 - A21]. bb[q, q]]
```
simplifica comple··· matriz inversa unidade imaginária

$$\left\{-\left(\left(2\,(i+2\,\omega_1)\,\left(-c1-c2+2\,i\,(1+c1+c2)\,\omega_1+\omega_1^2\right)\left(-c1^2+c2^2+2\,i\,(c1-c2)\,(c1+c2)\,\omega_1+3\,c2\,\omega_1^2\right)\right)\Big/\left(3\,c2\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)\right)\right),\ \left(2\,(i+2\,\omega_1)\,\left((c1-c2)\,(c1+c2)^2+\omega_1\,\left(-2\,i\,(c1-c2)\,(c1+c2)\,(1+c1+c2)+\omega_1\,\left(-c1^2+c2^2+6\,i\,c2\,\omega_1\right)\right)\right)\right)\Big/\left(3\,c2\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)\right)\right\}$$

```
F21 = FullSimplify[bb[qb, F2]]
```
simplifica completamente

$$\left\{\frac{1}{3\,c2\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)}4\,(1+c1+c2)\,(i+2\,\omega_1)\left((c1-c2)\,(c1+c2)^2+\omega_1\,\left(-i\,(c1-c2)\,(c1+c2)\,(2+3\,c1+3\,c2)+\omega_1\,\left(-(c1+c2)\,\left(c1\,(3+2\,c1)-2\,c2^2\right)+i\,\omega_1\,\left(6\,c2+(c1+c2)\,(c1+5\,c2)+3\,i\,c2\,\omega_1\right)\right)\right)\right),\right.$$

$$\frac{1}{3\,c2\,\omega_1^2\,\left(i\,(c1+c2)+2\,(c1+c2)\,\omega_1+3\,i\,\omega_1^2\right)}\left(-4\,i\,(c1-c2)\,(c1+c2)^2+\right.$$

$4\,\omega_1\,\left(-\left(c1-c2\right)\,\left(c1+c2\right)\,\left(2+5\,c1+5\,c2\right)+\omega_1\,\left(i\,\left(c1+c2\right)\,\left(c1\,\left(7+8\,c1\right)-4\,c2\,\left(1+2\,c2\right)\right)+\omega_1\,\left(6\,c2+\left(c1+c2\right)\,\left(c1\,\left(7+4\,c1\right)+\left(5-4\,c2\right)\,c2\right)-i\,\left(9\,c2+2\,\left(c1+c2\right)\,\left(c1+5\,c2\right)\right)\,\omega_1+6\,c2\,\omega_1^2\right)\right)\right)\}$

```
(*Valor de Z1*)
Z1 = Simplify[ cc[q, q, qb] - 2 D11 + F11]
```
simplifica

$$\left\{\frac{1}{3\,c1\,\omega_1^2\,\left(i\,\left(c1+c2\right)+2\,\left(c1+c2\right)\,\omega_1+3\,i\,\omega_1^2\right)}2\,\left(1+c1+c2\right)\,\left(i+\omega_1\right)\left(10\,\left(c1-c2\right)\,\left(c1+c2\right)^2-4\,i\,\left(4\,c1^3+c2^2-4\,c1\,c2^2-4\,c2^3+c1^2\,\left(-1+4\,c2\right)\right)\,\omega_1+\left(8\,c1^3-2\,c2^2\,\left(23+4\,c2\right)-c1\,c2\,\left(9+8\,c2\right)+c1^2\,\left(37+8\,c2\right)\right)\,\omega_1^2+2\,i\,\left(7\,c1^2+2\,c2^2+c1\,\left(6+9\,c2\right)\right)\,\omega_1^3-3\,c1\,\omega_1^4\right),\right.$$

$$\left.\frac{1}{3\,c1\,\omega_1^2\,\left(c1+c2-2\,i\,\left(c1+c2\right)\,\omega_1+3\,\omega_1^2\right)}2\,i\,\left(i+\omega_1\right)\left(10\,\left(c1-c2\right)\,\left(c1+c2\right)^2-4\,i\,\left(4\,c1^3+c2^2-4\,c1\,c2^2-4\,c2^3+c1^2\,\left(-1+4\,c2\right)\right)\,\omega_1+\left(8\,c1^3-2\,c2^2\,\left(23+4\,c2\right)-c1\,c2\,\left(9+8\,c2\right)+c1^2\,\left(37+8\,c2\right)\right)\,\omega_1^2+2\,i\,\left(7\,c1^2+2\,c2^2+c1\,\left(6+9\,c2\right)\right)\,\omega_1^3-3\,c1\,\omega_1^4\right)\right\}$$

```
(*Valor de Z2*)
Z2 = Simplify[ cc[q, q, qb] - 2 D21 + F21]
```
simplifica

$$\left\{-\left(\left(2\,\left(1+c1+c2\right)\,\left(i+\omega_1\right)\left(10\,\left(c1-c2\right)\,\left(c1+c2\right)^2-4\,i\,\left(4\,c1^3+c2^2-4\,c1\,c2^2-4\,c2^3+c1^2\,\left(-1+4\,c2\right)\right)\,\omega_1+\left(8\,c1^3+c1\,\left(9-8\,c2\right)\,c2-c2^2\,\left(37+8\,c2\right)+c1^2\,\left(46+8\,c2\right)\right)\,\omega_1^2-2\,i\,\left(2\,c1^2+9\,c1\,c2+c2\,\left(6+7\,c2\right)\right)\,\omega_1^3+3\,c2\,\omega_1^4\right)\right)\Big/\left(3\,c2\,\omega_1^2\,\left(i\,\left(c1+c2\right)+2\,\left(c1+c2\right)\,\omega_1+3\,i\,\omega_1^2\right)\right)\right),\right.$$

$$\left.-\frac{1}{3\,c2\,\omega_1^2\,\left(c1+c2-2\,i\,\left(c1+c2\right)\,\omega_1+3\,\omega_1^2\right)}2\,i\,\left(i+\omega_1\right)\left(10\,\left(c1-c2\right)\,\left(c1+c2\right)^2-4\,i\,\left(4\,c1^3+c2^2-4\,c1\,c2^2-4\,c2^3+c1^2\,\left(-1+4\,c2\right)\right)\,\omega_1+\left(8\,c1^3+c1\,\left(9-8\,c2\right)\,c2-c2^2\,\left(37+8\,c2\right)+c1^2\,\left(46+8\,c2\right)\right)\,\omega_1^2-2\,i\,\left(2\,c1^2+9\,c1\,c2+c2\,\left(6+7\,c2\right)\right)\,\omega_1^3+3\,c2\,\omega_1^4\right)\right\}$$

```
(*Calculando L1*)
s1 = Simplify[ComplexExpand[(c1^3) * (pb.Z1)], ω1 > 0]
```
simplifica　　expande funções complexas

$$-\frac{1}{3\,\omega_1^2\,\left(c1+c2-2\,i\,\left(c1+c2\right)\,\omega_1+3\,\omega_1^2\right)}2\,c1^2\,\left(i+\omega_1\right)^2\left(10\,\left(c1-c2\right)\,\left(c1+c2\right)^3-2\,i\,\left(c1+c2\right)^2\,\left(3\,c1+8\,c1^2-c2\,\left(3+8\,c2\right)\right)\,\omega_1+\left(8\,c1^4+4\,c1^2\,\left(1+3\,c2\right)+c1^3\,\left(21+16\,c2\right)-c1\,c2^2\,\left(39+16\,c2\right)-2\,c2^2\,\left(2+15\,c2+4\,c2^2\right)\right)\omega_1^2+i\,\left(6\,c1^3+2\,c2^2\,\left(23+6\,c2\right)+3\,c1\,c2\,\left(7+10\,c2\right)+c1^2\,\left(-25+24\,c2\right)\right)\,\omega_1^3+\left(11\,c1^2+4\,c2^2+3\,c1\,\left(4+5\,c2\right)\right)\,\omega_1^4+3\,i\,c1\,\omega_1^5\right)$$

```
s2 = Simplify[ComplexExpand[(c2^3) * (pb.Z2)], ω1 > 0]
```
simplifica　expande funções complexas

$$-\frac{1}{3\,\omega_1^2\left(c1+c2-2\,i\,(c1+c2)\,\omega_1+3\,\omega_1^2\right)}\,2\,c2^2\,(i+\omega_1)^2\Big(-10\,(c1-c2)\,(c1+c2)^3+2\,i\,(c1+c2)^2\,\left(3\,c1+8\,c1^2-c2\,(3+8\,c2)\right)\,\omega_1+\left(-8\,c1^4+4\,c1\,c2^2\,(3+4\,c2)-2\,c1^3\,(15+8\,c2)-c1^2\,(4+39\,c2)+c2^2\,(4+21\,c2+8\,c2^2)\right)\omega_1^2+i\left(12\,c1^3+c2^2\,(-25+6\,c2)+3\,c1\,c2\,(7+8\,c2)+c1^2\,(46+30\,c2)\right)\omega_1^3+\left(4\,c1^2+15\,c1\,c2+c2\,(12+11\,c2)\right)\omega_1^4+3\,i\,c2\,\omega_1^5\Big)$$

```
pi = FullSimplify[pb.q]
```
simplifica completamente

$$2\,\omega_1\,(i+\omega_1)$$

```
(*Valor de L1*)
L1 =
 FullSimplify[ComplexExpand[(1/(2*(c1+c2))) * Re[(1/pi) * (s1+s2)]], ω1 > 0]
```
simplifica comple···　expande funções complexas　parte real

$$-\Big(\Big(2\,(c1-c2)^2\,(c1+c2)^3\,(1+c1+c2)+(c1+c2)^2\,\big(21\,c1^3+8\,c1^4-3\,c1\,c2\,(11+8\,c2)+c2^2\,(1+c2)\,(13+8\,c2)+c1^2\,(13-8\,c2\,(3+2\,c2))\big)\,\omega_1^2+2\,(c1+c2)\,\big(2\,c1^4+c1^3\,(14-6\,c2)+2\,c2^2\,(1+c2)\,(6+c2)-c1\,c2\,(37+c2\,(37+6\,c2))+c1^2\,(12-c2\,(37+16\,c2))\big)\,\omega_1^4+\big(9\,c1^3+9\,c2^2\,(1+c2)+c1\,c2\,(-9+4\,c2)+c1^2\,(9+4\,c2)\big)\,\omega_1^6\Big)\Big/\Big(2\,\omega_1^2\,\big((c1+c2)^2+2\,(c1+c2)\,(3+2\,c1+2\,c2)\,\omega_1^2+9\,\omega_1^4\big)\Big)\Big)$$

```
(* Fazendo c1 = c2 = c *)
c1 := c
c2 := c
Z = Simplify[cc[q, q, qb] - 2 bb[q, Inverse[A].bb[q, qb]] +
   bb[qb, Inverse[(2 * I * ω1 * I2 - A)].bb[q, q]]]
```
simplifica　matriz inversa　matriz inversa　unidade imaginária

$$\left\{\left(2\,(i+\omega_1)\,\left(6\,i\,c\,(1+2\,c)+4\,(1+5\,c+6\,c^2)\,\omega_1+i\,(1+2\,c)\,\omega_1^2\right)\right)\Big/\left(2\,c-4\,i\,c\,\omega_1+3\,\omega_1^2\right),\ \frac{2\,(i+\omega_1)\,\left(6\,c-4\,i\,(1+3\,c)\,\omega_1+\omega_1^2\right)}{2\,i\,c+4\,c\,\omega_1+3\,i\,\omega_1^2}\right\}$$

```
(*Primeiro Coeficiente de Lyapunov para c1 = c2 = c*)
L1c = FullSimplify[ComplexExpand[(c^2/2) * Re[(1/pi) * (pb.Z)]], ω1 > 0]
```
simplifica comple···　expande funções complexas　parte real

$$\frac{c^2\,\left(4\,c^2\,(7+6\,c)+4\,c\,(13+46\,c+24\,c^2)\,\omega_1^2-(9+26\,c)\,\omega_1^4\right)}{8\,c^2+8\,c\,(3+4\,c)\,\omega_1^2+18\,\omega_1^4}$$

```
(*Teste*)
FullSimplify[L1 - L1c]
```
simplifica completamente

0

Sistemas Acoplados

Lema 3.2.5

```
A := {{-1 - 2 c , -b * (1 + 2 c)}, {1, 1}}

I0 := {{1, 0}, {0, 0}}

I4 := {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

I2 := {{1, 0}, {0, 1}}

A0 := A + (2 * c * I0)

J := {{-1 - c, -b * (1 + 2 c), -c, 0},
  {1, 1, 0, 0}, {-c, 0, -1 - c, -b * (1 + 2 c)}, {0, 0, 1, 1}}

f1 := Simplify[Inverse[x * I2 - A0]]
      simplifica  matriz inversa

f1
```

$$\left\{\left\{\frac{-1+x}{-1+b+2\,b\,c+x^2},-\frac{b\,(1+2\,c)}{-1+b+2\,b\,c+x^2}\right\},\left\{\frac{1}{-1+b+2\,b\,c+x^2},\frac{1+x}{-1+b+2\,b\,c+x^2}\right\}\right\}$$

```
f2 := f1.{{u1}, {u2}}

f2
```

$$\left\{\left\{-\frac{b\,(1+2\,c)\,u2}{-1+b+2\,b\,c+x^2}+\frac{u1\,(-1+x)}{-1+b+2\,b\,c+x^2}\right\},\left\{\frac{u1}{-1+b+2\,b\,c+x^2}+\frac{u2\,(1+x)}{-1+b+2\,b\,c+x^2}\right\}\right\}$$

```
f3 := Simplify[-Inverse[x * I2 - A0]]
      simplifica  matriz inversa

f3
```

$$\left\{\left\{\frac{1-x}{-1+b+2\,b\,c+x^2},\frac{b\,(1+2\,c)}{-1+b+2\,b\,c+x^2}\right\},\left\{-\frac{1}{-1+b+2\,b\,c+x^2},-\frac{1+x}{-1+b+2\,b\,c+x^2}\right\}\right\}$$

```
f4 := f3.{{u1}, {u2}}

f4
```

$$\left\{\left\{\frac{b\,(1+2\,c)\,u2}{-1+b+2\,b\,c+x^2}+\frac{u1\,(1-x)}{-1+b+2\,b\,c+x^2}\right\},\left\{-\frac{u1}{-1+b+2\,b\,c+x^2}-\frac{u2\,(1+x)}{-1+b+2\,b\,c+x^2}\right\}\right\}$$

```
U := FullSimplify[(x * I4 - J).
     simplifica completamente
```

$$\left\{\left\{-\frac{b\,(1+2\,c)\,u2}{-1+b+2\,b\,c+x^2}+\frac{u1\,(-1+x)}{-1+b+2\,b\,c+x^2}\right\},\left\{\frac{u1}{-1+b+2\,b\,c+x^2}+\frac{u2\,(1+x)}{-1+b+2\,b\,c+x^2}\right\},$$
$$\left\{\frac{b\,(1+2\,c)\,u2}{-1+b+2\,b\,c+x^2}+\frac{u1\,(1-x)}{-1+b+2\,b\,c+x^2}\right\},\left\{-\frac{u1}{-1+b+2\,b\,c+x^2}-\frac{u2\,(1+x)}{-1+b+2\,b\,c+x^2}\right\}\right\}]$$

```
U
{{u1}, {u2}, {-u1}, {-u2}}
```

Proposição 3.2.3

```
(* Componentes do campo de vetores com c = c1 = c2 *)

f1[x_, y_, z_, w_] := -a (x + b y + 2 x y + y^2 + x y^2) + c (x - z)

f2[x_, y_, z_, w_] := x + y + 2 x y + y^2 + x y^2

f3[x_, y_, z_, w_] := -a (z + b w + 2 z w + w^2 + z w^2) + c (z - x)

f4[x_, y_, z_, w_] := z + w + 2 z w + w^2 + z w^2



(* Ponto de equilíbrio *)

P0 = {0, 0, 0, 0};



(* Parte linear do campo de vetores *)

Df[{x_, y_, z_, w_}] :=
 {{Derivative[1, 0, 0, 0][f1][x, y, z, w], Derivative[0, 1, 0, 0][f1][x, y, z, w],
   derivação                               derivação
   Derivative[0, 0, 1, 0][f1][x, y, z, w], Derivative[0, 0, 0, 1][f1][x, y, z, w]},
   derivação                               derivação
  {Derivative[1, 0, 0, 0][f2][x, y, z, w], Derivative[0, 1, 0, 0][f2][x, y, z, w],
   derivação                               derivação
   Derivative[0, 0, 1, 0][f2][x, y, z, w], Derivative[0, 0, 0, 1][f2][x, y, z, w]},
   derivação                               derivação
  {Derivative[1, 0, 0, 0][f3][x, y, z, w], Derivative[0, 1, 0, 0][f3][x, y, z, w],
   derivação                               derivação
   Derivative[0, 0, 1, 0][f3][x, y, z, w], Derivative[0, 0, 0, 1][f3][x, y, z, w]},
   derivação                               derivação
  {Derivative[1, 0, 0, 0][f4][x, y, z, w], Derivative[0, 1, 0, 0][f4][x, y, z, w],
   derivação                               derivação
   Derivative[0, 0, 1, 0][f4][x, y, z, w], Derivative[0, 0, 0, 1][f4][x, y, z, w]}}
   derivação                               derivação



(* Valor de bifurcação *)
```

```
a = 2 c + 1;

(* Definindo b em função de ω0 e de a , onde ω0 = ω1 *)

b = (ω0^2 + 1)/a;



(* Autovetor q *)

q = {-1 + i ω0, 1, 1 - i ω0, -1};



(* Autovetor qb *)

qb = {-1 - i ω0, 1, 1 + i ω0, -1};


(* Autovetor pb normalizado*)

pb = Simplify[(ω0 - i)/(4 ω0 (ω0^2 + 1)) {1 - i ω0, ω0^2 + 1, -1 + i ω0, -ω0^2 - 1}]
     simplifica
```

$\left\{-\frac{i}{4\,\omega_0}, \frac{1}{4} - \frac{i}{4\,\omega_0}, \frac{i}{4\,\omega_0}, -\frac{1}{4} + \frac{i}{4\,\omega_0}\right\}$

```
(* Verificação da Normalização *)

FullSimplify[pb.q]
simplifica completamente

1

(* Funções multilineares simétricas B, C, D e E *)
                                        c··· der··· númer


                        (* Função B *)

bb[{x1_, x2_, x3_, x4_}, {y1_, y2_, y3_, y4_}] := {-2 a (x1 y2 + x2 y1 + x2 y2),
   2 (x1 y2 + x2 y1 + x2 y2), -2 a (x3 y4 + x4 y3 + x4 y4), 2 (x3 y4 + x4 y3 + x4 y4)}

                        (* Função C *)
                                    consta

cc[{x1_, x2_, x3_, x4_}, {y1_, y2_, y3_, y4_}, {u1_, u2_, u3_, u4_}] :=
 {-2 a (x2 y1 u2 + x2 y2 u1 + x1 y2 u2), 2 (x2 y1 u2 + x2 y2 u1 + x1 y2 u2),
  -2 a (x4 y3 u4 + x4 y4 u3 + x3 y4 u4), 2 (x4 y3 u4 + x4 y4 u3 + x3 y4 u4)}

                        (* Função D *)
                                    derivac
```

```
dd[{x1_, x2_, x3_, x4_}, {y1_, y2_, y3_, y4_},
  {u1_, u2_, u3_, u4_}, {v1_, v2_, v3_, v4_}] := {0, 0, 0, 0}

                                    (* Função E *)
                                              número

ee[{x1_, x2_, x3_, x4_}, {y1_, y2_, y3_, y4_}, {u1_, u2_, u3_, u4_},
  {v1_, v2_, v3_, v4_}, {w1_, w2_, w3_, w4_}] := {0, 0, 0, 0}
```

```
(* Parte linear do campo de vetores *)

A = Simplify[Df[P0]]
   simplifica
```

$$\left\{\left\{-1-c,\ -1-\omega_0^2,\ -c,\ 0\right\},\ \{1,\ 1,\ 0,\ 0\},\ \left\{-c,\ 0,\ -1-c,\ -1-\omega_0^2\right\},\ \{0,\ 0,\ 1,\ 1\}\right\}$$

```
(* Inversa da matriz A *)

AI = FullSimplify[Inverse[A]]
     simplifica comple··· matriz inversa
```

$$\left\{\left\{\frac{1}{2}\left(\frac{1}{\omega_0^2}+\frac{1}{-2\,c+\omega_0^2}\right),\ \frac{1}{2}\left(2+\frac{1}{\omega_0^2}+\frac{1+2\,c}{-2\,c+\omega_0^2}\right),\ \frac{c}{-2\,c\,\omega_0^2+\omega_0^4},\ \frac{c\left(1+\omega_0^2\right)}{-2\,c\,\omega_0^2+\omega_0^4}\right\},\right.$$

$$\left\{\frac{c-\omega_0^2}{-2\,c\,\omega_0^2+\omega_0^4},\ \frac{-c+(1+c)\,\omega_0^2}{2\,c\,\omega_0^2-\omega_0^4},\ \frac{c}{2\,c\,\omega_0^2-\omega_0^4},\ -\frac{c\left(1+\omega_0^2\right)}{-2\,c\,\omega_0^2+\omega_0^4}\right\},$$

$$\left\{\frac{c}{-2\,c\,\omega_0^2+\omega_0^4},\ \frac{c\left(1+\omega_0^2\right)}{-2\,c\,\omega_0^2+\omega_0^4},\ \frac{1}{2}\left(\frac{1}{\omega_0^2}+\frac{1}{-2\,c+\omega_0^2}\right),\ \frac{1}{2}\left(2+\frac{1}{\omega_0^2}+\frac{1+2\,c}{-2\,c+\omega_0^2}\right)\right\},$$

$$\left.\left\{\frac{c}{2\,c\,\omega_0^2-\omega_0^4},\ -\frac{c\left(1+\omega_0^2\right)}{-2\,c\,\omega_0^2+\omega_0^4},\ \frac{c-\omega_0^2}{-2\,c\,\omega_0^2+\omega_0^4},\ \frac{-c+(1+c)\,\omega_0^2}{2\,c\,\omega_0^2-\omega_0^4}\right\}\right\}$$

```
                              (* Matriz D2 = 2ⅈω0I *)
                                            unidad

D2 = 2 ⅈ ω0 IdentityMatrix[4]
            matriz identidade
```

$$\{\{2\,i\,\omega_0,\ 0,\ 0,\ 0\},\ \{0,\ 2\,i\,\omega_0,\ 0,\ 0\},\ \{0,\ 0,\ 2\,i\,\omega_0,\ 0\},\ \{0,\ 0,\ 0,\ 2\,i\,\omega_0\}\}$$

```
                              (* Matriz DA = 2ⅈω0I-A *)
                                            unidade ir

DA = D2 - A
```

$$\left\{\left\{1+c+2\,i\,\omega_0,\ 1+\omega_0^2,\ c,\ 0\right\},\ \{-1,\ -1+2\,i\,\omega_0,\ 0,\ 0\},\ \left\{c,\ 0,\ 1+c+2\,i\,\omega_0,\ 1+\omega_0^2\right\},\ \{0,\ 0,\ -1,\ -1+2\,i\,\omega_0\}\right\}$$

```
                              (* Inversa da matriz DA *)
```

```
DAI = FullSimplify[Inverse[DA]]
```
simplifica comple⋯ matriz inversa

$$\left\{\left\{\frac{(1-2\,\mathrm{i}\,\omega_0)\,(c-2\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)}{3\,\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ \frac{1}{6}\left(2+\frac{1}{\omega_0^2}+\frac{3-2\,c+4\,\mathrm{i}\,c\,\omega_0}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2}\right),\ \frac{c\,(\mathrm{i}+2\,\omega_0)^2}{3\,\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ -\frac{c\,(\mathrm{i}+2\,\omega_0)\,(1+\omega_0^2)}{3\,\omega_0^2\,(2\,\mathrm{i}\,c+4\,c\,\omega_0+3\,\mathrm{i}\,\omega_0^2)}\right\},\right.$$

$$\left\{-\frac{1}{6\,\omega_0^2}-\frac{1}{2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ -\frac{c+(3+7\,c+6\,\mathrm{i}\,\omega_0)\,\omega_0^2}{3\,\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ \frac{1}{6}\left(\frac{1}{\omega_0^2}-\frac{3}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2}\right),\ \frac{c+c\,\omega_0^2}{6\,c\,\omega_0^2-12\,\mathrm{i}\,c\,\omega_0^3+9\,\omega_0^4}\right\},$$

$$\left\{\frac{c\,(\mathrm{i}+2\,\omega_0)^2}{3\,\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ -\frac{c\,(\mathrm{i}+2\,\omega_0)\,(1+\omega_0^2)}{3\,\omega_0^2\,(2\,\mathrm{i}\,c+4\,c\,\omega_0+3\,\mathrm{i}\,\omega_0^2)},\ \frac{(1-2\,\mathrm{i}\,\omega_0)\,(c-2\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)}{3\,\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ \frac{1}{6}\left(2+\frac{1}{\omega_0^2}+\frac{3-2\,c+4\,\mathrm{i}\,c\,\omega_0}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2}\right)\right\},$$

$$\left.\left\{\frac{1}{6}\left(\frac{1}{\omega_0^2}-\frac{3}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2}\right),\ \frac{c+c\,\omega_0^2}{6\,c\,\omega_0^2-12\,\mathrm{i}\,c\,\omega_0^3+9\,\omega_0^4},\ -\frac{1}{6\,\omega_0^2}-\frac{1}{2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)},\ -\frac{c+(3+7\,c+6\,\mathrm{i}\,\omega_0)\,\omega_0^2}{3\,\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)}\right\}\right\}$$

```
(* Calculo do vetor complexo h20 *)

h20 = FullSimplify[DAI.bb[q, q]]
```
simplifica completamente

$$\left\{\frac{2\,\mathrm{i}\,(\mathrm{i}+2\,\omega_0)\,(-2\,c+\omega_0\,(2\,\mathrm{i}+4\,\mathrm{i}\,c+\omega_0))}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2},\ \frac{4\,\omega_0\,(\mathrm{i}+2\,\omega_0)}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2},\ \frac{2\,\mathrm{i}\,(\mathrm{i}+2\,\omega_0)\,(-2\,c+\omega_0\,(2\,\mathrm{i}+4\,\mathrm{i}\,c+\omega_0))}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2},\ \frac{4\,\omega_0\,(\mathrm{i}+2\,\omega_0)}{2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2}\right\}$$

```
(* Vetor complexo h20b *)

h20b = Simplify[ComplexExpand[Conjugate[h20]], ω0 ∈ Reals && c > 0]
```
simplifica expande funções ⋯ conjugado números reais

$$\left\{-\frac{2\,\mathrm{i}\,(-\mathrm{i}+2\,\omega_0)\,(-2\,c-2\,\mathrm{i}\,(1+2\,c)\,\omega_0+\omega_0^2)}{2\,c+4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2},\ \frac{4\,\omega_0\,(-\mathrm{i}+2\,\omega_0)}{2\,c+4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2},\ -\frac{2\,\mathrm{i}\,(-\mathrm{i}+2\,\omega_0)\,(-2\,c-2\,\mathrm{i}\,(1+2\,c)\,\omega_0+\omega_0^2)}{2\,c+4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2},\ \frac{4\,\omega_0\,(-\mathrm{i}+2\,\omega_0)}{2\,c+4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2}\right\}$$

```
(* Calculo do vetor complexo h11 *)

h11 = Simplify[-AI.bb[q, qb]]
```
simplifica

$$\{2,\ 0,\ 2,\ 0\}$$

```
(* Cálculo do número complexo G21 *)

G21 = FullSimplify[pb.(cc[q, q, qb] + 2 bb[q, h11] + bb[qb, h20])]
```
simplifica completamente

$$\frac{(2\,c-\mathrm{i}\,\omega_0)\,(\mathrm{i}+\omega_0)\,(6\,c-4\,\mathrm{i}\,(1+3\,c)\,\omega_0+\omega_0^2)}{\omega_0\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)}$$

```
(* Cálculo do número complexo G21b *)

G21b = Simplify[ComplexExpand[Conjugate[G21]], ω0 ∈ Reals && c > 0]
       simplifica  expande funções ⋯  conjugado             números reais
```

$((1 + i\,\omega_0)\,(-12\,i\,c^2 + 2\,c\,(7 + 12\,c)\,\omega_0 + 2\,i\,(2 + 5\,c)\,\omega_0^2 + \omega_0^3)) / (\omega_0\,(2\,c + 4\,i\,c\,\omega_0 + 3\,\omega_0^2))$

```
(* Cálculo da parte real do número complexo G21 *)

ReG21 = Simplify[ComplexExpand[Re[G21]], ω0 ∈ Reals && c > 0]
        simplifica  expande funções ⋯  parte real     números reais
```

$$\frac{4\,c^2\,(7 + 6\,c) + 4\,c\,(13 + 46\,c + 24\,c^2)\,\omega_0^2 - (9 + 26\,c)\,\omega_0^4}{4\,c^2 + 4\,c\,(3 + 4\,c)\,\omega_0^2 + 9\,\omega_0^4}$$

```
(* Cálculo de l1 *)

l1 = 1/2 Simplify[ReG21]
         simplifica
```

$$\frac{4\,c^2\,(7 + 6\,c) + 4\,c\,(13 + 46\,c + 24\,c^2)\,\omega_0^2 - (9 + 26\,c)\,\omega_0^4}{2\,(4\,c^2 + 4\,c\,(3 + 4\,c)\,\omega_0^2 + 9\,\omega_0^4)}$$

```
(* Primeiro coeficiente de Lyapunov simplificado *)

l1 = FullSimplify[l1]
     simplifica completamente
```

$$\frac{4\,c^2\,(7 + 6\,c) + 4\,c\,(13 + 46\,c + 24\,c^2)\,\omega_0^2 - (9 + 26\,c)\,\omega_0^4}{8\,c^2 + 8\,c\,(3 + 4\,c)\,\omega_0^2 + 18\,\omega_0^4}$$

```
(* Matriz D3 = 3iω0I *)
                   unidad

D3 = 3 i ω0 IdentityMatrix[4]
            matriz identidade
```

$\{\{3\,i\,\omega_0, 0, 0, 0\}, \{0, 3\,i\,\omega_0, 0, 0\}, \{0, 0, 3\,i\,\omega_0, 0\}, \{0, 0, 0, 3\,i\,\omega_0\}\}$

```
(* Matriz TA = 3iω0I-A *)
                   unidade ir

TA = D3 - A
```

$\{\{1 + c + 3\,i\,\omega_0, 1 + \omega_0^2, c, 0\}, \{-1, -1 + 3\,i\,\omega_0, 0, 0\},$
$\{c, 0, 1 + c + 3\,i\,\omega_0, 1 + \omega_0^2\}, \{0, 0, -1, -1 + 3\,i\,\omega_0\}\}$

```
(* Matriz inversa da matriz TA *)
```

```
TAI = Simplify[Inverse[TA]]
```
simplifica | matriz inversa

$$\left\{\left\{\frac{(1-3 i \omega_0)(c-3 i c \omega_0+8 \omega_0^2)}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, \frac{(1+\omega_0^2)(c-3 i c \omega_0+8 \omega_0^2)}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, \frac{i c (i+3 \omega_0)^2}{16 \omega_0^2 (i c+3 c \omega_0+4 i \omega_0^2)}, -\frac{c (i+3 \omega_0+i \omega_0^2+3 \omega_0^3)}{16 \omega_0^2 (i c+3 c \omega_0+4 i \omega_0^2)}\right\},\right.$$

$$\left\{-\frac{c-3 i c \omega_0+8 \omega_0^2}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, -\frac{c+(8+17 c) \omega_0^2+24 i \omega_0^3}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, \frac{c-3 i c \omega_0}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, \frac{c (1+\omega_0^2)}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}\right\},$$

$$\left\{\frac{i c (i+3 \omega_0)^2}{16 \omega_0^2 (i c+3 c \omega_0+4 i \omega_0^2)}, -\frac{c (i+3 \omega_0+i \omega_0^2+3 \omega_0^3)}{16 \omega_0^2 (i c+3 c \omega_0+4 i \omega_0^2)}, \frac{(1-3 i \omega_0)(c-3 i c \omega_0+8 \omega_0^2)}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, \frac{(1+\omega_0^2)(c-3 i c \omega_0+8 \omega_0^2)}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}\right\},$$

$$\left.\left\{\frac{c-3 i c \omega_0}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, \frac{c (1+\omega_0^2)}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, -\frac{c-3 i c \omega_0+8 \omega_0^2}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}, -\frac{c+(8+17 c) \omega_0^2+24 i \omega_0^3}{16 \omega_0^2 (c-3 i c \omega_0+4 \omega_0^2)}\right\}\right\}$$

```
(* Cálculo do vetor complexo h30 *)

h30 = FullSimplify[TAI.(3 bb[q, h20] + cc[q, q, q])]
```
simplifica completamente

$$\left\{\left(3 i (i+3 \omega_0)(-2 c+4 i (1+c) \omega_0+5 \omega_0^2)(-2 c+\omega_0 (3 i+6 i c+\omega_0))\right) / \left(4 \omega_0^2 (2 c-4 i c \omega_0+3 \omega_0^2)\right), -\frac{3 (2 c-3 i \omega_0)(i+3 \omega_0)(2 i c+(4+4 c-5 i \omega_0) \omega_0)}{4 \omega_0^2 (2 c-4 i c \omega_0+3 \omega_0^2)},\right.$$

$$\left.\left(3 (i+3 \omega_0)(2 i c+(4+4 c-5 i \omega_0) \omega_0)(-2 c+\omega_0 (3 i+6 i c+\omega_0))\right) / \left(4 \omega_0^2 (2 c-4 i c \omega_0+3 \omega_0^2)\right), \frac{3 (2 c-3 i \omega_0)(i+3 \omega_0)(2 i c+(4+4 c-5 i \omega_0) \omega_0)}{4 \omega_0^2 (2 c-4 i c \omega_0+3 \omega_0^2)}\right\}$$

```
(* Cálculo do vetor complexo h30b *)

h30b = FullSimplify[ComplexExpand[Conjugate[h30]], ω0 ∈ Reals && c > 0]
```
simplifica comple··· | expande funções ··· | conjugado | números reais

$$\left\{\left(3 (1+3 i \omega_0)(-2 i c+(3+6 c+i \omega_0) \omega_0)(-2 i c+(4+4 c+5 i \omega_0) \omega_0)\right) / \left(4 \omega_0^2 (2 c+4 i c \omega_0+3 \omega_0^2)\right), -\frac{3 (2 c+3 i \omega_0)(-i+3 \omega_0)(-2 i c+(4+4 c+5 i \omega_0) \omega_0)}{4 \omega_0^2 (2 c+4 i c \omega_0+3 \omega_0^2)},\right.$$

$$\left.\left(3 (-i+3 \omega_0)(-2 i c+(4+4 c+5 i \omega_0) \omega_0)(-2 c-3 i (1+2 c) \omega_0+\omega_0^2)\right) / \left(4 \omega_0^2 (2 c+4 i c \omega_0+3 \omega_0^2)\right), \frac{3 (2 c+3 i \omega_0)(-i+3 \omega_0)(-2 i c+(4+4 c+5 i \omega_0) \omega_0)}{4 \omega_0^2 (2 c+4 i c \omega_0+3 \omega_0^2)}\right\}$$

```
(* Matriz D1 = iω0I *)
```
unidad

```
D1 = i ω0 IdentityMatrix[4]
```
matriz identidade

$$\{\{i \omega_0, 0, 0, 0\}, \{0, i \omega_0, 0, 0\}, \{0, 0, i \omega_0, 0\}, \{0, 0, 0, i \omega_0\}\}$$

```
(* Matriz L = ⅈω0I-A *)
```
unidade ir

```
L = D1 - A
```
```
{{1 + c + ⅈ ω0, 1 + ω0^2, c, 0}, {-1, -1 + ⅈ ω0, 0, 0},
 {c, 0, 1 + c + ⅈ ω0, 1 + ω0^2}, {0, 0, -1, -1 + ⅈ ω0}}
```
```
q
```
```
{-1 + ⅈ ω0, 1, 1 - ⅈ ω0, -1}
```
```
pb
```
```
{-ⅈ/(4 ω0), 1/4 - ⅈ/(4 ω0), ⅈ/(4 ω0), -1/4 + ⅈ/(4 ω0)}
```
```
ⅈ ω0 IdentityMatrix[4] - A
```
matriz identidade

```
{{1 + c + ⅈ ω0, b (1 + 2 c), c, 0}, {-1, -1 + ⅈ ω0, 0, 0},
 {c, 0, 1 + c + ⅈ ω0, b (1 + 2 c)}, {0, 0, -1, -1 + ⅈ ω0}}
```
```
(* Matriz L21 = ((ⅈω0IdentityMatrix[4]-A  q
                           pb              0)) *)
```
```
L21 = {{a - c + ⅈ ω0, a b, c, 0, -1 + ⅈ ω0},
   {-1, -1 + ⅈ ω0, 0, 0, 1}, {c, 0, a - c + ⅈ ω0, a b, 1 - ⅈ ω0},
   {0, 0, -1, -1 + ⅈ ω0, -1}, {-ⅈ/(4 ω0), 1/4 - ⅈ/(4 ω0), ⅈ/(4 ω0), -1/4 + ⅈ/(4 ω0), 0}}
```
```
{{1 + c + ⅈ ω0, 1 + ω0^2, c, 0, -1 + ⅈ ω0},
 {-1, -1 + ⅈ ω0, 0, 0, 1}, {c, 0, 1 + c + ⅈ ω0, 1 + ω0^2, 1 - ⅈ ω0},
 {0, 0, -1, -1 + ⅈ ω0, -1}, {-ⅈ/(4 ω0), 1/4 - ⅈ/(4 ω0), ⅈ/(4 ω0), -1/4 + ⅈ/(4 ω0), 0}}
```
```
(* Inversa da matriz L21 *)
```
```
L21I = Simplify[Inverse[L21]];
```
simplifica   matriz inversa

```
(* Cálculo de R21 *)
```
```
{b11, b22, b33, b44} = Simplify[cc[q, q, qb] + bb[qb, h20] + 2 bb[q, h11] - G21 q];
```
simplifica

```
(* Cálculo de H21 *)
```
```
H21 = {b11, b22, b33, b44, 0};
```
```
(* Cálculo de h21 *)
```

```
{r21, r22, r23, r24, S} = Simplify[L21I.H21];
                          simplifica
```

(* Cálculo do vetor complexo h21 *)

```
h21 = FullSimplify[{r21, r22, r23, r24}]
      simplifica completamente
```

$$\left\{\frac{(-2\,i\,c+\omega_0)\,(6\,i\,c+(4+12\,c+i\,\omega_0)\,\omega_0)\,(1+\omega_0^2)}{2\,\omega_0^2\,(2\,c-4\,i\,c\,\omega_0+3\,\omega_0^2)},\right.$$
$$\frac{(2\,c+i\,\omega_0)\,(i+\omega_0)\,(6\,i\,c+(4+12\,c+i\,\omega_0)\,\omega_0)}{2\,\omega_0^2\,(2\,c-4\,i\,c\,\omega_0+3\,\omega_0^2)},$$
$$\frac{i\,(2\,c+i\,\omega_0)\,(6\,i\,c+(4+12\,c+i\,\omega_0)\,\omega_0)\,(1+\omega_0^2)}{2\,\omega_0^2\,(2\,c-4\,i\,c\,\omega_0+3\,\omega_0^2)},$$
$$\left.\frac{1}{6}\left(1+\frac{18\,c}{\omega_0^2}-\frac{3\,i\,(1+6\,c)}{\omega_0}+\frac{4\,(3-2\,c+4\,i\,c\,\omega_0)}{2\,c-4\,i\,c\,\omega_0+3\,\omega_0^2}\right)\right\}$$

(* Cálculo do vetor complexo h21b *)

```
h21b = FullSimplify[ComplexExpand[Conjugate[h21]], ω0 ∈ Reals && c > 0]
       simplifica comple···  expande funções ···  conjugado     números reais
```

$$\left\{\frac{(2\,c-i\,\omega_0)\,(1+\omega_0^2)\,(6\,c+4\,i\,(1+3\,c)\,\omega_0+\omega_0^2)}{2\,\omega_0^2\,(2\,c+4\,i\,c\,\omega_0+3\,\omega_0^2)},\right.$$
$$\frac{1}{6}\left(-1-\frac{18\,c}{\omega_0^2}-\frac{3\,i\,(1+6\,c)}{\omega_0}+\frac{4\,(-3+2\,c+4\,i\,c\,\omega_0)}{2\,c+4\,i\,c\,\omega_0+3\,\omega_0^2}\right),$$
$$\frac{i\,(1+\omega_0^2)\,(12\,i\,c^2-2\,c\,(1+12\,c)\,\omega_0+2\,i\,(2+7\,c)\,\omega_0^2+\omega_0^3)}{2\,\omega_0^2\,(2\,c+4\,i\,c\,\omega_0+3\,\omega_0^2)},$$
$$\left.\frac{1}{6}\left(1+\frac{18\,c}{\omega_0^2}+\frac{3\,i\,(1+6\,c)}{\omega_0}+\frac{4\,(3-2\,c-4\,i\,c\,\omega_0)}{2\,c+4\,i\,c\,\omega_0+3\,\omega_0^2}\right)\right\}$$

(* Matriz 4iω0I *)
unidad

```
D4 = Simplify[4 i ω0 IdentityMatrix[4]]
     simplifica       matriz identidade
```

$$\{\{4\,i\,\omega_0,0,0,0\},\{0,4\,i\,\omega_0,0,0\},\{0,0,4\,i\,\omega_0,0\},\{0,0,0,4\,i\,\omega_0\}\}$$

(* Matriz QA = 4iω0I-A *)
unidade ir

```
QA = Simplify[D4 - A]
     simplifica
```

$$\{\{1+c+4\,i\,\omega_0,1+\omega_0^2,c,0\},\{-1,-1+4\,i\,\omega_0,0,0\},$$
$$\{c,0,1+c+4\,i\,\omega_0,1+\omega_0^2\},\{0,0,-1,-1+4\,i\,\omega_0\}\}$$

(* Inversa da matriz QA *)

```
QAI = Simplify[Inverse[QA]]
```

simplifica matriz inversa

$$\left\{\left\{\frac{(1-4\,\mathrm{i}\,\omega_0)\,(c-4\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ \frac{(1+\omega_0^2)\,(c-4\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ \frac{\mathrm{i}\,c\,(\mathrm{i}+4\,\omega_0)^2}{15\,\omega_0^2\,(2\,\mathrm{i}\,c+8\,c\,\omega_0+15\,\mathrm{i}\,\omega_0^2)},\ -\frac{c\,(\mathrm{i}+4\,\omega_0+\mathrm{i}\,\omega_0^2+4\,\omega_0^3)}{15\,\omega_0^2\,(2\,\mathrm{i}\,c+8\,c\,\omega_0+15\,\mathrm{i}\,\omega_0^2)}\right\},\right.$$

$$\left\{-\frac{c-4\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ -\frac{c+(15+31\,c)\,\omega_0^2+60\,\mathrm{i}\,\omega_0^3}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ \frac{c\,(\mathrm{i}+4\,\omega_0)}{15\,\omega_0^2\,(2\,\mathrm{i}\,c+8\,c\,\omega_0+15\,\mathrm{i}\,\omega_0^2)},\ \frac{c\,(1+\omega_0^2)}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}\right\},$$

$$\left\{\frac{\mathrm{i}\,c\,(\mathrm{i}+4\,\omega_0)^2}{15\,\omega_0^2\,(2\,\mathrm{i}\,c+8\,c\,\omega_0+15\,\mathrm{i}\,\omega_0^2)},\ -\frac{c\,(\mathrm{i}+4\,\omega_0+\mathrm{i}\,\omega_0^2+4\,\omega_0^3)}{15\,\omega_0^2\,(2\,\mathrm{i}\,c+8\,c\,\omega_0+15\,\mathrm{i}\,\omega_0^2)},\ \frac{(1-4\,\mathrm{i}\,\omega_0)\,(c-4\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ \frac{(1+\omega_0^2)\,(c-4\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}\right\},$$

$$\left.\left\{\frac{c\,(\mathrm{i}+4\,\omega_0)}{15\,\omega_0^2\,(2\,\mathrm{i}\,c+8\,c\,\omega_0+15\,\mathrm{i}\,\omega_0^2)},\ \frac{c\,(1+\omega_0^2)}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ -\frac{c-4\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)},\ -\frac{c+(15+31\,c)\,\omega_0^2+60\,\mathrm{i}\,\omega_0^3}{15\,\omega_0^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)}\right\}\right\}$$

```
(* Vetor complexo h40 *)

h40 = FullSimplify[ComplexExpand[
    QAI.(3 bb[h20, h20] + 4 bb[q, h30] + 6 cc[q, q, h20] + dd[q, q, q, q])]]
```

simplifica comple··· expande funções complexas

$$\Big\{\Big(6\,(\mathrm{i}+4\,\omega_0)\,\big(-2\,c+\omega_0\,(4\,\mathrm{i}+8\,\mathrm{i}\,c+\omega_0)\big)\,\Big(8\,\mathrm{i}\,c^3+\omega_0\,\Big(28\,c^2\,(1+2\,c)+\omega_0\,\Big(-4\,\mathrm{i}\,c\,(6+c\,(39+32\,c))+\omega_0\,\Big(-12\,c\,(5+2\,c)\,(1+4\,c)+\omega_0\,\big(2\,\mathrm{i}\,(-26+c\,(-21+64\,c))+(-145-114\,c+91\,\mathrm{i}\,\omega_0)\,\omega_0\big)\Big)\Big)\Big)\Big)\Big)\Big/\Big(\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)\Big),$$

$$\Big(24\,(\mathrm{i}+4\,\omega_0)\,\Big(8\,c^3+\omega_0\,\Big(-28\,\mathrm{i}\,c^2\,(1+2\,c)+\omega_0\,\Big(-4\,c\,(6+c\,(39+32\,c))+\omega_0\,\Big(12\,\mathrm{i}\,c\,(5+2\,c)\,(1+4\,c)+\omega_0\,\big(-52+2\,c\,(-21+64\,c)+\omega_0\,(145\,\mathrm{i}+114\,\mathrm{i}\,c+91\,\omega_0)\big)\Big)\Big)\Big)\Big)\Big)\Big/\Big(\omega_0\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)\Big),$$

$$\Big(6\,(\mathrm{i}+4\,\omega_0)\,\big(-2\,c+\omega_0\,(4\,\mathrm{i}+8\,\mathrm{i}\,c+\omega_0)\big)\,\Big(8\,\mathrm{i}\,c^3+\omega_0\,\Big(28\,c^2\,(1+2\,c)+\omega_0\,\Big(-4\,\mathrm{i}\,c\,(6+c\,(39+32\,c))+\omega_0\,\Big(-12\,c\,(5+2\,c)\,(1+4\,c)+\omega_0\,\big(2\,\mathrm{i}\,(-26+c\,(-21+64\,c))+(-145-114\,c+91\,\mathrm{i}\,\omega_0)\,\omega_0\big)\Big)\Big)\Big)\Big)\Big)\Big/\Big(\omega_0^2\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)\Big),$$

$$\Big(24\,(\mathrm{i}+4\,\omega_0)\,\Big(8\,c^3+\omega_0\,\Big(-28\,\mathrm{i}\,c^2\,(1+2\,c)+\omega_0\,\Big(-4\,c\,(6+c\,(39+32\,c))+\omega_0\,\Big(12\,\mathrm{i}\,c\,(5+2\,c)\,(1+4\,c)+\omega_0\,\big(-52+2\,c\,(-21+64\,c)+\omega_0\,(145\,\mathrm{i}+114\,\mathrm{i}\,c+91\,\omega_0)\big)\Big)\Big)\Big)\Big)\Big)\Big/\Big(\omega_0\,(2\,c-4\,\mathrm{i}\,c\,\omega_0+3\,\omega_0^2)^2\,(2\,c-8\,\mathrm{i}\,c\,\omega_0+15\,\omega_0^2)\Big)\Big\}$$

```
(* Vetor complexo h40b *)
```

```
h40b = FullSimplify[ComplexExpand[Conjugate[h40]], ω0 ∈ Reals && c > 0]
       simplifica comple··· expande funções ··· conjugado          números reais
```

```
{(6 (1 + 4 ⅈ ω0) (-2 ⅈ c + (4 + 8 c + ⅈ ω0) ω0)
    (8 ⅈ c^3 + ω0 (-28 c^2 (1 + 2 c) + ω0 (-4 ⅈ c (6 + c (39 + 32 c)) + ω0 (12 c (5 + 2 c) (1 + 4 c) +
            ω0 (2 ⅈ (-26 + c (-21 + 64 c)) + (145 + 114 c + 91 ⅈ ω0) ω0)))))) /
  (ω0^2 (2 c + 4 ⅈ c ω0 + 3 ω0^2)^2 (2 c + 8 ⅈ c ω0 + 15 ω0^2)), (24 (-ⅈ + 4 ω0)
    (8 c^3 + ω0 (28 ⅈ c^2 (1 + 2 c) + ω0 (-4 c (6 + c (39 + 32 c)) + ω0 (-12 ⅈ c (5 + 2 c)
            (1 + 4 c) + ω0 (-52 + 2 c (-21 + 64 c) - ⅈ (145 + 114 c) ω0 + 91 ω0^2)))))) /
  (ω0 (2 c + 4 ⅈ c ω0 + 3 ω0^2)^2 (2 c + 8 ⅈ c ω0 + 15 ω0^2)), (6 (1 + 4 ⅈ ω0)
    (-2 ⅈ c + (4 + 8 c + ⅈ ω0) ω0)
    (8 ⅈ c^3 + ω0 (-28 c^2 (1 + 2 c) + ω0 (-4 ⅈ c (6 + c (39 + 32 c)) + ω0 (12 c (5 + 2 c) (1 + 4 c) +
            ω0 (2 ⅈ (-26 + c (-21 + 64 c)) + (145 + 114 c + 91 ⅈ ω0) ω0)))))) /
  (ω0^2 (2 c + 4 ⅈ c ω0 + 3 ω0^2)^2 (2 c + 8 ⅈ c ω0 + 15 ω0^2)), (24 (-ⅈ + 4 ω0)
    (8 c^3 + ω0 (28 ⅈ c^2 (1 + 2 c) + ω0 (-4 c (6 + c (39 + 32 c)) + ω0 (-12 ⅈ c (5 + 2 c)
            (1 + 4 c) + ω0 (-52 + 2 c (-21 + 64 c) - ⅈ (145 + 114 c) ω0 + 91 ω0^2)))))) /
  (ω0 (2 c + 4 ⅈ c ω0 + 3 ω0^2)^2 (2 c + 8 ⅈ c ω0 + 15 ω0^2))}
```

(* Cálculo do vetor complexo h31 *)

```
h31 = FullSimplify[DAI.(dd[q, q, q, qb] + 3 cc[q, q, h11] +
      simplifica completamente
    3 cc[q, qb, h20] + 3 bb[h20, h11] + bb[qb, h30] + 3 bb[q, h21] - 3 G21 h20)]
```

```
{1/(2 ω0^2 (2 c - 4 ⅈ c ω0 + 3 ω0^2)^3)
   3 (-80 c^4 + ω0 (8 ⅈ c^3 (-7 + 62 c) + ω0 (8 c^2 (-29 + 6 c (-27 + 20 c)) + ω0 (-4 ⅈ c (-28 +
              c (-593 + 2 c (-793 + 32 c))) + ω0 (8 c (97 + 2 c (447 + c (769 + 64 c))) +
              ω0 (-2 ⅈ (-20 + c (581 + 2 c (2069 + 24 c (105 + 8 c)))) +
                 ω0 (-2 (-83 + 2 c (47 + 16 c (52 + 43 c))) + ω0
                    (ⅈ (-69 + 2 c (-73 + 120 c)) + (49 + 68 c - 10 ⅈ ω0) ω0))))))),
 1/(ω0 (2 c - 4 ⅈ c ω0 + 3 ω0^2)^3) 3 (8 ⅈ c^3 + ω0 (4 c^2 (19 + 18 c) + ω0 (-4 ⅈ c (14 + 5 c) (1 + 12 c) +
           ω0 (-4 c (91 + 492 c + 88 c^2) + ω0 (2 ⅈ (-10 + c (227 + 16 c (67 + 6 c))) +
              ω0 (-85 - 70 c + 736 c^2 + ⅈ (25 + 156 c + 22 ⅈ ω0) ω0)))))),
 1/(2 ω0^2 (2 c - 4 ⅈ c ω0 + 3 ω0^2)^3) 3 (-80 c^4 + ω0 (8 ⅈ c^3 (-7 + 62 c) +
        ω0 (8 c^2 (-29 + 6 c (-27 + 20 c)) + ω0 (-4 ⅈ c (-28 + c (-593 + 2 c (-793 + 32 c))) +
              ω0 (8 c (97 + 2 c (447 + c (769 + 64 c))) +
              ω0 (-2 ⅈ (-20 + c (581 + 2 c (2069 + 24 c (105 + 8 c)))) +
                 ω0 (-2 (-83 + 2 c (47 + 16 c (52 + 43 c))) + ω0
                    (ⅈ (-69 + 2 c (-73 + 120 c)) + (49 + 68 c - 10 ⅈ ω0) ω0))))))),
 1/(ω0 (2 c - 4 ⅈ c ω0 + 3 ω0^2)^3) 3 (8 ⅈ c^3 + ω0 (4 c^2 (19 + 18 c) + ω0 (-4 ⅈ c (14 + 5 c) (1 + 12 c) +
           ω0 (-4 c (91 + 492 c + 88 c^2) + ω0 (2 ⅈ (-10 + c (227 + 16 c (67 + 6 c))) +
              ω0 (-85 - 70 c + 736 c^2 + ⅈ (25 + 156 c + 22 ⅈ ω0) ω0))))))}
```

```
(* Cálculo do vetor complexo h22 *)

h22 =
 Simplify[-AI.(dd[q, q, qb, qb] + 4 cc[q, qb, h11] + cc[qb, qb, h20] + cc[q, q, h20b] +
 simplifica
      2 bb[h11, h11] + 2 bb[q, h21b] + 2 bb[qb, h21] + bb[h20b, h20] - 4 h11 l1)]
```

$$\Big\{\Big(4\left(48c^4+4c^2\left(11+50c+72c^2\right)\omega_0^2+4c\left(13+62c+116c^2+96c^3\right)\omega_0^4-\left(13+166c+360c^2+192c^3\right)\omega_0^6+(29+52c)\,\omega_0^8\right)\Big)\Big/\Big(\omega_0^2\left(-2c+\omega_0^2\right)\left(2c-4\,i\,c\,\omega_0+3\omega_0^2\right)\left(2c+4\,i\,c\,\omega_0+3\omega_0^2\right)\Big),$$
$$\frac{4\left(4c^2(7+6c)+4c\left(13+46c+24c^2\right)\omega_0^2-(9+26c)\,\omega_0^4\right)}{8c^3+4c^2(5+8c)\,\omega_0^2+2(3-8c)\,c\,\omega_0^4-9\omega_0^6},$$
$$\Big(4\left(48c^4+4c^2\left(11+50c+72c^2\right)\omega_0^2+4c\left(13+62c+116c^2+96c^3\right)\omega_0^4-\left(13+166c+360c^2+192c^3\right)\omega_0^6+(29+52c)\,\omega_0^8\right)\Big)\Big/\Big(\omega_0^2\left(-2c+\omega_0^2\right)\left(2c-4\,i\,c\,\omega_0+3\omega_0^2\right)\left(2c+4\,i\,c\,\omega_0+3\omega_0^2\right)\Big),$$
$$\frac{4\left(4c^2(7+6c)+4c\left(13+46c+24c^2\right)\omega_0^2-(9+26c)\,\omega_0^4\right)}{8c^3+4c^2(5+8c)\,\omega_0^2+2(3-8c)\,c\,\omega_0^4-9\omega_0^6}\Big\}$$

```
(* Componentes do número complexo G32 *)

G32 =
  pb.(6 bb[h11, h21] + bb[h20b, h30] + 3 bb[h20, h21b] + 3 bb[q, h22] + 2 bb[qb, h31] +
     6 cc[q, h11, h11] + 3 cc[q, h20, h20b] + 3 cc[q, q, h21b] + 6 cc[q, qb, h21] +
     6 cc[qb, h20, h11] + cc[qb, qb, h30] + dd[q, q, q, h20b] +
     6 dd[q, q, qb, h11] + 3 dd[q, qb, qb, h20] + ee[q, q, q, qb, qb]);

ReG32 = Simplify[ComplexExpand[Re[G32]], ω0 ∈ Reals && c > 0]
        simplifica  expande funções ⋯  parte real  números reais
```

$$\frac{1}{\omega_0^2\left(-2c+\omega_0^2\right)\left(4c^2+4c(3+4c)\,\omega_0^2+9\omega_0^4\right)^3}\Big(6\Big(256c^8(7+6c)+128c^6\left(35+296c+364c^2+144c^3\right)\omega_0^2+128c^5\left(173+2084c+4132c^2+2592c^3+576c^4\right)\omega_0^4+32c^4\left(1065+17560c+62180c^2+70128c^3+27392c^4+3072c^5\right)\omega_0^6+32c^3\left(558+6099c+43254c^2+106560c^3+89216c^4+22528c^5\right)\omega_0^8-8c^2\left(-1125+24104c+178188c^2+350992c^3+228608c^4+41984c^5\right)\omega_0^{10}+8c\left(1269+13680c+54620c^2+76320c^3+29248c^4\right)\omega_0^{12}-2\left(81+216c+7236c^2+8816c^3\right)\omega_0^{14}+135(-3+2c)\,\omega_0^{16}\Big)\Big)$$

```
Clear[b]
apaga

(* Valor de ω0 = ω1 *)

ω0 = √(a b - 1)
```

$$\sqrt{-1+b\,(1+2c)}$$

```
(* Primeiro coeficiente de Lyapunov em função dos parâmteros b e c *)
```

```
l1 = FullSimplify[l1]
     simplifica completamente
```

$$\left(4\,c^2\,(7+6\,c)-(9+26\,c)\,(-1+b+2\,b\,c)^2+4\,c\,(-1+b+2\,b\,c)\,\left(13+46\,c+24\,c^2\right)\right)\Big/\left(8\,c^2+8\,c\,(3+4\,c)\,(-1+b+2\,b\,c)+18\,(-1+b+2\,b\,c)^2\right)$$

```
(* Segundo coeficiente de Lyapunov em função dos parâmteros b e c *)
```

$$l2 = \text{FullSimplify}\left[\frac{1}{12}\,\text{ReG32}\right]$$

simplifica completamente

$$\Big(256\,c^8\,(7+6\,c)+135\,(-3+2\,c)\,(-1+b+2\,b\,c)^8+128\,c^6\,(-1+b+2\,b\,c)\,(35+4\,c\,(74+c\,(91+36\,c)))-2\,(-1+b+2\,b\,c)^7\,(81+4\,c\,(54+c\,(1809+2204\,c)))+128\,c^5\,(-1+b+2\,b\,c)^2\,(173+4\,c\,(521+c\,(1033+72\,c\,(9+2\,c))))+8\,c\,(-1+b+2\,b\,c)^6\,(1269+4\,c\,(3420+c\,(13\,655+8\,c\,(2385+914\,c))))+32\,c^4\,(-1+b+2\,b\,c)^3\,(1065+4\,c\,(4390+c\,(15\,545+4\,c\,(4383+16\,c\,(107+12\,c)))))-8\,c^2\,(-1+b+2\,b\,c)^5\,(-1125+4\,c\,(6026+c\,(44\,547+4\,c\,(21\,937+16\,c\,(893+164\,c)))))+32\,c^3\,(-1+b+2\,b\,c)^4\,(558+c\,(6099+2\,c\,(21\,627+32\,c\,(1665+2\,c\,(697+176\,c))))))\Big)\Big/\Big(2\,(-1+b)\,(1+2\,c)\,(-1+b+2\,b\,c)\left(4\,c^2+4\,c\,(3+4\,c)\,(-1+b+2\,b\,c)+9\,(-1+b+2\,b\,c)^2\right)^3\Big)$$