Federal University of Itajubá
Institute of Physics and Chemistry
Postgraduate Program in Physics

Modified Theories of Gravity: The Horndeski Models

Ricardo Alexandre Fernandes Filho

Itajubá - MG, 26th February of 2024

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Master's Thesis presented to the Postgraduate Program in Physics at UNIFEI as part of the requirements necessary to obtain the Master's Degree in Physics.

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Advisor: Eduardo Henrique Silva Bittencourt

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Modified Theories of Gravity: The Horndeski Models/ Ricardo Alexandre Fernandes Filho. – Itajubá - MG, 26th February of 2024-

34 p. : il. (algumas color.) ; 30 cm.

Advisor: Eduardo Henrique Silva Bittencourt / Co-Advisor: Francesco Bajardi

Dissertation (Master degree) – Federal University of Itajubá - MG
Institute of Physics and Chemistry
Master Program in Physics, 26th February of 2024.

1. Keyword 1. 2. Keyword 2. 3. Keyword 3. 4. General Relativity. I. Eduardo Henrique Silva Bittencourt. II. Francesco Bajardi. III. Federal University of Itajubá. IV. Institute of Physics and Chemistry. V. Modified theories of gravity: The Horndeski Models.

CDU 02:141:005.7

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Itajubá - MG
26th February of 2024

Acknowledgements

I would like to dedicate these short lines to those who helped me tread this path. I appreciate:

- To Professor Eduardo Bittencourt for supervision, patience during scientific debates, and especially for helping me take advantage of every opportunity that arose.
- To the great friends who were present in my daily life, Carlos Augusto, André Felipe, Maria Eduarda, Mateus Ferreira, Luan Carlos, Luiz Eduardo, and Thales Willian. The support network they provided and the enjoyable moments kept me standing.
- I also thank the friends who had to leave Itajubá to pursue their own paths. Even from afar, they continued to accompany me, each in their own way.
- To my family, who has been by my side even without fully understanding what I do. Juliana Fernandes, my mother; Ricardo Alexandre Fernandes, my father; Ana Carla Fernandes, my sister; will always be a part of this achievement.
- To those who were with me during my internship in Italy, Alexandra Turmina, Caio Graciani, Micol Benetti, Francesco Bajardi and Salvatore Capozziello. I thank them for assisting me professionally and personally when everything seemed new and mysterious. Where the
- To the Federal University of Itajubá for hosting my work and providing financial support for my professional growth.
- To the Scuola Superiore Meridionale in Naples for hosting me during my internship.
- To CAPES for the grant.

Resumo

Neste manuscrito, é proposto uma análise qualitativa sobre como as teorias relacionadas a gravidade mudaram com o passar do tempo até chegar as conhecidas teorias modificadas da gravitação, mostrando o poderio que as mesmas guardam, com o intuito de servir como uma porta de entrada ao assunto e sendo base para trabalhos mais complexos no futuro. Começando com o que se entendia por gravidade até chegar a grande revolução de Albert Einstein e a teoria da Relatividade Geral, a qual explicou alguns pontos não alcançados pela teoria newtoniana, mas ainda assim mantinha inexplicado observações feitas em larga escalas. Após Einstein os esforços não se baseavam mais em encontrar uma nova teoria, mas sim fazer extensões da mesma, dando espaço então para as conhecidas teorias modificadas da gravitação. Estas teorias surgem de várias formas na literatura, porém aqui a atenção será voltada a aquelas que são construídas a partir da adição de um campo escalar à teoria, tomando o nome de teorias escalares-tensoriais. Dentre as teorias modificadas da gravitação escalares-tensoriais será restringida a discussão sobre aquelas que conservam as equações de movimento em segunda ordem, sendo a mais geral possível conhecida como a teoria de Horndeski, que será a teoria principal aqui tratada. Após destrinchar a teoria de Horndeski, analisando suas equações de movimento e casos particulares, será também aproveitado o aspecto geral desta teoria para se derivar teorias já conhecidas a partir dela, como Brans-Dicke, Cubic Galileon e $f(R)$. Por fim, serão dedicados esforços para adentrar à cosmologia quantica usando a teoria de Horndeski.

Palavras-chave: Gravitação modificada, Teorias escalares-tensoriais, Teoria de Horndeski.

Abstract

In this thesis, a qualitative analysis is proposed on how theories related to gravity have changed over time until reaching the well-known modified theories of gravitation, demonstrating the power they hold. The aim is to serve as an entry point to the subject and provide a basis for more complex works in the future. Starting with the understanding of gravity and progressing to Albert Einstein's groundbreaking theory of General Relativity, which explained some aspects not addressed by Newtonian theory but still left unexplained observations on large scales. After Einstein, efforts have been mostly focused on constructing extensions of General Relativity, paving the way for the well-known modified theories of gravitation. These theories manifest in various forms in the literature, but here the focus will be on those constructed by the inclusion of a scalar field into the theory, giving rise to the so called scalar-tensor theories. Among the modified scalar-tensor theories, the discussion will be restricted to those that preserve second-order equations of motion, with the most general known as the Horndeski theory, which will be the main theory addressed here. After dissecting the Horndeski theory, we analyze its equations of motion and apply the latter to certain cases of interest. The general aspect of the Horndeski theory will also be exploited to derive, from it, already known theories, such as Brans-Dicke, Cubic Galileon, and $f(R)$ models. The final part of the work will be dedicated to delving into quantum cosmology using the Horndeski theory.

Keywords: Modified gravity, Scalar-tensor theories, Horndeski theory.

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Notation

- All indexes in this work are Latin, representing space-time indexes $a, b, c... = 0, 1, 2, 3$.
- Signature of space-time: $(-, +, +, +)$.
- Planck units are used, such that $c = \hbar = 1$.
- Partial derivative: $\frac{\partial}{\partial x}$ or A_x, where x is a coordinate.
- Christoffel symbol: $\Gamma^a{}_{bc} \equiv \frac{1}{2} g^{ad} (g_{bd,c} + g_{dc,b} - g_{cb,d})$.
- Covariant derivative: $\nabla_b \xi^a \equiv \xi^a{}_{,b} + \Gamma^a{}_{bc} \xi^c$.
- Riemann tensor: $R^a{}_{bcd} \equiv \Gamma^a{}_{db,c} - \Gamma^a{}_{cb,d} + \Gamma^e{}_{db} \Gamma^a{}_{ce} - \Gamma^e{}_{cb} \Gamma^a{}_{de}$.
- Ricci tensor: $R_{ab} = R^c{}_{acb} = g^{cd} R_{cadb}$.
- Ricci scalar: $R = R^a{}_a = g^{ab} R_{ba}$.
- Einstein Field Equation: $G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = \frac{1}{M_P^2} T^{(m)}_{ab}$.
- Planck mass: $M_P \equiv \frac{1}{\sqrt{8\pi G}}$.
- Stress-energy tensor: $T^{(m)}_{ab} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{ab}}$.
- 4-velocity of the fluid: $u^a = (1, 0, 0, 0)$
- Symmetrization operator: $A_{(bc)} = \frac{1}{2} (A_{bc} + A_{bc})$.
- Antisymmetrization operator: $A_{[bc]} = \frac{1}{2} (A_{bc} - A_{cb})$.

1 Introduction

Gravity has intrigued thinkers throughout history, even before being known by this name. In the 4th century BC, Aristotle pondered why stones, when thrown, always returned to the ground. His explanation was simpler than attributing an attractive interaction between bodies; he theorized that every body has its “natural place” [1] and when removed from it tends to return. This explained why massive bodies always returned to the ground, as it was their natural place. Seventeen centuries later, another renowned thinker, Galileo Galilei, sought a more robust explanation for the observed events. He aimed to describe mathematically what he experienced, taking the first steps towards what we now call the scientific method. Isaac Newton eventually achieved the ambition of providing a mathematical description of the attraction between bodies [2]. He coined the term “gravity” and proposed the universal law of gravitation, applicable to all massive bodies in the universe. This law stated that the gravitational interaction was proportional to the inverse square of the distance between the bodies, with the gravitational constant $\boldsymbol{G}$ and the product of the masses involved. Newton’s description allowed the explanation of any body’s motion under the influence of a gravitational field.

The first challenge to Newtonian gravity came in the late 19th century with Ernst Mach, who introduced the Mach principle (see the book [3]). It opposed an absolute reference frame for describing physics (which Newton used) and suggested that the entire universe’s configuration influenced local physical laws. This principle laid the groundwork for more robust gravitational theories. Besides conflicting ideas with Newton’s theory, there were also incompleteness issues, the most famous being the inability to mathematically describe the known perihelion precession of Mercury’s orbit accurately. Some even considered the existence of an undetectable planet interior to Mercury’s orbit, influencing its orbit gravitationally and explaining the discrepancy between data and theory. At this point, a more comprehensive theory of gravitation was expected.

In 1905, Albert Einstein published a groundbreaking work that marked the beginning of the gravitational theory revolution. He introduced what we now know as Special Relativity, paving the way for the 1915 publication of General Relativity (The latest works and other valuable works of the time can be found in Ref. [4]). This theory, in certain aspects, completed Newtonian gravity, providing a purely mathematical framework relating gravity to the geometry of spacetime. With General Relativity, the perihelion precession of Mercury was successfully explained, aligning with observational data. The intense gravitational field around Mercury caused a perihelion precession that Newtonian theory couldn’t describe, and General Relativity accounted for it as a correction term. The development of General Relativity does not invalidate Newtonian physics but rather shows that it is a limit of General Relativity for weak fields.

At this point, we have a theory that perfectly describes the solar system, a prerequisite for a gravitational theory to be considered viable—passing tests within the solar system. However, General Relativity's greatness extends beyond solar system scales; it also made several predictions later confirmed by data, such as the existence of singularities (black holes), the emission of gravitational waves [5], and the deflection of light due to intense fields [6]. Despite these successes, when we move to galactic scales, there are inconsistencies. Observations of galaxy rotation speeds do not match theory predictions, and the most accepted explanation involves the presence of exotic matter interacting with known baryonic matter, causing the deviation between measurement and theory. However, adding an exotic matter term to General Relativity's field equations is nontrivial, revealing the theory's incompleteness on galactic scales. Going further to cosmological scales, General Relativity does not align well with observations of the current universe expansion scenario. In addition to these two shortcomings, another one arises when considering the primordial universe, at the epoch in which the latter was hot and dense. As expected, Einstein's gravity fails on the Planck scale, where an immensely energetic scenario and high matter interaction fields prevail. In this high-energy scenario, it is expected that quantum contributions may arise, and a theory that combines gravity with quantum physics becomes the most complete, with Einstein's General Relativity serving as a limit for large temporal scales. In other words, this occurs when the universe has expanded enough for quantum contributions to gravity to become negligible.

Thus, it is evident that we have not obtained the final theory of gravity yet. Still, certain criteria can be raised to guide its search. A good theory of gravity should have three pillars: (*i*) Respect classical tests within the solar system; (*ii*) Align with observations of galaxy behavior; (*iii*) Reproduce cosmological observations, such as the universe's expansion rate. General Relativity fails to simultaneously satisfy these three pillars, necessitating the implementation of a more robust theory, which is what modified theories of gravitation (also understood as extended theories of gravity) aim to achieve.

So, in the mid-20th century, there were already efforts to find a more comprehensive theory, and one can point to the class of modified theories known as scalar-tensor theories. In this class, gravity is no longer described solely by the geometry of spacetime but rather with the addition of a scalar field in this description. A pioneering scalar-tensor theory was presented by Carl Brans and Henry Dicke [7], where they proposed a theory intending to be more faithful to Mach's principle. This aspect was introduced by considering the gravitational constant G no longer as a constant but as a function of the mass of the entire universe.

Another class of modified gravity theories is known as $f(R)$ gravity [8]. This class stems from the effort to consider the action S, assigned to gravity, as a function of the

Ricci scalar, which can take polynomial forms used to describe different moments in the history of the universe, allowing for a more flexible description of gravity.

Two other theories that have gained prominence in the literature are the Cubic Galileon model gravity [9, 10] and the Horndeski theory [11]. The first is valued for not requiring the addition of dark energy to the theory for the acceleration of the universe to match the collected data. The second stands out for being the most general modified theory of gravity that maintains second-order system equations of motion. By construction, the equations of the theory's action have four free functions of the scalar field and its first derivative, known as Horndeski parameters.

Here, we will go into more detail about each of the mentioned theories, with an emphasis on the Horndeski Theory. A choice of Horndeski parameters will be made to recover already known theories to demonstrate the power of such a theory. The resulting equations of motion from the Horndeski action will also be shown, and we will delve into quantum cosmology using the Horndeski theory and its consequences.

2 From General Relativity to Second-Order Scalar-Tensor Theories

Before we begin discussing how modified theories of gravitation have been improving over the years, let us delve a little deeper into Einstein's theory, the precursor to all others.

For this study, we will delve into the matter of actions and the variational principle. Every mechanical system is endowed with an action, which is a scalar that holds information about the balance of energies within the system. With the system's action in hand, we can invoke the Principle of Least Action to determine the system's equations of motion. By treating the universe as a physical system and seeking to describe its early, middle, and late times, we first must determine an action that will be altered from theory to theory. As mentioned earlier, we will commence this investigation with Einstein's General Relativity, where its action is known as the Einstein-Hilbert action

$$\mathcal{S}_{GR} \sim \frac{M_P^2}{2}\int d^4x\sqrt{-g}R + \mathcal{S}^{(m)}, \tag{2.1}$$

where M_P is the Planck mass, g the metric determinant, R the Ricci curvature scalar and $\mathcal{S}$ the matter action. The latter is contained in what we call the matter action and denoted as $S^{(m)}$. The choice of linearity in the Ricci scalar arises from the fact that this mathematical object is the simplest one that retains information about the curvature of the spacetime in question, *i.e.*, gravity, and maintains second-order equations of motion. To find the equations of motion, we have two paths to choose from (i) Use the principle of least action, extremizing the action with respect to the metric g_{ab}; (ii) When the coordinate system is known, or when the metric in which the work will be done is known, the Lagrangian formalism can be used. At this moment, we will use the variational method, but the Lagrangian formalism will be revisited shortly. The equation of motion will be given by considering the variation

$$\delta S_{GR} = \int d^4x \frac{\delta\left(\frac{M_P^2}{2}\sqrt{-g}R + \mathcal{S}^{(m)}\right)}{\delta g^{ab}}\delta g^{ab} = 0, \tag{2.2}$$

which leads to

$$\int d^4x\frac{M_P^2}{2}\left(R\frac{\delta\sqrt{-g}}{\delta g^{ab}} + \sqrt{-g}\frac{\delta R}{\delta g^{ab}}\right)\delta g^{ab} = -\int d^4x\frac{\delta\mathcal{S}^{(m)}}{\delta g^{ab}}\delta g^{ab}. \tag{2.3}$$

Taking the appropriate variations (which can be found in great detail in [12]), we find

$$\int d^4x\frac{M_P^2}{2}\sqrt{-g}\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab} = -\int d^4x\frac{\delta\mathcal{S}^{(m)}}{\delta g^{ab}}\delta g^{ab}. \tag{2.4}$$

By definition,

$$T_{ab}^{(m)} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{S}^{(m)}}{\delta g^{ab}}, \tag{2.5}$$

known as the stress-energy tensor, which is a mathematical object that holds information about the matter and energy contained in the system. The stress-energy tensor respects the continuity equation

$$\nabla_a T^{ab(m)} = 0. \tag{2.6}$$

Of course, in a vacuum, Eq. (2.6) is identically zero. Finally, we get the famous Einstein's Field Equations

$$R_{ab} - \frac{1}{2}R g_{ab} = \frac{1}{M_P^2}T_{ab}^{(m)}. \tag{2.7}$$

Eq. (2.7) can describe all the motion equations of any depicted system, depending only on the metric that describes the studied spacetime. In total, up to 10 differential equations can be determined to describe a system, but not all these equations will yield different results, in other words, some results of these equations will not be linearly independent. With the Einstein Field Equations in hand, the challenge is to find solutions for the metric tensors that describe the dynamics of the universe. The first and revolutionary one was proposed by Alexander Friedmann and later refined and better understood by Georges Lemaître, Howard Robertson, and Arthur Walker, the famous Friedmann-Lemaître-Robertson-Walker metric that describes a homogeneously and isotropically expanding universe, given by

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{2.8}$$

where $a(t)$ is the scale factor. Knowing the matter, we can define the energy-momentum tensor. Here, we will choose the cosmic fluid as a perfect fluid, which has a stress-energy tensor described by

$$T^{ab(m)} = (\rho + p)\, u^a u^b + p g^{ab}, \tag{2.9}$$

where ρ is the energy density of the matter content and p is its thermodynamic pressure. Now, by considering the 0-0 component of Einstein's field equations, we have the following equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_P^2}\rho, \tag{2.10}$$

also known as the first Friedmann equation, where H is the Hubble parameter. The second equation is obtained from the trace of equation (2.7):

$$\mathrm{Tr}\left[R_{ab} - \frac{1}{2}R g_{ab} = \frac{1}{M_P^2}T_{ab}^{(m)}\right] \Rightarrow -R = \frac{1}{M_P^2}T, \tag{2.11}$$

being $T \equiv \mathrm{Tr}\,[T_{ab}] = -\rho + 3p$, thus

$$\frac{\ddot{a}}{a} = -\frac{2}{3M_P^2}\left(\rho + 3p\right), \tag{2.12}$$

where we used the first Friedmann equation for simplification. This equation was also determined by Friedmann (more information can be found in basic or advanced cosmology books [13, 14]). The latter equation brings forth an interesting piece of information: The expansion of the universe was expected to be decelerated. However, based on experimental evidence [15], the universe is experiencing an accelerated expansion, requiring the proposition that the cosmic fluid has negative pressure, thus making the acceleration equation (2.12) consistent with the data.

However, for Einstein, the universe should be static and not expanding. This idea arose from the scientist's beliefs that the universe should be perfect and unchanging. For a long time, scientists focused on finding static solutions for a universe with matter, but they faced the instability of these models, where the gravity due to matter would cause it to collapse in on itself. It was then that Einstein artificially introduced a constant term into his field equations, nowadays known as the cosmological constant, which, according to him, had no real reason to be in the equations. The sole purpose of this constant was to allow for a static universe with matter.

Years later, Edwin Hubble gathered data on the velocity of galaxies relative to Earth and he irrefutably noticed that the universe was expanding. This finding was further supported by the redshift in the observed spectrum of distant objects. Thus, Einstein and other scientists attempting to find viable solutions for a static universe (such as Willem de Sitter) had more than enough evidence that these solutions did not describe the universe as it truly is.

Even after discarding the cosmological constant as an adjustment to the theory to prevent the gravitational collapse of the universe, nowadays it has been useful in another aspect. As mentioned several times, Einstein's theory is unable to theoretically reproduce observed results, among which in particular the apparent accelerated expansion of the universe. However, cosmologists attribute explanations for this phenomenon to a type of energy known as dark energy (the term "dark" comes from the feature that this source is undetectable by current instruments). The way they found to incorporate this exotic component into the equations of motion was precisely through the cosmological constant. Now, it is no longer a forced term in the equations of motion, but rather a term that describes the presence of dark energy. Thus, the most comprehensive current form of General Relativity action is as follows:

$$\mathcal{S}_{GR} \sim \frac{M_P^2}{2} \int d^4x \sqrt{-g}\, [R - 2\Lambda] + \mathcal{S}^{(m)}, \tag{2.13}$$

being easy to see that Einstein's Field Equation for this theory are

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{1}{M_P^2} T_{ab}^{(m)}. \tag{2.14}$$

However, there is still a yearning to find a modified theory of gravitation that can more coherently replicate the data without the addition of a term of exotic nature in its formulation. In the early second half of the 20th century, some theories began to take shape, such as scalar-tensor theories, with their main characteristic being the description of gravity not only as a geometric effect measured from the mathematical object g_{ab} but also described by a scalar field. This is the explanation for the name given to this class of theories. Now, the gravitational part of the action becomes a function of two variables (g_{ab}, ϕ). The first theory that caught attention was formulated by Brans and Dicke in 1961 [7], where they sought a model that might be more coherent with Mach's Principle. In other words, the entire distribution of matter in the universe would influence a given physical system, making Newton's universal gravitational constant a function of this distribution. Due to the dynamics of this distribution, Brans and Dicke determined that the gravitational constant G had to be a function dependent on time, denoted as $G(t)$. To construct the Brans-Dicke action, they took the old gravitational constant as inversely proportional to a scalar field, that is,

$$G \sim \frac{1}{\phi(t)}, \tag{2.15}$$

in addition to adding a kinetic term for this scalar field, ensuring its dynamics, and a potential term. Thus, replacing the last consideration about G in the action (2.1) and adding the kinetic and potential terms of the scalar field, the Brans-Dicke action takes the form:

$$\mathcal{S}_{BD} \sim \frac{M_P^2}{2} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega}{\phi} \nabla_a \phi \nabla^a \phi - V(\phi) \right] + \mathcal{S}^{(m)}, \tag{2.16}$$

where ω is a dimensionless parameter known as the Brans-Dicke parameter. The entire process of describing different frames involves a conformal transformation of the metric and the rescaling of the scalar field (a thorough discussion on frames can be found in [16]). The Einstein Field Equation of this theory is

$$G_{ab} = \frac{1}{M_P^2 \phi} T_{ab} + \frac{1}{2\phi} \left(\Box\phi - V \right) g_{ab} + \frac{1}{\phi} \nabla_a \nabla_b \phi + \frac{\omega}{\phi^2} \nabla_a \phi \nabla_b \phi, \tag{2.17}$$

and the equation of motion for the scalar field is

$$(3 + 2\omega) \Box\phi + \phi \frac{dV}{d\phi} - 2V = 0, \tag{2.18}$$

which is nothing but the Klein-Gordon equation, obtainable by varying the action with respect to the scalar field. From there, it is expected that the new modified theory theoretically reproduces what is already confirmed by General Relativity and seeks to explain the observations that Einstein's theory fails to reproduce. To do so, note that there is a value for ω that simplifies the field equation (2.18). Taking $\omega = -\frac{3}{2}$, the latter equation becomes

$$\phi \frac{dV}{d\phi} - 2V = 0, \tag{2.19}$$

leading to a degeneracy in the value of ϕ. By defining the shape of the potential as a function of the scalar field, an algebraic function for ϕ is found. However, by setting the potential identically to zero, the scalar field can take any value, and the above equation will still be satisfied. Several solutions to these equations can be proposed, but the key to the success of a theory is passing the tests of the solar system. When attempting to recover the values of the solar system, it is noticed that the value that generates degeneracy is not a good value for the parameter. Along this same path, it is observed that the theory becomes more closely adjusted to the data as the value of ω increases, approaching General Relativity as $\omega \to \infty$ [17, 18].

Even succeeding in replicating General Relativity for a sufficiently large value of ω, this fact already brings counterpoints to this theory. This is because the Brans-Dicke parameter is explained as how coupled the scalar field would be to gravitational effects and should be able to assume random values to describe the data. Therefore, the limit approaching infinite values brings divergences to the theory. Another issue with the Brans-Dicke theory is the fact that it still requires the addition of dark energy to explain the apparent accelerated expansion of the universe.

Another theory that stands out in the literature, seeking explanations beyond what Einstein achieved, even if not necessarily a scalar-tensor theory, is the well-known $f(R)$ theory. As the name itself suggests, the Ricci curvature scalar present in the Einstein-Hilbert action (2.1) is replaced by a function of R, thus opening up new possibilities such as increasing the degrees of freedom of the theory with fourth-order equations of motion. As mentioned numerous times, this proposal should provide faithful results with experiments on the accelerated expansion of the universe, tests of the solar system, explanations about the origin of dark sources, and, furthermore, due to the greater freedom that a function of the curvature scalar entails, $f(R)$ theories can be a bit more ambitious and attempt to describe the continuity believed to exist in the transition between past eras of the universe and the most recent ones. This class of theories proves to be more robust than the previous one extending General Relativity.

Some interesting aspects arise from these theories. The general action of an $f(R)$ theory is [8]

$$\mathcal{S}_{f(R)} \sim \frac{M_P^2}{2} \int d^4x\sqrt{-g}f(R) + \mathcal{S}^{(m)}, \tag{2.20}$$

with the corresponding field equations of the form

$$G_{ab} = F(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - \nabla_a\nabla_b F(R) + g_{ab}\Box F(R) = \frac{1}{M_P^2}T_{ab}^{(m)}, \tag{2.21}$$

where $F(R) = \frac{df(R)}{dR}$ and the trace is

$$3\Box F(R) + F(R)R - 2f(R) = -\frac{1}{M_P^2}T. \tag{2.22}$$

Note that by taking $f(R) = R$, one indeed returns to General Relativity, as expected, and the Ricci scalar is proportional to the matter distribution. In modified theories, the term $3\Box F(R)$ is not zero, and there is an additional degree of freedom in the theory. Due to the functional form of (2.22), one can consider that $F(R)$ has dynamic characteristics and adopt the definition $F(R) \equiv \phi$ (we give the name "scalaron" to this field). In this way, Eq. (2.22) becomes an equation for the dynamics of a propagating scalar field.

This theory can also explain the period of inflationary expansion, As a matter of fact, if one considers De Sitter's proposal, which consisted of finding a static solution for the vacuum ($T = 0$), along with the fact that $F(R)$ does not have propagating behavior ($\Box F(R) = 0$), then Eq. (2.22) takes the form

$$F(R)R - 2f(R) = 0. \tag{2.23}$$

whose solution is $f(R) = \alpha R^2$. $f(R)$ gravity can be a powerful tool when trying to model the period of the universe known as the inflationary period [19]. With $f(R)$ being a power-law of the form

$$f(R) = R + \alpha R^n, \tag{2.24}$$

where α is a positive coupling parameter and n is a real number, not one. In the limit as $n \longrightarrow 2$, this model is equal to the Starobinsky model. This model considers that the polynomial term is dominant over the linear term during the inflationary period, and after this period, the polynomial term becomes negligible, returning the theory to the Einstein-Hilbert action.

In an attempt to describe dark energy, some works rely on the form

$$f(R) = R - \frac{\alpha}{R^n}, \quad n > 0, \tag{2.25}$$

however, this choice of the Ricci scalar function results in a negative mass for the scalaron field, which is clearly non-physical. In an attempt to address this issue, some works have been developed, studying the dynamic variables of the universe and their responses to new forms of the $f(R)$ function (see about dynamic variables in $f(R)$ theory, stability, and other models in [8, 20, 21, 22]). A model that has attracted attention is

$$f(R) = R - \alpha R^n, \quad \alpha > 0, \quad \text{and} \quad 0 < n < 1. \tag{2.26}$$

However, even though it is promising, there still exists a critical value that the curvature scalar can take, potentially causing issues in the comparison between data and theory.

A theory that has been proving its worth is known as the Galileon Model, named for the invariance under shift symmetry in flat spacetime, i.e., $\partial_a\phi \longrightarrow \partial_a\phi + v_a$, where v_a is a constant vector. It has gained prominence in the literature due to the natural emergence of a cosmological constant in its theory, without the need to impose it forcibly. Another success of the theory is that Galileon mimics dark energy. In addition to what has already been mentioned, the theory requires the messenger particle of gravity, the graviton, to be massive. This fact has sparked some discussion in the literature. however, the Vainshtein mechanism indicates that the massive nature of the graviton does not invalidate the theory. This is because, according to the mentioned mechanism, on small scales General Relativity can be recovered again (on small scales, the theory may be non-linear) and, on large scales, defined as greater than the Vainshtein radius, the theory will be linear, leading to changes that Einstein's theory does not predict (for more details, see [23]).

Two successes of the Galileon Model are mentioned above, but the following will be emphasized, focusing specifically on the mimicry of dark energy. The natural appearance of a cosmological constant and even the expansion itself can be found in the following work [24]. Initially, we consider the action of a Cubic Model known as Cubic Galileon [9, 10], whose action reads

$$\mathcal{S}_{CG} \sim \frac{M_P^2}{2} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{k_1}{2} \tilde{\nabla}_a \phi \tilde{\nabla}^a \phi - \frac{k_2}{2M^2} \tilde{\nabla}_a \phi \tilde{\nabla}^a \phi \tilde{\Box} \phi \right] + \mathcal{S} \left[\chi_m, g_{ab} \right], \tag{2.27}$$

where the symbol tilde denotes that the mathematical objects involved are described in the Einstein frame, related to the Jordan frame (which represents non-minimal coupling with gravity, unlike the Einstein frame) by a conformal transformation of the metric of the form $g_{ab} = e^{2\alpha\phi} \tilde{g}_{ab}$, where α is interpreted as the coupling parameter between the scalar field and matter. k_1 and k_2 are theory parameters, and M is a length scale parameter, with the Hubble horizon proportional to M^{-1}. The last term in Eq. (2.27) corresponds to matter, where χ_m is the matter field, and the metric is related to the Jordan frame. This becomes clear when taking $\alpha = 0$ so that both frames are described in the same geometry, with no coupling between matter and gravity.

When it is stated that the Galileon mimics dark energy, it is understood that due to its presence, in cosmological applications, there is an observable expansion of the universe caused by something non-material. In other words, even in an empty universe, the Galileon will cause the universe to expand. To understand this assertion, one can look at the equations of motion of the theory

$$\begin{aligned} M_P^2 G_{ab} = T_{ab}^{(m)} &+ k_1 \left[2 \left(\nabla_a \phi \right) \left(\nabla_b \phi \right) - g_{ab} (\nabla \phi)^2 \right] + \frac{k_2}{M^2} \left[2 \left(\nabla_a \phi \right) \left(\nabla_b \phi \right) \nabla_c \nabla^c \phi \right. \\ &\left. + g_{ab} \left(\nabla_c \phi \right) \nabla^c (\nabla \phi)^2 - \left(\nabla_{(a} \phi \right) \nabla_{b)} (\nabla \phi)^2 \right]. \end{aligned} \tag{2.28}$$

Applying to the cosmological case, using $\phi = \phi(t)$ and considering the material content described by a perfect fluid (2.9), the equation arising from the "00" component of the modified Friedmann equations will be

$$H^2 = \frac{1}{3M_P^2} \left[\rho(t) + k_1 \dot{\phi}^2 - \frac{6k_2 \dot{\phi}^3 H}{M^2} \right]. \tag{2.29}$$

Note that the first term on the right-hand side of this equation is notably the one from the Friedmann equation (2.10). Therefore, this equation is commonly referred to as the modified Friedmann equation, where each extended theory of gravity presents a unique form due to the contribution from the scalar field. As mentioned earlier, we want to determine if there will be expansion in the universe without material content. Thus, let us take $\rho(t) = 0$ and determine the roots for H

$$H = -\frac{6k_2\dot{\phi}^3}{M^2} \pm \frac{6k_2\dot{\phi}^3}{M^2}\left(1 + \frac{M_P^2 M^4 k_1}{3k_2^2\dot{\phi}^4}\right)^{\frac{1}{2}}. \tag{2.30}$$

It is clear that one of the conditions for determining expansive solutions must be

$$\frac{M_P^2 M^4 k_1}{3k_2^2\dot{\phi}^4} > -1, \tag{2.31}$$

in order to ensure that H is real. Note also that if $\frac{M_P^2 M^4 k_1}{3k_2^2\dot{\phi}^4} = 0$, it causes the Hubble parameter to become zero when the positive sign is used, and there will be no expansion. The conditions related to the choice of the ± sign are:

- If "+", $\left(1 + \frac{M_P^2 M^4 k_1}{3k_2^2\dot{\phi}^4}\right)^{\frac{1}{2}} > 1$,
- If "−", $k_2\dot{\phi}^3 < 0$.

So the possibility of an expansive solution for the universe becomes evident, even without the addition of matter.

It is also possible to choose a solution for the scalar field in such a way as to demonstrate that the de Sitter solution is an attractor. Using the linear *ansatz* for the field

$$\phi(t) = \phi_0 + \phi_1 t, \tag{2.32}$$

it becomes clear that by adding this solution to the equation (2.30), the Hubble parameter becomes constant and leads to the static universe described by de Sitter. Once this condition holds, the universe no longer expands (or even collapses).

In the next chapter, we will delve into the most general second-order modified theory of gravity, known as Horndeski's theory.

3 Horndeski Theory

Gregory Horndeski, an American physicist, in 1974, after his work on second-order scalar-tensor theories [25], formulated what we now know as the Horndeski Theory. It is the most general scalar-tensor theory that maintains second-order derivative equations of motion in the phase space coordinates [11]. In the pursuit of the most general second-order scalar-tensor theory, Horndeski decomposed the action of his model into four Lagrangian densities, as shown below

$$\mathcal{S}_H = \frac{M_P^2}{2} \int d^4x \sqrt{-g}\, (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) + S^{(\mathrm{m})}, \tag{3.1}$$

and each Lagrangian density is

$$\begin{aligned} \mathcal{L}_2 &= G_2(\phi, X), \quad \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_a\nabla_b\phi)^2\right], \\ \mathcal{L}_5 &= G_5(\phi, X)G_{ab}\nabla^a\nabla^b\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi\,(\nabla_a\nabla_b\phi)^2 + 2\,(\nabla_a\nabla_b\phi)^3\right], \end{aligned} \tag{3.2}$$

with $X \equiv -\frac{1}{2}\nabla^c\phi\nabla_c\phi$. The Horndeski parameters are $G_i (i = 2, 3, 4, 5)$, and we denote their derivatives as $G_{i\phi} \equiv \partial G_i/\partial\phi$, $G_{iX} \equiv \partial G_i/\partial X$, $(\nabla_a\nabla_b\phi)^2 \equiv \nabla_a\nabla_b\phi\nabla^a\nabla^b\phi$, $(\nabla_a\nabla_b\phi)^3 \equiv \nabla_\mathrm{a}\nabla_c\phi\nabla^c\nabla^d\phi\nabla_d\nabla^a\phi$, $\Box = g^{ab}\nabla_a\nabla_b$ and R is the Ricci scalar. The physical meaning of each parameter may vary or be only a mathematical object designed to reproduce a theory or provide more precision between theory and observation. Taking, for example, the case of non-minimal coupling in the form $f(\phi)R$, as mentioned in [26], $G_4 = f(\phi)$ assume the role of introducing non-minimal coupling. In a possible limit where $G_4 = const$, we recover the term present in the Einstein-Hilbert action (2.1). In the same case, G_2 takes the form of the term present in k-inflation theories [27], and G_3 appears as the term present in k-essence theories [28].

The equations of motion for this theory, as expected, take complex forms, but they can be derived conventionally using variational calculus, that is (we consider only the Horndeski sector here, setting $\mathcal{S}^{(m)} = 0$)

$$\delta\mathcal{S}_H = \delta\left(\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i\right) = \frac{M_P^2}{2}\sqrt{-g}\left[\sum_{i=2}^{5}\mathcal{G}^i_{ab}\delta g^{ab} + \sum_{i=2}^{5}\left(P^i_\phi - \nabla^a J^i_a\right)\delta\phi\right], \tag{3.3}$$

where total derivatives are omitted. Taking the variation above equal to zero one finds the equation of motion for the metric tensor and the scalar field, respectively, as

$$\sum_{i=2}^{5}\mathcal{G}^i_{ab} = 0, \tag{3.4}$$

and

$$\nabla^a \left(\sum_{i=2} J_a^i \right) = \sum_{i=2}^{5} P_\phi^i, \tag{3.5}$$

where J_a^i can be identified as a 4-current. If the matter is present in the starting action through a non-vanishing contribution, Eq. (3.4) is non-null, but proportional to the stress-energy tensor of matter. The auxiliary functions for the scalar field motion equations P_ϕ^i and J_a^i are

$$\begin{aligned}
&P_\phi^2 = G_{2\phi}, \\
&P_\phi^3 = \nabla_a G_{3\phi} \nabla^a \phi, \\
&P_\phi^4 = G_{4\phi} R + G_{4\phi X} \left[(\Box\phi)^2 - (\nabla_a \nabla_b \phi)^2 \right], \\
&P_\phi^5 = -\nabla_a G_{5\phi} G^{ab} \nabla_b \phi - \frac{1}{6} G_{5\phi X} \left[(\Box\phi)^3 - 3\Box\phi \left(\nabla_a \nabla_b \phi\right)^2 + 2 \left(\nabla_a \nabla_b \phi\right)^3 \right], \\
&J_a^2 = -\mathcal{L}_{2X} \nabla_a \phi, \\
&J_a^3 = -\mathcal{L}_{3X} \nabla_a \phi + G_{3X} \nabla_a X + 2 G_{3\phi} \nabla_a \phi, \\
&J_a^4 = -\mathcal{L}_{4X} \nabla_a \phi + 2 G_{4X} R_{ab} \nabla^b \phi - 2 G_{4XX} \left(\Box\phi \nabla_a X - \nabla^b X \nabla_a \nabla_b \phi \right) \\
&-2 G_{4\phi X} \left(\Box\phi \nabla_a \phi + \nabla_a X \right), \\
&J_a^5 = -\mathcal{L}_{5X} \nabla_a \phi - 2 G_{5\phi} G_{ab} \nabla^b \phi \\
&-G_{5X} \left[G_{ab} \nabla^b X + R_{ab} \Box\phi \nabla^b \phi - R_{bc} \nabla^b \phi \nabla^c \nabla_a \phi - R_{daeb} \nabla^b \phi \nabla^d \nabla^e \phi \right] \\
&+G_{5XX} \left\{ \frac{1}{2} \nabla_a X \left[(\Box\phi)^2 - (\nabla_d \nabla_e \phi)^2 \right] - \nabla_b X \left(\Box\phi \nabla_a \nabla^b \phi - \nabla_d \nabla_a \phi \nabla^d \nabla^b \phi \right) \right\} \\
&+G_{5\phi X} \left\{ \frac{1}{2} \nabla_a \phi \left[(\Box\phi)^2 - (\nabla_d \nabla_e \phi)^2 \right] + \Box\phi \nabla_a X - \nabla^b X \nabla_b \nabla_a \phi \right\}.
\end{aligned} \tag{3.6}$$

For the metric tensor, the $\mathcal{G}_{ab}^i$'s are

$$\begin{aligned}
&\mathcal{G}_{ab}^2 = -\frac{1}{2} G_{2X} \nabla_a \phi \nabla_b \phi - \frac{1}{2} G_2 g_{ab}, \\
&\mathcal{G}_{ab}^3 = \frac{1}{2} G_{3X} \Box\phi \nabla_a \phi \nabla_b \phi + \nabla_{(a} G_3 \nabla_{b)} \phi - \frac{1}{2} g_{ab} \nabla_c G_3 \nabla^c \phi, \\
&\mathcal{G}_{ab}^4 = G_4 G_{ab} - \frac{1}{2} G_{4X} R \nabla_a \phi \nabla_b \phi - \frac{1}{2} G_{4XX} \left[(\Box\phi)^2 - (\nabla_d \nabla_e \phi)^2 \right] \nabla_a \phi \nabla_b \phi \\
&-G_{4X} \Box\phi \nabla_a \nabla_b \phi + G_{4X} \nabla_c \nabla_a \phi \nabla^c \nabla_b \phi + 2 \nabla_c G_{4X} \nabla^c \nabla_{(a} \phi \nabla_{b)} \phi \\
&-\nabla_c G_{4X} \nabla^c \phi \nabla_a \nabla_b \phi + g_{ab} \left(G_{4\phi} \Box\phi - 2 X G_{4\phi\phi} \right) \\
&+g_{ab} \left\{ -2 G_{4\phi X} \nabla_d \nabla_e \phi \nabla^d \phi \nabla^e \phi + G_{4XX} \nabla_d \nabla_c \phi \nabla_e \nabla^c \phi \nabla^d \phi \nabla^e \phi \right. \\
&\left. + \frac{1}{2} G_{4X} \left[(\Box\phi)^2 - (\nabla_d \nabla_e \phi)^2 \right] \right\} + 2 \left[G_{4X} R_{c(a} \nabla_{b)} \phi \nabla^c \phi - \nabla_{(a} G_{4X} \nabla_{b)} \phi \Box\phi \right] \\
&-g_{ab} \left[G_{4X} R^{de} \nabla_d \phi \nabla_e \phi - \nabla_c G_{4X} \nabla^c \phi \Box\phi \right] + G_{4X} R_{adbe} \nabla^d \phi \nabla^e \phi \\
&-G_{4\phi} \nabla_a \nabla_b \phi - G_{4\phi\phi} \nabla_a \phi \nabla_b \phi + 2 G_{4\phi X} \nabla^c \phi \nabla_c \nabla_{(a} \phi \nabla_{b)} \phi \\
&-G_{4XX} \nabla^d \phi \nabla_d \nabla_a \phi \nabla^e \phi \nabla_e \nabla_b \phi,
\end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&\mathcal{G}^5_{ab} = G_{5X}R_{de}\nabla^d\phi\nabla^e\nabla_{(a}\phi\nabla_{b)}\phi - G_{5X}R_{d(a}\nabla_{b)}\phi\nabla^d\phi\Box\phi - \frac{1}{2}G_{5X}R_{adbe}\nabla^d\phi\nabla^e\phi\Box\phi\\
&-\frac{1}{2}G_{5X}R_{de}\nabla^d\phi\nabla^e\phi\nabla_a\nabla_b\phi + G_{5X}R_{dce(a}\nabla_{b)}\phi\nabla^c\phi\nabla^d\nabla^e\phi\\
&-\frac{1}{2}\nabla_{(a}\left[G_{5X}\nabla^d\phi\right]\nabla_d\nabla_{b)}\phi\Box\phi + \frac{1}{2}\nabla_{(a}\left[G_{5\phi}\nabla_{b)}\right]\Box\phi - \nabla_c\left[G_{5\phi}\nabla_{(a}\phi\right]\nabla_{b)}\nabla^c\phi\\
&+\frac{1}{2}\left[\nabla_c\left(G_{5\phi}\nabla^c\phi\right) - \nabla_d\left(G_{5X}\nabla_e\phi\right)\nabla^d\nabla^e\phi\right]\nabla_a\nabla_b\phi + \nabla^d G_5\nabla^e\phi R_{d(ab)e}\\
&+\frac{1}{2}\nabla_{(a}G_{5X}\nabla_{b)}\phi - \left[(\Box\phi)^2 - (\nabla_d\nabla_e\phi)^2\right] + G_{5X}R_{dce(a}\nabla_{b)}\nabla^c\phi\nabla^d\phi\nabla^e\phi\\
&-\nabla^c G_5 R_{c(a}\nabla_{b)}\phi + \nabla_d\left[G_{5X}\nabla_e\phi\right]\nabla^d\nabla_{(a}\phi\nabla^e\nabla_{b)}\phi - \nabla_{(a}G_5 G_{b)c}\nabla^c\phi\\
&-\nabla_e G_{5X}\left[\Box\phi\nabla^e\nabla_{(a}\phi - \nabla^d\nabla^e\phi\nabla_d\nabla_{(a}\phi\right]\nabla_{b)}\phi\\
&+\frac{1}{2}\nabla^d\phi\nabla_d G_{5X}\left[\Box\phi\nabla_a\nabla_b\phi - \nabla_e\nabla_a\phi\nabla^e\nabla_b\phi\right] + \frac{1}{2}\nabla_c G_5 G_{ab}\nabla^c\phi\\
&-\frac{1}{2}G_{5X}G_{de}\nabla^d\nabla^e\phi\nabla_a\phi\nabla_b\phi - \frac{1}{2}G_{5X}\Box\phi\nabla_d\nabla_a\phi\nabla^d\nabla_b\phi\\
&+\frac{1}{2}G_{5X}(\Box\phi)^2\nabla_a\nabla_b\phi + \frac{1}{12}G_{5XX}\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_d\nabla_e\phi\right)^2 + 2\left(\nabla_d\nabla_e\phi\right)^3\right]\nabla_a\nabla_b\phi\\
&+g_{ab}\left\{-\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_d\nabla_e\phi\right)^2 + 2\left(\nabla_d\nabla_e\phi\right)^3\right] + \nabla_d G_5 R^{de}\nabla_e\phi -\right.\\
&-\frac{1}{2}\nabla_d\left(G_{5\phi}\nabla^d\phi\right)\Box\phi + \frac{1}{2}\nabla_d\left(G_{5\phi}\nabla_{e\phi}\phi\right)\nabla^d\nabla^e\phi - \frac{1}{2}\nabla_d G_{5X}\nabla^d X\Box\phi\\
&+\frac{1}{2}\nabla_d G_{5X}\nabla_e X\nabla^d\nabla^e\phi - \frac{1}{4}\nabla^c G_{5X}\nabla_c\phi\left[(\Box\phi)^2 - (\nabla_d\nabla_e\phi)^2\right]\\
&\left.+\frac{1}{2}G_{5X}R_{de}\nabla^d\phi\nabla^e\phi\Box\phi - \frac{1}{2}G_{5X}R_{dce\rho}\nabla^d\nabla^e\phi\nabla^c\phi\nabla^\rho\phi\right\}.
\end{aligned}
$$

In this work, it will be used the so-called *Viable Class* for Horndeski gravity, where it is considered the gravitational wave propagation with a velocity equal to c, according to observations from binary black holes mergers [5]. This class is helpful because it avoids the instabilities and admits an Einstein frame description [26]. Mathematically, this class is represented assuming G_5 and G_{4X} null. By varying (3.1) with respect to the metric tensor, the modified Einstein equations of the theory are determined, and by varying the same action with respect to the scalar field, the corresponding field equation for ϕ is determined. Yet, applying the viable class, Eqs. (3.6) and (3.7) reduce to

$$
\begin{aligned}
&G_4 G_{ab} - \nabla_a\nabla_b G_4 + \left[\Box G_4 - \frac{G_2}{2} - \frac{1}{2}\nabla_c\phi\nabla^c G_3\right]g_{ab}\\
&+\frac{1}{2}\left[G_{3X}\Box\phi - G_{2X}\right]\nabla_a\phi\nabla_b\phi + \nabla_{(a}\phi\nabla_{b)}G_3 = T^{(m)}_{ab},
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&G_{4\phi}R + G_{2\phi} + G_{2X}\Box\phi + \nabla_c\phi\nabla^c G_{2X}\\
&-G_{3X}(\Box\phi)^2 - \nabla_e\phi\nabla^c G_{3X}\Box\phi - G_{3X}\nabla^c\phi\Box\nabla_c\phi\\
&+G_{3X}R_{ab}\nabla^a\phi\nabla^b\phi - \Box G_3 - G_{3\phi}\Box\phi = 0,
\end{aligned}
\tag{3.9}
$$

where, as mentioned above, $T^{(m)}_{ab}$ is the stress-energy tensor for the matter and G_{ab} is the conventional Einstein tensor.

Now we will recast these equations for the cosmological case, where the metric g_{ab} will be taken as that described in Eq. (2.8), for a homogeneous and isotropic universe with a scale factor $a(t)$, in Cartesian coordinates. The scalar field will also be considered as a function of time only, $\phi = \phi(t)$. With all this information, we can calculate various important terms, such as the Ricci scalar of the geometry

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \tag{3.10}$$

and some other important objects (see [29])

$$X = \frac{1}{2}\dot{\phi}^2, \Box\phi = -\left(\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}\right), \ (\nabla_a\nabla_b\phi)^2 = \ddot{\phi}^2 + 3\frac{\dot{a}^2}{a^2}\dot{\phi}^2, \ (\nabla_a\nabla_b\phi)^3 = -\ddot{\phi}^3 - \frac{3\dot{a}^3}{a^3}\dot{\phi}^3. \tag{3.11}$$

Having all these objects in hand, it becomes simpler to use the Lagrange method to determine the equations of motion. Thus, by incorporating all the considerations made so far into Eq. (3.1), it is possible to obtain the Horndeski Lagrangian plus the matter term. Taking due care during the integration by parts, it is possible to find the canonical Lagrangian of Horndeski theory, where all total derivative terms are neglected, which is[1]

$$\begin{aligned}\mathbf{L} = {} & a^3G_2 - a^2\dot{a}\dot{\phi}^2G_{3\dot{\phi}} - \frac{1}{3}a^3\dot{\phi}^3G_{3\dot{\phi}\phi} - a^3\dot{\phi}^2G_{3\phi} - 6a\dot{a}^2G_4 - 6a^2\dot{a}\dot{\phi}G_{4\phi} \\ & -3a\dot{a}^2\left(2G_{4\dot{\phi}}\dot{\phi} + G_{5\phi}\dot{\phi}^2\right) - \dot{a}^3\dot{\phi}^2G_{5\dot{\phi}} + \rho_0 a^{-3\omega},\end{aligned} \tag{3.12}$$

with the notation $\dot{x} = \frac{dx}{dt}$, where x stands for a or ϕ. The last term in the Horndeski Lagrangian is the way that we introduce matter in our theory [30]. Here ρ_0 is a constant related to the density energy of the universe and the state equation of the perfect fluid is $p = \omega\rho$. Usually, ω can take one of three significant values: $\omega = 0$ (dust matter), $\omega = \frac{1}{3}$ (radiation), and $\omega = -1$ (cosmological constant). Another important remark on this is that $G_3(\phi, X)$ must assume the particular form

$$G_3 = f(\phi)X + g(\phi), \tag{3.13}$$

otherwise, the above Lagrangian would contain terms proportional to second-order derivatives of the scalar field, losing its canonicity and consequently bringing *ghosts* to the theory. Furthermore, this choice makes it easier to obtain other modified theories from Horndeski's theory. It is important to note that here, the derivatives with respect to X have been replaced by derivatives with respect to $\dot{\phi}$, using the definition $X = -\frac{1}{2}\dot{\phi}^2$. Hence,

$$\frac{\partial G_i}{\partial X} = \frac{\partial G_i}{\partial\dot{\phi}}\frac{\partial\dot{\phi}}{\partial X} = -\frac{1}{\dot{\phi}}\frac{\partial G_i}{\partial\dot{\phi}}. \tag{3.14}$$

Such a change is justified for the greater ease in using the Euler-Lagrange formalism since the generalized coordinates will be $(a(t), \phi(t))$.

[1] Note that it is for the general case and not only the viable class. We return to it afterward.

It is worth mentioning that the variation formalism was used to find the equations of motion for any given geometry because this is the only method available when only the action S is known and no specific geometry is determined. However, restricting the study to the Friedmann-Lemaître-Robertson-Walker geometry, the Lagrangian can be obtained in terms of the generalized coordinates. Consequently, the Euler-Lagrange equations are the most intuitive means to derive the equations of motion. Even when utilizing two different resources, the results obtained in both formalisms must be equivalent. In the cosmological context, the energy condition $H = \frac{\partial L}{\partial \dot{a}}\dot{a} + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L = 0$ is equivalent to the $(0,0)$ component of the Einstein equations (analog to (2.10)). The Euler-Lagrange equation assigned to the scale factor $a(t)$ is equivalent to the second Friedmann equation or the $Tr\left[G_{ab}\right]$ (analog to (2.12)). Finally, the Euler-Lagrange equation with respect to the scalar field ϕ is the Klein-Gordon equation of the system. Therefore, in the cosmological context, using the viable class Horndeski theory, the equations of motion for $a(t)$,$\phi(t)$, and the energy condition are, respectively,

$$\begin{aligned}&-\ddot{a}\, a^2\dot{\phi}^2 G_{3\dot{\phi}\dot{\phi}} - 2\ddot{\phi}\, a^2\dot{\phi} G_{3\dot{\phi}} + 6\left(a^2\dot{\phi}^2 G_{4\phi\phi} + \ddot{\phi}\, a^2 G_{4\phi} + \dot{a}^2 G_4\right)\\ &-12\left(a\dot{a}\dot{\phi} G_{4\phi} + a\ddot{a} G_4\right) - 3\left(a^2 G_2 - a^2\dot{\phi}^2 G_{3\phi}\right) + 3\omega\rho_0 a^{-(1+3\omega)} = 0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}&-4\ddot{\phi}\, a^2\dot{a}\dot{\phi} G_{3\dot{\phi}\phi} - \ddot{\phi}\, a^2\dot{a}\,\dot{\phi}^2 G_{3\dot{\phi}\dot{\phi}\phi} - \frac{a^3\dot{\phi}^4}{3} G_{3\phi\phi\dot{\phi}\dot{\phi}} + \dot{\phi}\, a^3 G_{2\phi\dot{\phi}} + \ddot{\phi}\, a^3 G_{2\dot{\phi}\dot{\phi}}\\ &-\frac{5a^3\dot{\phi}^3}{3} G_{3\phi\phi\dot{\phi}} - a^3\dot{\phi}^2 G_{3\phi\phi} - a^2\dot{\phi}^2\ddot{a} G_{3\dot{\phi}\dot{\phi}} - 7a^2\dot{a}\,\dot{\phi}^2 G_{3\phi\dot{\phi}} - 6\ddot{\phi}\, a^3\dot{\phi} G_{3\phi\dot{\phi}}\\ &-3\ddot{\phi}\, a^3\dot{\phi}^2 G_{3\phi\dot{\phi}\dot{\phi}} - \frac{\ddot{\phi}\, a^3\dot{\phi}^3}{3} G_{3\phi\dot{\phi}\dot{\phi}\dot{\phi}} - 2G_{3\dot{\phi}\dot{\phi}} a\,\dot{a}^2\dot{\phi}^2 - 2a^2\dot{a}\,\dot{\phi}^3 G_{3\phi\dot{\phi}}\\ &+\left(3a^2\dot{a} - a^3\right) G_{2\phi} - 6\left(a^2\ddot{a} + a\,\dot{a}^2\right) G_{4\phi} - 2\left(\ddot{\phi}\, a^2\dot{a} + 2a\,\dot{a}^2\dot{\phi} + a^2\dot{\phi}\ddot{a}\right) G_{3\dot{\phi}}\\ &-\left(6a^2\dot{a}\dot{\phi} + 2\ddot{\phi}\, a^3\right) G_{3\phi} = 0,\end{aligned} \tag{3.16}$$

$$\begin{aligned}&-\frac{a^3\dot{\phi}^4 G_{3\phi\dot{\phi}\dot{\phi}} - 5a^3\dot{\phi}^3 G_{3\phi\dot{\phi}}}{3} - a\dot{a}\,\dot{\phi}^3 G_{3\dot{\phi}\dot{\phi}} - a^3\dot{\phi}^2 G_{3\phi} - 2a^2\dot{a}\,\dot{\phi}^2 G_{3\dot{\phi}}\\ &+\dot{\phi}\, a^3 G_{2\dot{\phi}} - 6a^2\dot{a}\dot{\phi} G_{4\phi} - a^3 G_2 - 6a\dot{a}^2 G_4 - \rho_0 a^{-3\omega} = 0.\end{aligned} \tag{3.17}$$

The description of the Horndeski theory is notably more flexible compared to others due to the inclusion of free functions. These various arbitrary functions within the theory facilitate adjustments to fit observational data more easily. Another advantage is its ability to explain the accelerated expansion of the universe without the need to introduce dark energy. However, it heavily depends on a complex mathematical framework, which can pose computational challenges crucial for simulations based on the theory. It's also important to note the purely mathematical development of Horndeski's theory, lacking consideration for philosophical or conceptual issues, aspects that any robust theory in physics should address. This could potentially lead to difficulties later on in interpreting phenomena. Other aspects of the theory emerge when considering perturbations in the Horndeski theory. One of them is the possibility of finding constraints within the theory in order to reconcile it with observations. As discussed in [31], by employing a linear parametrization of the perturbation, done with the parameters α_i, one can relate the constraints of the cosmological parameters (arising from observations) to the linearization parameters and determine the best form of α_i. See also [32] for further insights into the observational context. It is also possible to treat perturbation theory in the Horndeski theory using an anisotropic background metric (such as Bianchi type-I metrics), as discussed in the reference [33], which addresses the construction of the problem and manages to demonstrate stable solutions for the anisotropic case. Lastly, within the perturbation theory, it is possible to describe gravitational waves with the Horndeski theory, where the development resembles general relativity but in the present context, the polarization modes will be dependent on the Horndeski parameters and consequently, the scalar field, potentially imposing constraints on the theory (such as the viable class), see [34].

4 Modified Theories Derived From the Horndeski Theory

The determination of the functions $G_i(\phi, X)$ can be done arbitrarily. However, certain choices may be more convenient for specific cases under study (as mentioned above, here the choice referring to *Viable Class* will be taken). We will collect the Horndeski Lagrangian through the functions of a and $\dot{a}$, obtaining:

$$\mathbf{L} = -a^3\left(\frac{1}{3}\dot{\phi}^3 G_{3\dot{\phi}\phi} + \dot{\phi}^2 G_{3\phi} - G_2\right) - a^2\dot{a}\left(\dot{\phi}^2 G_{3\dot{\phi}} + 6\dot{\phi}G_{4\phi}\right) - 6a\dot{a}^2 G_4, \tag{4.1}$$

for simplicity, we evaluate the cosmological dynamics without the inclusion of matter fields, namely in vacuum. Though the universe can be thought of as filled with dust matter, this *ansatz* is still physically viable, as we are interested in cosmological epochs in which geometric contributions were predominant with respect to matter ones.

Now a mapping will be made between the Horndeski Lagrangian and the Lagrangians of already known modified theories. To do so, the Horndeski parameters will be found with algebraic relationships for each of the theories. It is worth noting that this mapping is taking into account only the cosmological context, that is, restricting the study only to the Friedmann-Lemaître-Robertson-Walker metric.

4.1 Brans-Dicke gravity

Starting from the action suggested by Brans and Dicke (2.16),

$$\mathcal{S}_{BD} \sim \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_a\phi\nabla^a\phi - V(\phi)\right] + \mathcal{S}^{(m)}, \tag{4.2}$$

in order to avoid a term with dependence ϕ^{-1}, let us rescale the scalar field as follows (see [35])

$$\phi = f(\varphi), \tag{4.3}$$

$$\mathcal{S} \sim \int d^4x\sqrt{-g}\left[f(\varphi)R - \frac{\omega(f(\varphi))}{f(\varphi)}\nabla_a f(\varphi)\nabla^a f(\varphi) - V(\varphi)\right] + \mathcal{S}^{(m)}, \tag{4.4}$$

where we also redefine ω as $\omega(f(\varphi)) = \frac{1}{2}\frac{f(\varphi)}{f'(\varphi)^2}$ (note that, because of this definition, $f(\varphi)$ cannot take on constant values). Thus, the action will take on a more complete form because the coupling factor between gravity and the scalar field, which was previously just a scalar, becomes a function of the scalar field

$$\mathcal{S} \sim \int d^4x\sqrt{-g}\left[f(\varphi)R - \frac{1}{2}\nabla_a\varphi\nabla^a\varphi - V(\varphi)\right] + \mathcal{S}^{(m)}. \tag{4.5}$$

Therefore, the Lagrangian for this theory, in a vacuum, is

$$L = -12f(\varphi)a\dot{a}^2 - 12\dot{a}f'(\varphi)\dot{\varphi}a^2 - a^3\left(\dot{\varphi}^2 + 2V(\varphi)\right). \tag{4.6}$$

Comparing the Lagrangian (4.6) with the Lagrangian (4.1), the following equations are found

$$\begin{aligned}
&6G_4 = 12f(\varphi),\\
&\frac{1}{3}\dot{\varphi}G_{3\dot{\varphi}\varphi} + \varphi G_{3\varphi} - G_2 = \varphi^2 + 2V(\varphi),\\
&\dot{\varphi}^2G_{3\dot{\varphi}} + 6\dot{\varphi}G_{4\varphi} = 12f'(\varphi)\dot{\varphi},
\end{aligned}$$

where Horndeski's parameters that solve these equations are

$$G_2 = \varphi g'(\varphi) - 2V(\varphi) - \dot{\varphi}^2, \qquad G_3 = g(\varphi), \qquad \text{and} \qquad G_4 = 2f(\varphi). \tag{4.7}$$

$g(\varphi)$ is an arbitrary function that comes from the fact that $G_{3\dot{\varphi}} = 0$ for these equations.

The following are the equations of motion arising from the action (4.4), respecting the entire rescaling of the scalar field. Note that $V(\varphi)$ can be chosen to add a cosmological constant to the theory, with this choice being $V(\varphi) = \Lambda\varphi$, however, at this moment, the potential term will be set to zero. For simplicity we define $f(\varphi) = \varphi$

$$f(\varphi)\left[\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\right] + f'(\varphi)\left[2\frac{\dot{a}}{a}\dot{\varphi} + \ddot{\varphi}\right] + \dot{\varphi}f''(\varphi) - \frac{1}{4}\dot{\varphi}^2 = 0, \tag{4.8}$$

$$6f'(\varphi)\left[\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right] + 3\frac{\dot{a}}{a}\dot{\varphi} + \ddot{\varphi} = 0, \tag{4.9}$$

$$12f(\varphi)a\dot{a}^2 + 12\dot{a}f'(\varphi)\dot{\varphi}a^2 + a^3\dot{\varphi}^2 = 0. \tag{4.10}$$

Therefore, the Brans-Dicke theory is a particular case of the Horndeski theory and, consequently, all the improvements of the former concerning the gravitational tests at large scales can be also reproduced by the latter.

4.2 $f(R)$ Gravity

To recover the $f(R)$ theory from the Horndeski theory, we can choose any function of the Ricci scalar. However, here we will make the choice that assimilates the Brans-Dicke theory, as it allows us to make comparisons with previous results and avoid falling into terms with very large powers of R, thus introducing complications beyond the scope of this work. Let us assume a potential term $V(\phi)$ and that R has a functional dependence $R(\phi)$, such that we can define $f'(R) \equiv \phi$. Consequently, we can find an equivalent form of the previous action for this theory (2.20) as follows

$$S \sim \int d^4x\sqrt{-g}\left[\phi R - V(\phi)\right]. \tag{4.11}$$

The result is very similar to the Brans-Dicke theory, differing only in the absence of the kinetic term for the scalar field ϕ.

Now, we need to recover $f(R)$ gravity within the framework of Horndeski's theory. Consider the $f(R)$ Lagrangian given by Eq. (4.11) for a FLRW metric:

$$L = -6a\dot{a}^2\phi - 6\dot{a}a^2\dot{\phi} - a^3V(\phi). \tag{4.12}$$

Comparing with (4.1), we obtain

$$\begin{aligned}
&6G_4 = 6\phi,\\
&\frac{1}{3}\dot{\phi}G_{3\dot{\phi}\phi} + \phi G_{3\phi} - G_2 = V(\phi),\\
&\dot{\phi}^2G_{3\dot{\phi}} + 6\dot{\phi}G_{4\varphi} = 6\dot{\phi},
\end{aligned}$$

resulting in

$$G_2 = \phi f'(\phi)V(\phi), \qquad G_3 = f(\phi), \qquad \text{and} \qquad G_4 = \phi, \tag{4.13}$$

The equations of motion generated by the Lagrangian are, respectively

$$\frac{\dot{a}^2}{a^2}\phi + \frac{2\dot{a}}{a}\dot{\phi} + \ddot{\phi} - 3a^2V(\phi) = 0, \tag{4.14}$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{1}{6}\frac{\partial V(\phi)}{\partial\phi}, \tag{4.15}$$

$$\frac{\dot{a}^2}{a^2}\phi + \frac{\dot{a}}{a}\dot{\phi} = \frac{V(\phi)}{6}. \tag{4.16}$$

Note that, if we consider the potential equal to $V(\phi) = V_0\phi$, we return to General Relativity with a cosmological constant and the De Sitter solution can be recovered.

4.3 Cubic Galileon model gravity

The results found here will be of great utility in the next section. So, let us recall the action that brings forth the class of Cubic Galileon theories

$$\mathcal{S} \sim \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{k_1}{2}\tilde{\nabla}_a\phi\tilde{\nabla}^a\phi - \frac{k_2}{2M^2}\tilde{\nabla}_a\phi\tilde{\nabla}^a\phi\tilde{\Box}\phi\right]. \tag{4.17}$$

After some integration by parts, the Lagrangian linked to the last action in vacuum is

$$L = -6a\dot{a}^2 + \left(\frac{k_1}{2}a^3 - \frac{k_2}{M^2}a^2\dot{a}\right)\dot{\phi}, \tag{4.18}$$

Proceeding along the same steps as before, and comparing the Lagrangians (4.1) and (4.18), we end up with

$$\begin{aligned}
&6G_4 = 6,\\
&\frac{1}{3}\dot{\phi}G_{3\dot{\phi}\phi} + \phi G_{3\phi} - G_2 = -\frac{k_1}{2}\dot{\phi},\\
&\dot{\phi}^2 G_{3\dot{\phi}} + 6\dot{\phi}G_{4\phi} = \frac{k_2}{M^2}\dot{\phi},
\end{aligned}$$

where the Horndeski parameters that solve these equations are

$$G_2 = \frac{k_1}{2}\dot{\phi}, \qquad G_3 = \frac{k_2}{M^2}\ln(\dot{\phi}), \quad \text{and} \quad G_4 = 1. \tag{4.19}$$

The Euler-Lagrange equations and the zero energy condition determine the following equations of motion

$$\frac{3}{2}k_1\frac{\dot{a}}{a} - \frac{k_2}{M^2}\left(2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = 0, \tag{4.20}$$

$$12\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\right) + \frac{k_2}{M^2}\left(2\frac{a}{\dot{a}}\dot{\phi} + \ddot{\phi} - \dot{\phi}\right) = 0, \tag{4.21}$$

$$\frac{6\dot{a}^2}{a^2} - \frac{k_2}{M^2}\frac{\dot{a}^2}{a}\dot{\phi} = 0. \tag{4.22}$$

In principle, this system can be easily solved for $\phi(t)$ and $a(t)$ for certain values of k_1, k_2, and M since we have three equations and only two variables.

4.4 String Motivated Gravity

Finally, it is worth bringing up an attempt to address the possible quantum nature of gravity. We will delve into the details of this attempt in the next chapter, but it is worth mentioning here a theory that has gained prominence in this endeavor. String gravity has been proving to be the bridge for the encounter between quantum mechanics and gravitation, having its space in the attempt to find alternatives to theories of gravitation, as well as the others discussed here.

The Lagrangian of the respective theory takes the form [29]

$$L = e^{-2\phi}\left[12a^2\dot{a}\dot{\phi} - 6a\dot{a}^2 - a^3\left(4\dot{\phi}^2 + V(\phi)\right)\right]. \tag{4.23}$$

Note that the global term $e^{-2\phi}$ indicates that the theory is described in the Jordan frame. In order for the study to be consistent, a conformal transformation must be performed on the metric with the intention of finding the Lagrangian described in the Einstein frame [36].

The action responsible for (4.23) is

$$S \sim \int d^4x\sqrt{-g}e^{-2\phi}\left[R - 4\nabla_a\phi\nabla^a\phi - V(\phi)\right]. \tag{4.24}$$

By taking the transformation

$$g_{\mu\nu} = e^{\phi}\tilde{g}_{ab} \tag{4.25}$$

the system will go from the Einstein frame, represented by $\tilde{g}_{ab}$, to the Jordan frame $g_{\mu\nu}$. Applying the inverse transformation to the action, rescaling the quantities involved, and considering $\phi = \phi(t)$, we get

$$\tilde{R} = e^{-2\phi}R, \quad \dot{\tilde{\phi}} = e^{-\phi}\dot{\phi}, \quad \tilde{V}(\tilde{\phi}) = e^{-2\phi}V(\phi). \tag{4.26}$$

Thus, the Lagrangian in the Einstein frame is determined to be

$$L = -6a\dot{a}^2 - a^3\left(4\dot{\tilde{\phi}}^2 + \tilde{V}(\tilde{\phi})\right). \tag{4.27}$$

Comparing it with (4.1), we obtain

$$\begin{aligned} &6G_4 = 6, \\ &\frac{1}{3}\dot{\phi}G_{3\dot{\phi}\phi} + \phi G_{3\phi} - G_2 = 4\dot{\tilde{\phi}}^2 + \tilde{V}(\tilde{\phi}), \\ &\dot{\phi}^2 G_{3\dot{\phi}} + 6\dot{\phi}G_{4\phi} = 0, \end{aligned}$$

and, therefore,

$$G_2 = \tilde{\phi}f'(\tilde{\phi}) - 4\dot{\tilde{\phi}} - \tilde{V}(\tilde{\phi}), \qquad G_3 = f(\tilde{\phi}), \quad \text{and} \quad G_4 = 1, \tag{4.28}$$

where $f(\tilde{\phi})$ is a general function of the rescaled scalar field.

Again, from the Euler-Lagrange equations and the null energy condition we find the following equations

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} - \frac{1}{2}\left(4\dot{\tilde{\phi}}^2 + V_0\right) = 0, \tag{4.29}$$

$$\ddot{\phi} = 24\frac{\dot{a}}{a}\dot{\tilde{\phi}}^2, \tag{4.30}$$

$$6\frac{\dot{a}^2}{a^2} + 4\dot{\tilde{\phi}}^2 = 0. \tag{4.31}$$

Once again, it was possible to map a scalar-tensor theory into a particular case of the Horndeski models.

5 Towards quantum cosmology in Horndeski models

In this final chapter, we will briefly delve into the area known as quantum cosmology. We shall start by studying the exciting model called Cubic Galileon, as shown before, and then, we will prepare some mathematical tools to expand it for the case of Horndeski models, in a forthcoming work.

The attempt to quantize gravity arises from the need to explain the primordial universe. Just as very large scales are referred to as cosmological scales, the primordial universe occurs on the well-known Planck scale. At this scale, it is understood that the universe was extremely hot and dense, with unimaginable energies, thus bringing about quantum effects that conventional gravity theories are not prepared to foresee. The fact that gravity is an exclusively attractive interaction requires greater care and hinders the easy formulation of a quantum theory. Other problems arise when the quantum description of gravitation comes into play; the notion of time is lost, and another quantity must be found to serve as a parameter for the system's evolution (we will go into more details shortly). Another detail about the theory still under discussion is the spin of the theory. It is believed that the graviton, the messenger particle of gravitational interaction, has spin 2. However, some studies allow for greater degrees of freedom in its description, consequently accepting theories with a spin different from 2.

Here, we will address the 3+1 decomposition found in the ADM formalism, which generates constraints and equations for a quantum theory of gravity. We will apply the Horndeski theory, generating the Wheeler-DeWitt equation for the theory. Also, it is worth mentioning the change in our notation. Throughout the manuscript, we have used Latin letters as space-time indices. However, here, as we will separate space and time, it will be useful to introduce Greek letters as space-time indices and Latin letters as purely spatial indices. For clarity, $\mu, \nu, \alpha ... = 0, 1, 2, 3$ and $a, b, c ... = 1, 2, 3$. The entire procedure of the 3+1 split and more can be found in the references [37, 38].

The ADM formalism is based on the Hamiltonian description of a system, developed by R. Arnowitt, S. Deser, and C. W. Misner in 1959 [39]. In this formalism, the notion of time as just another coordinate of space-time is ignored, and a complete separation of spatial hypersurfaces from the "time" parameter is formulated, known as 3+1. In other words, we take the 4-metric and separate it into spatial hypersurfaces parametrized with tangent vectors $\partial_\mu t$. With this separation, it is possible to express the metric $g_{\mu\nu}$ of any space-time in terms of a 3-metric h_{ij}, the lapse function N, and the shift vector N^i.

Our endeavor consists of finding the Ricci scalar in this new formalism, considering that it will be used in all theories of gravity, whether in the Einstein-Hilbert action or modified theories. The key is to identify the constraints and restrictions that will apply to any theory. Therefore, we define the 3-metric in its 4-dimensional form

$$h^{\mu\nu} = g^{\mu\nu} + n^{\mu}n^{\nu}, \tag{5.1}$$

where $n_\mu = N\delta^0_\mu$ with the lapse function arising as a normalization term that makes the norm of n^μ equal to 1. The non-zero matrix presented above refers to the projector onto the spatial hypersurfaces and can be expressed as

$$h^{ij} = g^{ij} + N^2 g^{0i}g^{0j}. \tag{5.2}$$

Now we can define the metric in terms of N, N^i, and h_{ij}

$$g^{\mu\nu} = \begin{pmatrix} 1/N^2 & N^i/N^2 \\ N^j/N^2 & h^{ij} - N^iN^j/N^2 \end{pmatrix}. \tag{5.3}$$

With the metric tensor in hand, it is possible to express the line element for any metric in the 3+1 split

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right). \tag{5.4}$$

To have all the mathematical objects describing space-like surfaces, we need to define the extrinsic curvature, which, as the name suggests, measures the curvature of hypersurfaces with respect to the space-time in which they are embedded

$$K_{ij} \equiv h^a_i h^b_j \nabla_a n_b. \tag{5.5}$$

So, to reproduce the gravitational part of a theory, i.e., $\sqrt{-g}R$, first, it is necessary to find the determinant of the metric tensor in the 3+1 formalism. That is

$$\sqrt{-g} = N\sqrt{h}. \tag{5.6}$$

Then, the corresponding curvature scalar, which after an extensive calculation, results in

$$R = {}^{(3)}R - K^2 + K_{ij}K^{ij}, \tag{5.7}$$

where ${}^{(3)}R$ is the Ricci scalar corresponding to the 3-metric and $K = h^{ij}K_{ij}$. It is worth noting that here it is not considered surface terms that would contribute to the Ricci scalar. Consequently, the action for the gravitational sector without modifications is given by

$$\mathcal{S}_{EH-ADM} = \int d^4x N\sqrt{h}\frac{M_P^2}{2}\left[{}^{(3)}R - K^2 + K_{ij}K^{ij}\right] \tag{5.8}$$

To achieve a quantization of the theory and find the Wheeler-DeWitt equation, it is necessary to determine the Hamiltonian of the system, which in turn arises from the conventional Legendre transformation $H = \pi_q \dot{q} - L$, where q stands for the canonical coordinates of the system, and π_q represents their canonically conjugate momenta, defined as follows

$$\pi_q = \frac{\delta L}{\delta \dot{q}}, \tag{5.9}$$

where dot means the derivative with respect to the evolution parameter that is not necessarily the time coordinate. In this case, the canonically conjugate momenta would be

$$\pi^{ij} = \frac{\delta L}{\delta \dot{h}_{ij}}, \tag{5.10}$$

$$\pi^{i} = \frac{\delta L}{\delta \dot{N}_{i}}, \tag{5.11}$$

$$\pi^{0} = \frac{\delta L}{\delta \dot{N}}. \tag{5.12}$$

By performing the necessary calculations, it is possible to arrive at the following result

$$\pi^{ij} = \frac{M_P^2}{2}\sqrt{h}\left(K^{ij} - h^{ij}K\right), \tag{5.13}$$

$$\pi^{i} = 0, \tag{5.14}$$

$$\pi^{0} = 0. \tag{5.15}$$

We can then conclude that the lapse function and the shift vector are not dynamic variables of the system but rather constraints and appear as Lagrange multipliers. Now we are ready to obtain the Hamiltonian, which after a Legendre transformation assumes the following form

$$H = \int d^3x \left[N\mathcal{H} + N_i\mathcal{H}^i\right], \tag{5.16}$$

where

$$\mathcal{H} = \frac{1}{2M_P^2}\left(K^{ij}K_{ij} - K^2 - {}^{(3)}R\right)\sqrt{h}, \tag{5.17}$$

$$\mathcal{H}^i = \frac{1}{M_P^2}D_j\left(K^{ij} - h^{ij}K\right)\sqrt{h}. \tag{5.18}$$

From this, we can extract two important links for the quantization of the theory. Taking the functional derivatives with respect to the lapse function and the shift vector, we find the following identities

$$\frac{\delta H}{\delta N} = \mathcal{H} = 0, \tag{5.19}$$

$$\frac{\delta H}{\delta N_i} = \mathcal{H}^i = 0. \tag{5.20}$$

These relations constitute constraints of the theory, and by construction, they are valid for any theory, playing a crucial role in Dirac's quantization scheme. In particular, the equation (5.19) called Hamiltonian constraint, and the equation (5.20) is the momentum constraint[1].

[1] In several previous parts of this manuscript, the equation (5.19) was referred to as the energy condition.

This implies that, when we take the quantization of the theory, that is, imposing that the Poisson brackets become commutators and the variables become operators, we will have the following Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = \hat{H}\Psi. \tag{5.21}$$

There will be no temporal dependence, as $\hat{H}\Psi = 0$, and what we interpret as time truly disappears; we will not have dynamics in the system. It remains to incorporate the notion of time into another parameter, as we expected from the beginning, and this will become clearer later on.

To add a scalar field that may be related to gravity, as seen in modified theories, or it could be a matter field commonly used to describe a universe with matter, the action for this addition can be seen as

$$\mathcal{S}_\phi = \int d^4x\sqrt{-g}L_\phi\left(\phi, \partial_\mu\phi, g_{\mu\nu}\right) = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{5.22}$$

where it presents a kinetic term for the field and a potential term, thus allowing for dynamics for this field. Writing the last equation in terms of N, N^i and h^{ij}, we get

$$\mathcal{S}_\phi = \int N\sqrt{h}\left[-\frac{1}{2}\left(g^{00}\dot{\phi}^2 + 2g^{i0}\dot{\phi}\partial_i\phi + g^{ij}\partial_i\phi\partial_j\phi\right) - V(\phi)\right]dx^4 \tag{5.23}$$

$$= \int dx^4 N\sqrt{h}\left[-\frac{1}{2}\left(-N^{-2}\dot{\phi}^2 + 2N^{-2}N^i\dot{\phi}\partial_i\phi\right.\right. \tag{5.24}$$
$$\left.\left. + \left(h^{ij} - N^{-2}N^iN^j\right)\partial_i\phi\partial_j\phi\right) - V(\phi)\right].$$

Therefore, the Lagrangian of a theory that modifies general relativity by adding a scalar field is

$$\mathcal{S} = \int d^4xN\sqrt{h}\left\{\frac{M_P^2}{2}\left[{}^{(3)}R - K^2 + K_{ij}K^{ij}\right] - \frac{1}{2}\left[-N^{-2}\dot{\phi}^2 + 2N^{-2}N^i\dot{\phi}\partial_i\phi\right.\right. \tag{5.25}$$
$$\left.\left. + \left(h^{ij} - N^{-2}N^iN^j\right)\partial_i\phi\partial_j\phi\right) - V(\phi)\right].$$

Even taking a different form, we still have the energy condition valid, thus allowing us to proceed with the canonical quantization of the system.

Our proposal now is to find the Hamiltonian for the Horndeski theory. This procedure could start with the Lagrangian (3.12), but since it will be necessary to take many derivatives with respect to the scalar field and its derivative, we choose to explicitly express the Horndeski parameters to get an understanding of the process and in future works return to the complete theory. This parameterization choice is contained in equation (4.19), which ensures the reproduction of the Lagrangian (4.18) related to the Cubic Galileon theory, one of the most promising modified theories currently. In this point do not worry about the matter term in (3.12).

The canonical momentum of each coordinate $(a(t), \phi(t))$, yet in Horndeski Lagrangian, are

$$\pi_a = \frac{\partial L}{\partial \dot{a}} = -a \left[a\dot{\phi}^2 G_{3\dot{\phi}} + 6a\dot{\phi}G_{4\phi} + 12\dot{a}G_4 \right], \tag{5.26}$$

$$\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{a}{3} \left[a^2\dot{\phi}^3 G_{3\phi\dot{\phi}\dot{\phi}} + 3a\dot{a}\dot{\phi}^2 G_{3\dot{\phi}\dot{\phi}} + 6a^2\dot{\phi}^2 G_{3\phi\dot{\phi}} + 6a^2\dot{\phi}G_{3\phi} \right. \tag{5.27}$$
$$\left. +18a\dot{a}\phi G_{4\phi\dot{\phi}} + 6a\dot{a}\dot{\phi}G_{3\dot{\phi}} + 18\dot{a}^2 G_{4\dot{\phi}} + 18a\dot{a}G_{4\phi} - 3a^2 G_{2\dot{\phi}} \right],$$

where the last one can be simplified with $G_{4\dot{\phi}} = 0$ because we are in viable class of Horndeski theories. Thus,

$$\pi_\phi = -\frac{a}{3} \left[a^2\dot{\phi}^3 G_{3\phi\dot{\phi}\dot{\phi}} + 3a\dot{a}\dot{\phi}^2 G_{3\dot{\phi}\dot{\phi}} + 6a^2\dot{\phi}^2 G_{3\phi\dot{\phi}} + 6a^2\dot{\phi}G_{3\phi} \right. \tag{5.28}$$
$$\left. +6a\dot{a}\dot{\phi}G_{3\dot{\phi}} + 18a\dot{a}G_{4\phi} - 3a^2 G_{2\dot{\phi}} \right].$$

As already discussed, using the parameters found in (4.19), we obtain

$$\pi_a = -a\left(\frac{a\dot{\phi}k_2}{M^2} + 12\dot{a}\right), \tag{5.29}$$

$$\pi_\phi = -\frac{a^2}{3}\left(\frac{\dot{a}k_2}{M^2} - \frac{ak_1}{2}\right). \tag{5.30}$$

The two equations above form a system of equations that will be used to determine $\dot{a}$ and $\dot{\phi}$ in terms of the canonically conjugate momenta. Therefore

$$\dot{a} = \frac{M^2}{k_2}\left(\frac{ak_1}{2} - \frac{\pi_\phi}{a^2}\right), \tag{5.31}$$

$$\dot{\phi} = \frac{M^2}{k_2}\left[-\frac{12M^2}{k_2}\left(\frac{k_1}{2} - \frac{\pi_\phi}{a^3}\right) - \frac{\pi_a}{a^2}\right]. \tag{5.32}$$

With the temporal derivatives of the coordinates in hand, one can write the Lagrangian in terms of the coordinates and their conjugate momenta

$$L(a, \phi, \pi_a, \pi_\phi) = \frac{6M^4\pi_\phi^2}{a^3k_2^2} - \frac{M^2\pi_a\pi_\phi}{a^2k_2} - \frac{3M^4k_1^2a^3}{2k_2}, \tag{5.33}$$

and with the corresponding Legendre transformation, $H = \dot{a}\pi_a + \dot{\phi}\pi_\phi - L$, the Hamiltonian of the system is determined

$$H = \frac{3M^4k_1^2a^3}{2k_2^2} + \frac{6M^4\pi_\phi^2}{k_2^2a^3} + \frac{M^2k_1a\pi_a}{2k_2} - \frac{M^2\pi_a\pi_\phi}{k_2a^2} - \frac{6M^4k_1\pi_\phi}{k_2^2}. \tag{5.34}$$

Before we take canonical quantization, we will add the matter term, corresponding to a perfect fluid of action

$$\mathcal{S}_{matter} = \int d^4x\sqrt{-g}p, \tag{5.35}$$

where p is the pressure of the fluid, which is related to the energy density ρ through the equation of state $p = \omega\rho$. Using Schutz's formalism [40] and some thermodynamic arguments, we can write the well-known super-Hamiltonian of matter as [41, 42, 43]

$$H_{matter} = -\frac{\pi_T}{a^{3\omega}}, \tag{5.36}$$

where π_T is the canonical momentum related to the entropy of the system. As the entropy of a system never decreases, we can take the parameter related to T as the new temporal parameter, leaving behind the problem of time. Thus, adding the super-Hamiltonian of matter to the previously found Hamiltonian, we have

$$H = \frac{3M^4k_1^2a^3}{2k_2^2} + \frac{6M^4\pi_\phi^2}{k_2^2a^3} + \frac{M^2k_1a\pi_a}{2k_2} - \frac{M^2\pi_a\pi_\phi}{k_2a^2} - \frac{6M^4k_1\pi_\phi}{k_2^2} - \frac{\pi_T}{a^{3\omega}}. \tag{5.37}$$

With the energy condition determined in (5.19), specifying that each of the canonically conjugate momenta behaves as operators and follows the relation $\hat{\pi}_q \longrightarrow -i\frac{\partial}{\partial q}$, we obtain

$$\left[\frac{3M^2k_1^2a^3}{2k_2} - \frac{6M^2}{k_2a^3}\frac{\partial^2}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial^2}{\partial a\partial\phi} - i\left(\frac{k_1a}{2}\frac{\partial}{\partial a} - \frac{6M^2k_1}{k_2}\frac{\partial}{\partial\phi}\right)\right]\Psi\left(a,\phi,T\right) \tag{5.38}$$
$$= -i\frac{k_2}{M^2a^{3\omega}}\frac{\partial}{\partial T}\Psi\left(a,\phi,T\right),$$

Finally, this is the Wheeler-DeWitt equation for the system. Yet, the Hamiltonian (5.37) can be simplified, as follows

$$H = \frac{6M^4}{k_2^2a^2}\left(\pi_\phi - \frac{k_1a^3}{2}\right)^2 - \frac{M^2\pi_a}{k_2a^2}\left(\pi_\phi - \frac{k_1a^3}{2}\right) - \frac{\pi_T}{a^{3\omega}}, \tag{5.39}$$

and defining $\pi_\Phi \equiv \pi_\phi - \frac{k_1a^3}{2}$ a more compact form of the Wheeler-DeWitt equation

$$\left(-\frac{6M^2}{k_2}\frac{\partial^2}{\partial\Phi^2} + \frac{\partial^2}{\partial\Phi\partial a}\right)\Psi\left(a,\Phi,T\right) = -i\frac{k_2a^{2-3\omega}}{M^2}\frac{\partial}{\partial T}\Psi\left(a,\Phi,T\right). \tag{5.40}$$

For the near future, our goal will be to solve this equation and try to interpret its solutions in the context of a very hot and dense Universe, close to the Planck scale. Previous works on this issue have indicated that nonsingular solutions for the wave function of the universe can be easily found [37], healing the initial singularity problem of cosmology. After completing the procedure, now with the experience gained in dealing with the subject in the light of Cubic Galileon theory, we can return to the challenge of obtaining the Wheeler-DeWitt equation for Horndeski theory and finding physically meaningful solutions.

6 Conclusion

The theory of gravitation has been evolving over time, where theory and observation have been closely intertwined. The agreement between the two branches was successful while studies were limited to everyday scales, as instruments were not yet capable of measuring at high energy or large distance scales. However, at the beginning of the 20th century, the technological revolution had an incredibly impactful effect on the understanding of gravity, leading to the need for a new way to conceptualize gravity: General Relativity. The harmony between theory and observation was disrupted due to the enhanced accuracy of equipment, and it didn't take long for Einstein's theory to be strongly tested beyond the solar system, revealing incompleteness.

As a result, in less than half a century, a set of theories extending the understanding of General Relativity using a scalar field as an additional degree of freedom emerged, giving rise to scalar-tensor theories. Within this set, a class can be determined that maintains second-order equations of motion. Within this class, the Brans-Dicke theory was discussed, which accurately represents General Relativity when its Brans-Dicke parameter tends to infinity but still falls short in explaining phenomena unexplained by relativity. The discussion continues to $f(R)$ theories, where choosing a nonlinear function of the Ricci tensor introduces more degrees of freedom, useful for explaining moments in the primordial universe or the current scenario of cosmic expansion. This theory is versatile and extensively debated in the literature. When seeking a theory that doesn't require theorizing exotic matter to explain galaxy rotation and exotic energy for the accelerated expansion of the universe, the Galileon theory, particularly the Cubic Galileon, stands out. However, when talking about a second-order scalar-tensor theory, it is necessary to mention the most general one, theorized by Gregory Horndeski, which involves scalar field functions and their derivatives, making it possible to recover any other second-order scalar-tensor theory and significantly increasing versatility for theoretically describing data. This theory is one of the most widely used currently to address the Hubble constant tension problem.

It is clear that a definitive theory for gravitation does not yet exist, and many are in search of it. Much debate surrounds the form a theory should take to describe the current universe and its phase of accelerated expansion, as well as the form of the theory for the inflationary universe. However, on very small scales, smaller than an atom, there is a certain agreement that gravitational theory must accept quantization. Yet, more problems arise when attempting to quantize gravity, such as boundary problems in the quantum/classical transition and the loss of the notion of temporal coordinate, as discussed in the previous chapter on quantum cosmology. With this loss of the interpretation of what the system's evolution parameter would be, null probabilities could arise when interpreting the wave function generated by canonical quantization of the system. However, using thermodynamics, it is possible to determine the temporal arrow using the concept of entropy, thereby reintroducing an evolution parameter into the theory and allowing for the correct determination of probabilities and observables. Future efforts should precisely follow the path of quantum cosmology, applying the techniques discussed here to delve into deeper issues.

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