Federal University of Itajubá
Institute of Physics and Chemistry
Postgraduate Program in Physics

The Fell-Heisenberg Warp Drive and the horizon problem

Victor Manuel Neyra Salvador

Itajubá - MG, 22nd February of 2024

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Master's Thesis presented to the Postgraduate Program in Physics at UNIFEI as part of the requirements necessary to obtain the Master's Degree in Physics.

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For Bianca and Anguely

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Resumo

O princípio da localidade é respeitado pela teoria de relatividade especial e geral. Em consequência, a velocidade da luz é a máxima velocidade que a informação pode viajar. No entanto, pela lei de Hubble-Lemaitre nos sabemos que dois observadores comoveis podem se separar com velocidades maiores que a velocidade da luz. Além disso, a teoria da inflação cosmica tambem implica o movimento superluminal de observadores comoveis no primeiros instantes do nosso Universo [8]. Com esta motivação física, Alcubierre [2] construiu uma solução da relatividade geral que permite viagens superluminais sem violar a causalidade. Sua solução inspirou a definição matemática geral de warp drives dada por Natario [22] e o comportamento solitônico de Fell-Heisenberg [10]. Nesta tese, focaremos em duas questões: primeiro, com base na definição de Natario, estudamos o "problema do observador" em warp drives, e segundo, estabelecemos restrições para vetores de deslocamento do tipo gradiente para representar warp drives físicos na métrica de Fell-Heisenberg.

Palavras-chave: relatividade geral, 3+1 formalismo, *warp drives*.

Abstract

The locality principle is respected by the special and general theory of relativity. Consequently, the speed of light is the maximum speed at which information can travel. However, by the Hubble-Lemaitre law, we know that two moving observers can separate at speeds greater than the speed of light. Furthermore, the theory of cosmic inflation also implies the superluminal movement of comoving observers in the first moments of our Universe [8]. With this physical motivation, Alcubierre [2] constructed a solution of general relativity that allows superluminal travels without violating causality. His solution inspired the general mathematical definition of warp drives given by Natario [22] and the solitonic behavior of Fell-Heisenberg [10]. In this thesis, we will focus on two questions: first, based on Natario's definition, we study the "observer problem" in warp drives, and second, we establish constraints for gradient shift vectors to represent physical warp drives in the Fell-Heisenberg metric.

Keywords: general relativity, 3+1 formalism, warp drives.

List of Figures

Contents

1 General relativity and its mathematical tools

The theory of general relativity postulated by Einstein in 1915 is a highly successful theory. To date, this theory is supported by numerous observational tests [1] [8] [13] [21] [27] [31]. The existence of black holes and the existence of gravitational waves are some of its most successful predictions of the theory of general relativity [21]. In other words, the most accurate and predictive theory of gravity that exists to date is the theory of general relativity (also known simply as general relativity) [8] [13] [21] [27] [31].

As we also know, weak gravitational fields can be studied very well (with a high level of precision) by the Newtonian theory of gravitation. However, if we are dealing with strong gravitational fields (e.g. gravitational fields generated by black holes, and neutron stars) general relativity is necessary. In this thesis, we will study hypothetical objects known as warp drives. And these warp drives are related to strong gravitational fields [2] [3]. This is the reason why the study of general relativity will be necessary.

Also, it is important to mention that in this thesis the so-called "geometric units" will be used, where the speed of light c and Newton's universal gravitational constant G are both equal to unity.

1.1 Mathematical tools

General relativity (as we will see later) is based on a series of principles and postulates. Furthermore, as we will also show later, general relativity is a geometric theory of gravity [21]. Therefore, it will use sophisticated concepts of tensor calculus and differential geometry [13] [31].

Differential geometry and tensor calculus itself are very extensive and complex [8] [13] [21] [27] [31]. However, in this thesis, we will only mention the essential concepts to understand the topics of general relativity that we will need. Also, we will always take the component 0 as the one that refers to time, and the components $\{1, 2, 3\}$ those that refer to space. We will also use the usual convention: Latin indices take values from 1 to 3, while, Greek indices take values from 0 to 3 [1] [8] [21] [27] [30] [31]. Furthermore, we will assume that the reader already has some familiarity with some basic concepts of differential geometry, tensor calculus, and special relativity.

1.1.1 Metric tensor

On a differentiable manifold M we can define the tensor $g_{\mu\nu}$. To do this in the most natural way, let's look at the vector $\vec{v}$ defined within the manifold M. This vector can be expressed as follows (remembering the Einstein's summation convention):

$$\vec{v} = v^{\alpha} e_{\alpha} = v_{\alpha} e^{\alpha}, \tag{1.1}$$

where v^{α} and v_{α} are the contravariant and covariant components of vector $\vec{v}$, respectively. Likewise, e_{α} is a contravariant basis defined in $T_P M$ (tangent space at point $P \in M$). And in the same way, e^{α} is a covariant basis defined in $T_P^* M$ (cotangent space at point $P \in M$) [8]. The basis of the covariant vectors e^{α} and the basis of the contravariant vectors e_{α} are related as follows:

$$e^{\alpha} \cdot e_{\beta} = \delta^{\alpha}_{\beta}. \tag{1.2}$$

Furthermore, if we have a point $P \in M$ expressed in two different coordinates, in this case x^{μ} and $\bar{x}^{\mu}$, then v^{α} and v_{α} obey the following transformation rules of a tensor $(1,0)$ and $(0,1)$, respectively [1]:

$$\bar{v}^{\mu}(\bar{x}) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} v^{\nu}(x), \tag{1.3}$$

$$\bar{v}_{\mu}(\bar{x}) = \frac{\partial x^{\nu}}{\partial \bar{x}^{\mu}} v_{\nu}(x). \tag{1.4}$$

Now let's consider the inner product of two vectors $\vec{v}$ and $\vec{u}$. This inner product is defined in the following way:

$$\vec{v} \cdot \vec{u} = g_{\alpha\beta} v^{\alpha} u^{\beta}, \tag{1.5}$$

where $g_{\alpha\beta}$ is called the "metric tensor". Also, from the expression (1.1) we also have the following:

$$\vec{v} \cdot \vec{u} = (v^{\alpha} e_{\alpha}) \cdot (v^{\beta} e_{\beta}) = v^{\alpha} v^{\beta} (e_{\alpha} \cdot e_{\beta}) \tag{1.6}$$

Comparing the expressions (1.5) and (1.6), we have:

$$e_{\alpha} \cdot e_{\beta} \equiv g_{\alpha\beta}. \tag{1.7}$$

It is important to mention that the basis vectors e_{α} do not need to be exclusively orthonormal. However, for convenience, we will assume that these basis vectors are orthonormal (to avoid mathematical complications that do not allow us to clearly see the physics of the problems) [8].

If we have two events with coordinates x^{α} and $x^{\alpha} + dx^{\alpha}$ infinitesimally close to each other, it is possible to define the following:

$$d\vec{s} \equiv dx^{\alpha} e_{\alpha}. \tag{1.8}$$

Now, we can do the following:

$$ds^2 = d\vec{s} \cdot d\vec{s} = (dx^{\alpha} e_{\alpha}) \cdot (dx^{\beta} e_{\beta}) = (e_{\alpha} \cdot e_{\beta}) dx^{\alpha} dx^{\beta} \tag{1.9}$$

Replacing (1.7) into (1.9), then:

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}, \tag{1.10}$$

In the expression (1.10) ds^2 is called "invariant distance". The invariant distance ds^2 is invariant under Lorentz transformations and is an absolute quantity that does not depend on the coordinate system used to describe spacetime [1].

In the expression (1.10), $g_{\alpha\beta}$ is the metric tensor. Also, we can express the metric tensor $g_{\alpha\beta}$ in a matrix form:

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}. \tag{1.11}$$

Also, from the expression (1.10) we can see that the metric tensor is symmetric, that is:

$$g_{\mu\nu} = g_{\nu\mu}. \tag{1.12}$$

It is important to mention that on a differentiable manifold M, the metric tensor $g_{\mu\nu}$ can change from one point to another. Furthermore, the eigenvalues of the metric tensor have the following signature: $(-,+,+,+)$. Here, the three positive eigenvalues $(+,+,+)$ are related to space and the negative eigenvalue $(-)$ is associated to time.

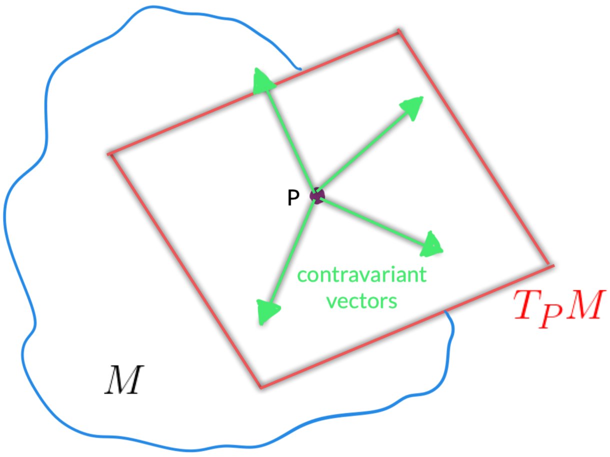

Figure 1 – Tangent space at point P on manifold M.

In a similar way to the covariant metric tensor $g_{\alpha\beta}$, the contravariant metric tensor $g^{\alpha\beta}$ is defined using the basis of the covariant vectors e^{α}, as follows:

$$e^{\alpha} \cdot e^{\beta} \equiv g^{\alpha\beta}. \tag{1.13}$$

Furthermore, the contravariant metric tensors $g^{\alpha\beta}$ and the covariant metric tensor $g_{\alpha\beta}$ can relate the vector basis $\{e^{\mu}\}$ and $\{e_{\mu}\}$, as follows:

$$e_{\beta} = g_{\beta\mu} e^{\mu}, \tag{1.14}$$

$$e^{\alpha} = g^{\alpha\mu} e_{\mu}. \tag{1.15}$$

It is possible to show that the contravariant metric tensor $g^{\alpha\beta}$ and the covariant metric tensor $g_{\alpha\beta}$ are related, Then, from (1.14) and (1.15) we have:

$$e^{\alpha} \cdot e_{\beta} = (g^{\alpha\mu} e_{\mu}) \cdot (g_{\beta\nu} e^{\nu}) = g^{\alpha\mu} g_{\beta\nu} (e_{\mu} \cdot e^{\nu}). \tag{1.16}$$

Replacing (1.2) into (1.16) we have:

$$\delta^{\alpha}_{\beta} = g^{\alpha\mu} g_{\beta\nu} \delta^{\nu}_{\mu}. \tag{1.17}$$

Indeed;

$$g^{\alpha\mu} g_{\beta\mu} = \delta^{\alpha}_{\beta}. \tag{1.18}$$

The expression (1.18) tells us that $g^{\alpha\beta}$ can be calculated as the inverse matrix of $g_{\alpha\beta}$.

Another important characteristic of the metric tensor $g_{\alpha\beta}(x)$ is that it allows relating covariant vectors v_{α} and contravariant vectors v^{α}. It does this in the following way:

$$v^{\alpha} = g^{\alpha\beta} v_{\beta}, \tag{1.19}$$

$$v_{\alpha} = g_{\alpha\beta} v^{\beta}, \tag{1.20}$$

The relations (1.19) and (1.20) can be generalized to (m,n) tensors as follows:

$$A^{a_1 a_2 \ldots a_m}_{b_1 b_2 \ldots b_n} = g^{a_1 c_1} g^{a_2 c_2} \ldots g^{a_m c_m} g_{b_1 d_1} g_{b_2 d_2} \ldots g_{b_n d_n} A^{d_1 d_2 \ldots d_n}_{c_1 c_2 \ldots c_m} \tag{1.21}$$

For two coordinates x and $\bar{x}$ defined at a certain point $P \in M$, the metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ follow the rules of transformation of the tensors of order (0,2) and (2,0), respectively:[1]

$$\bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} g_{\alpha\beta}(x), \tag{1.22}$$

$$\bar{g}^{\mu\nu}(\bar{x}) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x). \tag{1.23}$$

1.1.2 The geodesic equation

Geodesics within a given differentiable manifold M can be calculated in many ways. However, an elegant way to do it is by using the variational principle [31]. Let's define the following action I [8]:

$$I\left(x^{\alpha}, \frac{dx^{\alpha}}{d\lambda}\right) = \int_{\lambda_1}^{\lambda_2} f\left(x^{\alpha}, \frac{dx^{\alpha}}{d\lambda}\right) d\lambda, \tag{1.24}$$

where $x^{\alpha}(\lambda)$ represents a curve, parameterized with parameter λ. The values of λ_1 and λ_2 represent the extremes of curve $x^{\alpha}(\lambda)$. Now, let's consider a curve $\bar{x}^{\alpha}$ very close to curve x^{α} (see the figure (2)). Indeed:

$$\bar{x}^{\alpha} = x^{\alpha} + \varepsilon^{\alpha}, \tag{1.25}$$

where $\varepsilon^{\alpha} \ll 1$. The action I related to $\bar{x}^{\alpha}$ is given by:

[1] The first entrance in the ordered pair $(\ldots, \ldots)$ indicates the numbers of contravariant indexes while the second entrance corresponds to the number of covariant indexes.

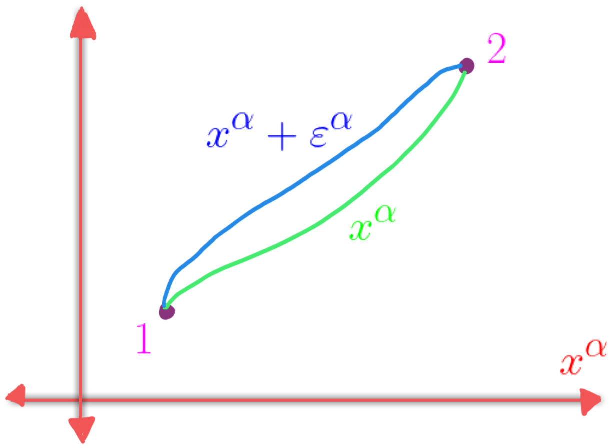

Figure 2 – Two infinitesimally close trajectories

$$I\left(x^\alpha+\varepsilon^\alpha, \frac{dx^\alpha}{d\lambda}+\frac{d\varepsilon^\alpha}{d\lambda}\right)=\int_{\lambda_1}^{\lambda_2} f\left(x^\alpha+\varepsilon^\alpha, \frac{dx^\alpha}{d\lambda}+\frac{d\varepsilon^\alpha}{d\lambda}\right)d\lambda. \tag{1.26}$$

Using the variational principle [8] [21] [29], we have:

$$\delta I = 0. \tag{1.27}$$

In our case, we have

$$\delta I = I\left(x^\alpha+\varepsilon^\alpha, \frac{dx^\alpha}{d\lambda}+\frac{d\varepsilon^\alpha}{d\lambda}\right) - I\left(x^\alpha, \frac{dx^\alpha}{d\lambda}\right)=0. \tag{1.28}$$

Replacing (1.24) and (1.26) into (1.28) we have the following:

$$\int_{\lambda_1}^{\lambda_2}\left[f\left(x^\alpha+\varepsilon^\alpha, \frac{dx^\alpha}{d\lambda}+\frac{d\varepsilon^\alpha}{d\lambda}\right)-f\left(x^\alpha, \frac{dx^\alpha}{d\lambda}\right)\right]d\lambda=0. \tag{1.29}$$

Taking into account that in the Taylor expansion that $(\varepsilon^\alpha)^2 \approx 0$ and $\left(\frac{d\varepsilon^\alpha}{d\lambda}\right)^2 \approx 0$ and using Taylor's theorem for $f\left(x^\alpha+\varepsilon^\alpha, \frac{dx^\alpha}{d\lambda}+\frac{d\varepsilon^\alpha}{d\lambda}\right)$, we have the following:

$$f\left(x^\alpha+\varepsilon^\alpha, \frac{dx^\alpha}{d\lambda}+\frac{d\varepsilon^\alpha}{d\lambda}\right)=f\left(x^\alpha, \frac{dx^\alpha}{d\lambda}\right)+\frac{\partial f}{\partial x^\alpha}\varepsilon^\alpha+\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\frac{d\varepsilon^\alpha}{d\lambda}. \tag{1.30}$$

Replacing (1.30) into (1.29), we have:

$$\int_{\lambda_1}^{\lambda_2}\left[\frac{\partial f}{\partial x^\alpha}\varepsilon^\alpha+\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\frac{d\varepsilon^\alpha}{d\lambda}\right]d\lambda=0. \tag{1.31}$$

Indeed:

$$\int_{\lambda_1}^{\lambda_2}\frac{\partial f}{\partial x^\alpha}\varepsilon^\alpha d\lambda+\int_{\lambda_1}^{\lambda_2}\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\frac{d\varepsilon^\alpha}{d\lambda}d\lambda=0. \tag{1.32}$$

From equation (1.32), we see the second integral. So:

$$\int_{\lambda_1}^{\lambda_2}\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\frac{d\varepsilon^\alpha}{d\lambda}d\lambda=\varepsilon^\alpha(\lambda_2)\frac{d\varepsilon^\alpha}{d\lambda}d\lambda-\varepsilon^\alpha(\lambda_1)\frac{d\varepsilon^\alpha}{d\lambda}d\lambda-\int_{\lambda_1}^{\lambda_2}\frac{d}{d\lambda}\left(\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\right)\varepsilon^\alpha d\lambda. \tag{1.33}$$

From figure (2), we can see that:

$$\varepsilon^\alpha(\lambda_1) = \varepsilon^\alpha(\lambda_2) = 0. \tag{1.34}$$

Therefore, replacing (1.34) into (1.33):

$$\int_{\lambda_1}^{\lambda_2} \frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)} \frac{d\varepsilon^\alpha}{d\lambda} d\lambda = -\int_{\lambda_1}^{\lambda_2} \frac{d}{d\lambda}\left(\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\right) \varepsilon^\alpha d\lambda. \tag{1.35}$$

Indeed, replacing (1.35) into (1.32), we have:

$$\int_{\lambda_1}^{\lambda_2} \left[\frac{\partial f}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\right)\right] \varepsilon^\alpha d\lambda = 0, \tag{1.36}$$

where

$$\frac{\partial f}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial f}{\partial\left(\frac{dx^\alpha}{d\lambda}\right)}\right) = 0. \tag{1.37}$$

The equations (1.37) are the so-called Euler-Lagrange equations. Now, using these equations we are going to calculate the geodesic equation. From equation (1.10) we can define our function:

$$f = f\left(x^\nu, \frac{dx^\nu}{d\lambda}\right) = \sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}, \tag{1.38}$$

where λ is an arbitrary parameter of the curve x^μ. Namely:

$$x^\alpha = x^\alpha(\lambda). \tag{1.39}$$

Also, we must remember that:

$$g_{\alpha\beta} = g_{\alpha\beta}(x^\mu). \tag{1.40}$$

Now, in order to make the calculations simpler, we are going to define:

$$u^\alpha \equiv \frac{dx^\alpha}{d\lambda}. \tag{1.41}$$

Replacing (1.41) into (1.38) we have:

$$f = \sqrt{g_{\alpha\beta}u^\alpha u^\beta}. \tag{1.42}$$

Let's replace the function (1.42) in the Euler-Lagrange equations. Now we will calculate each of its terms:

$$\frac{\partial f}{\partial u^\gamma} = \frac{1}{2}\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\frac{\partial}{\partial u^\gamma}\left[g_{\alpha\beta}u^\alpha u^\beta\right]. \tag{1.43}$$

Then:

$$\frac{\partial}{\partial u^\gamma}\left[g_{\alpha\beta}u^\alpha u^\beta\right] = g_{\alpha\beta}\left[\frac{\partial u^\alpha}{\partial u^\gamma}u^\beta + \frac{\partial u^\beta}{\partial u^\gamma}u^\alpha\right], \tag{1.44}$$

$$\frac{\partial}{\partial u^\gamma}\left[g_{\alpha\beta}u^\alpha u^\beta\right] = g_{\alpha\beta}\left[\delta^\alpha_\gamma u^\beta + \delta^\beta_\gamma u^\alpha\right] = 2g_{\gamma\delta}u^\delta. \tag{1.45}$$

Replacing (1.45) into (1.43):

$$\frac{\partial f}{\partial u^\gamma} = \frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}} g_{\gamma\delta} u^\delta. \tag{1.46}$$

Now we differentiate the expression (1.46) again with respect to λ:

$$\frac{d}{d\lambda}\left(\frac{\partial f}{\partial u^\gamma}\right) = \frac{d}{d\lambda}\left[\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\right] g_{\gamma\delta} u^\delta + \frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}} \frac{d}{d\lambda}\left[g_{\gamma\delta}\right] u^\delta + \frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}} g_{\gamma\delta} \frac{d}{d\lambda}\left[u^\delta\right]. \tag{1.47}$$

We can see the following:

$$\frac{dg_{\gamma\delta}}{d\lambda} = \frac{\partial g_{\gamma\delta}}{\partial x^\sigma}\frac{dx^\sigma}{d\lambda} = \frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma \tag{1.48}$$

Replacing (1.48) into (1.47) we have:

$$\frac{d}{d\lambda}\left(\frac{\partial f}{\partial u^\gamma}\right) = \frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma u^\delta + g_{\gamma\delta}\frac{du^\delta}{d\lambda}\right] + \frac{d}{d\lambda}\left[\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\right] g_{\gamma\delta} u^\delta. \tag{1.49}$$

Now we are going to calculate the other term of the Euler-Lagrange equations:

$$\frac{\partial f}{\partial x^\gamma} = \frac{\partial}{\partial x^\gamma}\left[\sqrt{g_{\alpha\beta}u^\alpha u^\beta}\right] = \frac{1}{2\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} u^\alpha u^\beta \tag{1.50}$$

Replacing (1.49) and (1.50) into Euler-Lagrange equations:

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} u^\alpha u^\beta = 2\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma u^\delta + 2 g_{\gamma\delta}\frac{du^\delta}{d\lambda} + 2\sqrt{g_{\alpha\beta}u^\alpha u^\beta}\frac{d}{d\lambda}\left[\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\right] g_{\gamma\delta} u^\delta, \tag{1.51}$$

$$g^{\gamma\omega} g_{\gamma\delta}\frac{du^\delta}{d\lambda} + \frac{g^{\gamma\omega}}{2}\left[2\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma u^\delta - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} u^\alpha u^\beta\right] = -\sqrt{g_{\alpha\beta}u^\alpha u^\beta}\frac{d}{d\lambda}\left[\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\right] g^{\gamma\omega} g_{\gamma\delta} u^\delta. \tag{1.52}$$

Also, from (1.12), we have:

$$2\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma u^\delta = \frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma u^\delta + \frac{\partial g_{\gamma\sigma}}{\partial x^\delta} u^\delta u^\sigma. \tag{1.53}$$

Replacing (1.53) into (1.52) we have the following:

$$\delta^\omega_\delta \frac{du^\delta}{d\lambda} + \frac{g^{\gamma\omega}}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} u^\sigma u^\delta + \frac{\partial g_{\gamma\sigma}}{\partial x^\delta} u^\delta u^\sigma - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} u^\alpha u^\beta\right] = -\sqrt{g_{\alpha\beta}u^\alpha u^\beta}\frac{d}{d\lambda}\left[\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\right] \delta^\omega_\delta u^\delta. \tag{1.54}$$

Indeed:

$$\frac{du^\omega}{d\lambda} + \frac{g^{\gamma\omega}}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} + \frac{\partial g_{\gamma\sigma}}{\partial x^\delta} - \frac{\partial g_{\sigma\delta}}{\partial x^\gamma}\right] u^\sigma u^\delta = -\sqrt{g_{\alpha\beta}u^\alpha u^\beta}\frac{d}{d\lambda}\left[\frac{1}{\sqrt{g_{\alpha\beta}u^\alpha u^\beta}}\right] u^\omega. \tag{1.55}$$

From the equation (1.55) we can define the so-called Christoffel symbol $\Gamma^\omega_{\sigma\delta}$, as follows:

$$\Gamma^\omega_{\sigma\delta} \equiv \frac{g^{\gamma\omega}}{2}\left[\frac{\partial g_{\gamma\delta}}{\partial x^\sigma} + \frac{\partial g_{\gamma\sigma}}{\partial x^\delta} - \frac{\partial g_{\sigma\delta}}{\partial x^\gamma}\right]. \tag{1.56}$$

Replacing (1.56) into (1.55) we have:

$$\frac{du^\omega}{d\lambda} + \Gamma^\omega_{\sigma\delta} u^\sigma u^\delta = -\sqrt{g_{\alpha\beta} u^\alpha u^\beta} \frac{d}{d\lambda} \left[\frac{1}{\sqrt{g_{\alpha\beta} u^\alpha u^\beta}} \right] u^\omega. \tag{1.57}$$

Also, replacing (1.41) into (1.57) finally we have:

$$\frac{d^2 x^\omega}{d\lambda^2} + \Gamma^\omega_{\sigma\delta} \frac{dx^\sigma}{d\lambda} \frac{dx^\delta}{d\lambda} = -\sqrt{g_{\alpha\beta} u^\alpha u^\beta} \frac{d}{d\lambda} \left[\frac{1}{\sqrt{g_{\alpha\beta} u^\alpha u^\beta}} \right] u^\omega. \tag{1.58}$$

The equation (1.58) represents the geodesic equation for a massive body [8]. This equation uses an arbitrary parameter λ. In general for an arbitrary parameter λ the term on the right-hand-side of equation (1.58) does not disappear. However, now we will find a parameter to make this term disappear.

The trajectory of a massive object is usually parameterized using the so-called "proper time" τ. This proper time is defined in the following way [8] [21] [27]:

$$d\tau^2 \equiv -ds^2. \tag{1.59}$$

Physically, proper time τ is related to the time interval measured by a clock attached to the object moving along the curve $x^\alpha(\tau)$ [8] [21] [27]. We should not confuse proper time τ with time coordinate t. The time coordinate t is just a "coordinate" to label a certain event. Indeed, from the equations (1.10) and (1.59) we have:

$$d\tau^2 = -g_{\alpha\beta} dx^\alpha dx^\beta. \tag{1.60}$$

Therefore:

$$i = \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}} = \sqrt{g_{\alpha\beta} u^\alpha u^\beta}. \tag{1.61}$$

So, if we change the parameter λ for the trajectory of a massive body, that is:

$$\lambda \to \tau. \tag{1.62}$$

Then, from (1.61) and (1.62), we will have:

$$\frac{d}{d\tau} \left[\frac{1}{\sqrt{g_{\alpha\beta} u^\alpha u^\beta}} \right] = 0. \tag{1.63}$$

Finally, replacing (1.62) and (1.63) into (1.58) we have:

$$\frac{d^2 x^\omega}{d\tau^2} + \Gamma^\omega_{\sigma\delta} \frac{dx^\sigma}{d\tau} \frac{dx^\delta}{d\tau} = 0. \tag{1.64}$$

The equation (1.64) is the equation of the geodesics that we find in books. Apart from being mathematically more elegant, equation (1.64) stores more physical information (see [8] [21] [27]).

Now, to find the null geodesics (corresponding to light or massless bodies) then we need to remember the following:

$$ds^2 = 0 \rightarrow \sqrt{g_{\alpha\beta} u^\alpha u^\beta} = 0 \tag{1.65}$$

For any parameter λ that parameterizes the light trajectory $x^\alpha(\lambda)$. Replacing (1.65) into (1.58), then, the null geodesic equations will be the same as the equation (1.64). Because in general relativity the differentiable manifold is Pseudoriemannian, the trajectory that the geodesic equation (1.64) will give will be an "extremal trajectory". It will not necessarily be the minimum trajectory due to the negative sign of one of the eigenvalues of the metric tensor $g_{\alpha\beta}$ (for more details see [8] [21] [27]). For example, in special relativity where we have the Minkowski metric, bodies and light follow straight paths. However, in any curved spacetime with a metric $g_{\alpha\beta}$, generally the extremal trajectories that the bodies and light will follow will not be a straight line [21].

1.1.3 Covariant derivative

The covariant derivative of a contravariant vector has the following definition (with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$):

$$\nabla_\nu V^\alpha \doteq \frac{\partial V^\alpha}{\partial x^\nu} + \Gamma^\alpha_{\mu\nu} V^\mu = \partial_\nu V^\alpha + \Gamma^\alpha_{\mu\nu} V^\mu \tag{1.66}$$

It can be shown (see [8] [21] [32]) that the covariant derivative of a covariant vector is defined as follows:

$$\nabla_\nu V_\alpha = \frac{\partial V_\alpha}{\partial x^\nu} - \Gamma^\lambda_{\nu\alpha} V_\lambda = \partial_\nu V_\alpha - \Gamma^\lambda_{\nu\alpha} V_\lambda \tag{1.67}$$

The same concept of covariant derivative can be extended to tensors of many components, where the rule is to add a term with a Christoffel symbol for each free index, with the appropriate sign depending on whether the index is contravariant (up) or covariant (down). For example,

$$\nabla_\alpha T^{\mu\nu} = \partial_\alpha T^{\mu\nu} + \Gamma^\mu_{\alpha\beta} T^{\beta\nu} + \Gamma^\nu_{\alpha\beta} T^{\mu\beta}, \tag{1.68}$$

or

$$\nabla_\alpha T_{\mu\nu} = \partial_\alpha T_{\mu\nu} - \Gamma^\beta_{\alpha\mu} T_{\beta\nu} - \Gamma^\beta_{\alpha\nu} T_{\mu\beta}. \tag{1.69}$$

The covariant derivative tells us that a tensor from one point to another not only changes its components, also changes the vector basis in which said tensor is defined. For example let's see the following:

$$\frac{\partial \vec{A}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}(A^\beta e_\beta) = \frac{\partial A^\beta}{\partial x^\alpha} e_\beta + A^\beta \frac{\partial e_\beta}{\partial x^\alpha} \tag{1.70}$$

Also we known (see [32]):

$$\frac{\partial e_\beta}{\partial x^\alpha} \equiv \Gamma^\gamma_{\beta\alpha} e_\gamma \tag{1.71}$$

Replacing (1.71) into (1.70) we have:

$$\frac{\partial \vec{A}}{\partial x^\alpha} = \frac{\partial A^\beta}{\partial x^\alpha} e_\beta + A^\beta \Gamma^\gamma_{\beta\alpha} e_\gamma = \frac{\partial A^\gamma}{\partial x^\alpha} e_\gamma + A^\beta \Gamma^\gamma_{\beta\alpha} e_\gamma \tag{1.72}$$

Indeed:

$$\frac{\partial \vec{A}}{\partial x^\alpha} = \left[\frac{\partial A^\gamma}{\partial x^\alpha} + \Gamma^\gamma_{\beta\alpha} A^\beta \right] e_\gamma \tag{1.73}$$

Replacing (1.66) into (1.73) then:

$$\frac{\partial \vec{A}}{\partial x^\alpha} = \left(\nabla_\alpha A^\beta \right) e_\beta \tag{1.74}$$

Vector $\vec{A}$ will be constant only if the covariant derivative is zero. Indeed:

$$\nabla_\alpha A^\beta = 0 \rightarrow \vec{A} = constant \tag{1.75}$$

The covariant derivative reduces to the partial derivative when the Christoffel symbols are zero, which occurs in flat space in Cartesian coordinates, but not in spherical coordinates. However, it is always possible to find a coordinate transformation for which the Christoffel symbols are equal to zero at a given point (but not at other points unless the spacetime is flat). This is because all curved space is locally flat: in a region infinitesimally close to every point the geometry approaches flat.

Using this rule it is possible to show that the covariant derivative of the metric tensor is zero, that is

$$\nabla_\alpha g_{\mu\nu} = 0. \tag{1.76}$$

1.1.4 Curvature

As we have seen, the space-time metric allows us to obtain the trajectory of objects. However, the metric tensor is not the most convenient way to describe the presence of a gravitational field. To see this, it suffices to note that even in flat space, one can change the shape of the metric tensor by a simple coordinate transformation.

We must then find a way to distinguish with certainty between a flat space and one that is not. The way to do this is through the so-called Riemann tensor of curvature. This tensor measures the change of a vector when transporting it around a circuit, always keeping it parallel to itself (parallel transport). In flat space, the vector does not change when one does this, while in curved space it does. In this sense, the Riemann tensor is defined, as follows (see [8] [21] [32]):

$$R^\sigma_{\mu\nu\rho} := \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\alpha_{\mu\rho} \Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho} \Gamma^\sigma_{\alpha\mu}, \tag{1.77}$$

where ∂_μ is an abbreviation for $\partial/\partial x^\mu$. We should note that the Riemann tensor has 4 indices, that is, 256 components. However, it has the symmetries

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta},$$

so it has only 20 independent components. It is possible to show that the Riemann tensor is equal to zero if and only if the space-time is flat. On the other hand, from the Riemann tensor, we can define the so-called Ricci tensor as

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}. \tag{1.78}$$

We should note that the fact that the Ricci tensor is zero does not mean that the space-time is flat.

1.1.5 Lie derivative

We have two points x^α and $\bar{x}^\alpha$ infinitesimally close to each other, and we have a vector field A^α. Points x^α and $\bar{x}^\alpha$ are related by the vector field A^α as follows [8] [32]:

$$\bar{x}^\alpha = x^\alpha - \varepsilon A^\alpha(x), \tag{1.79}$$

where $\varepsilon \ll 1$. Now, we want to compare another vector field B^α defined in x^α and $\bar{x}$. To do this, we are going to "drag" vector $B^\alpha(x)$ along the congruence of the curve of A^α an infinitesimal distance (see figure (3)).

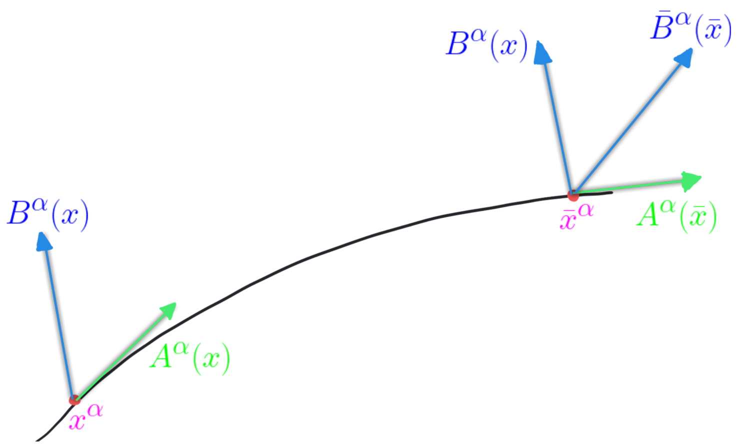

Figure 3 – The vector field B^α along the congruence of curve of A^α.

On the other hand, we also know that vectors $\bar{B}^\alpha(\bar{x})$ and $B^\beta(x)$ are related in the following way:

$$\bar{B}^\alpha(\bar{x}) = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} B^\beta(x). \tag{1.80}$$

Replacing (1.79) into (1.80):

$$\bar{B}^\alpha(\bar{x}) = \frac{\partial}{\partial x^\beta}\left[x^\alpha - \varepsilon A^\alpha(x)\right] B^\beta(x) = \left[\delta^\alpha_\beta - \varepsilon \frac{\partial A^\alpha(x)}{\partial x^\beta}\right] B^\beta(x), \tag{1.81}$$

$$\bar{B}^\alpha(\bar{x}) = B^\alpha(x) - \varepsilon \frac{\partial A^\alpha(x)}{\partial x^\beta} B^\beta(x). \tag{1.82}$$

Now, let's do a Taylor series expansion of $\bar{B}^\alpha(\bar{x})$ (only including the first order of ε) about x:

$$\bar{B}^\alpha(\bar{x}) = \bar{B}^\alpha(x) + (\bar{x}^\beta - x^\beta)\frac{\partial \bar{B}^\alpha(x)}{\partial x^\beta}. \tag{1.83}$$

Replacing (1.79) into (1.83):

$$\bar{B}^{\alpha}(\bar{x}) = \bar{B}^{\alpha}(x) - \varepsilon A^{\beta}(x)\frac{\partial \bar{B}^{\alpha}(x)}{\partial x^{\beta}}. \tag{1.84}$$

Now, equating the equations (1.82) and (1.84), we have:

$$B^{\alpha}(x) - \varepsilon\frac{\partial A^{\alpha}(x)}{\partial x^{\beta}}B^{\beta}(x) = \bar{B}^{\alpha}(x) - \varepsilon A^{\beta}(x)\frac{\partial \bar{B}^{\alpha}(x)}{\partial x^{\beta}}, \tag{1.85}$$

$$\frac{\bar{B}^{\alpha}(x) - B^{\alpha}(x)}{\varepsilon} = -B^{\beta}(x)\frac{\partial A^{\alpha}(x)}{\partial x^{\beta}} + A^{\beta}(x)\frac{\partial \bar{B}^{\alpha}(x)}{\partial x^{\beta}}. \tag{1.86}$$

We will now take the limit $\varepsilon \to 0$, then:

$$\lim_{\varepsilon\to 0} \bar{B}^{\beta}(x) = B^{\beta}(x). \tag{1.87}$$

Also:

$$\lim_{\varepsilon\to 0}\left(\frac{\bar{B}^{\alpha}(x) - B^{\alpha}(x)}{\varepsilon}\right) = \mathcal{L}_A B^{\alpha}, \tag{1.88}$$

where $\mathcal{L}_A B^{\alpha}$ is the Lie derivative. Replacing (1.86) and (1.87) into (1.88), we have:

$$\mathcal{L}_A B^{\alpha} = A^{\beta}(x)\frac{\partial B^{\alpha}(x)}{\partial x^{\beta}} - B^{\beta}(x)\frac{\partial A^{\alpha}(x)}{\partial x^{\beta}}. \tag{1.89}$$

Since the vector field $A^{\beta}(x)$ and $B^{\alpha}(x)$ is defined only at point x then for simplicity: $A^{\beta}(x) \equiv A^{\beta}$ and $B^{\beta}(x) \equiv B^{\beta}$. We have finally:

$$\mathcal{L}_A B^{\alpha} = A^{\beta}\partial_{\beta}B^{\alpha} - B^{\beta}\partial_{\beta}A^{\alpha}. \tag{1.90}$$

An analogous procedure can be applied for tensors of order $(m.n)$ $T^{a_1a_2\dots a_m}_{b_1b_2\dots b_n}$ [8], and the general Lie derivative is defined, as follows:

$$\begin{aligned}\mathcal{L}_X T^{a_1a_2\dots a_m}_{b_1b_2\dots b_n} = X^{\gamma}\nabla_{\gamma}T^{a_1a_2\dots a_m}_{b_1b_2\dots b_n} - T^{\gamma a_2\dots a_m}_{b_1b_2\dots b_n}\nabla_{\gamma}X^{a_1} - T^{a_1\gamma\dots a_m}_{b_1b_2\dots b_n}\nabla_{\gamma}X^{a_2}\dots - T^{a_1a_2\dots\gamma}_{b_1b_2\dots b_n}\nabla_{\gamma}X^{a_m} + \\ + T^{a_1a_2\dots a_n}_{\gamma b_2\dots b_n}\nabla_{b_1}X^{\gamma} + T^{a_1a_2\dots a_n}_{b_1\gamma\dots b_n}\nabla_{b_2}X^{\gamma} + \dots + T^{a_1a_2\dots a_n}_{b_1b_2\dots\gamma}\nabla_{b_n}X^{\gamma},\end{aligned} \tag{1.91}$$

where X^{α} is a vector field defined on the manifold M.

Let's look at another curious feature of the Lie derivative. Let's consider the following parameterized curve (see figure (4)):

$$x^{\alpha} = x^{\alpha}(\lambda). \tag{1.92}$$

Now, we want to go from point $x^{\alpha}(\lambda = 0)$ to point $x^{\alpha}(\lambda = \varepsilon)$ along curve $x^{\alpha}(\lambda)$, where $\varepsilon \ll 1$ (is a very small positive parameter). In this case using the Taylor series we have:

$$x^{\alpha}(\varepsilon) = x^{\alpha}(0) + \frac{dx^{\alpha}(0)}{d\lambda}\varepsilon + \frac{1}{2!}\frac{d^2x^{\alpha}(0)}{d\lambda^2}\varepsilon^2 + \dots, \tag{1.93}$$

$$x^{\alpha}(\varepsilon) = \left[1 + \varepsilon\frac{d}{d\lambda} + \frac{\varepsilon^2}{2!}\frac{d^2}{d\lambda^2} + \dots\right]x^{\alpha}(0). \tag{1.94}$$

Then:

$$x^\alpha(\varepsilon) = e^{\varepsilon \frac{d}{d\lambda}} x^\alpha(0). \tag{1.95}$$

Now, let's consider another curve. This curve will be parameterized with parameter ρ (see figure (4)). Then:

$$x^\alpha = x^\alpha(\rho). \tag{1.96}$$

Using the same reasoning above, we have:

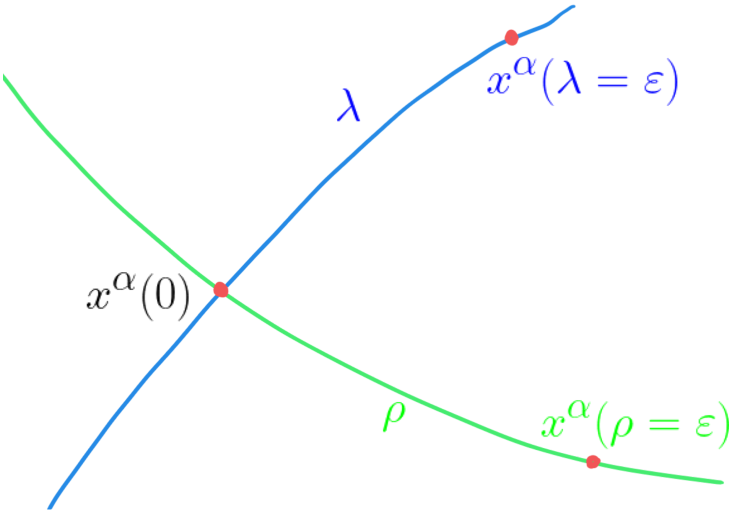

Figure 4 – The curves $x^\alpha(\lambda)$ and $x^\alpha(\rho)$.

$$x^\alpha(\varepsilon) = e^{\varepsilon \frac{d}{d\rho}} x^\alpha(0). \tag{1.97}$$

We should note that at each point of the integral curves $x^\alpha(\lambda)$ and $x^\alpha(\rho)$ the following vector fields are defined (see figure (5)):

$$A^\alpha = \frac{dx^\alpha}{d\lambda}, \tag{1.98}$$

$$B^\alpha = \frac{dx^\alpha}{d\rho}. \tag{1.99}$$

Now, let's see from figure (5) the following: Using path $1 \to 2 \to 3$ we arrive at point $x^\alpha_{(123)}$. On the other hand, using path $1 \to 4 \to 3$ we arrive at point $x^\alpha_{(143)}$. These points are defined, as follows:

$$x^\alpha_{(123)} = e^{\varepsilon \frac{d}{d\rho}} e^{\varepsilon \frac{d}{d\lambda}} x^\alpha(0), \tag{1.100}$$

$$x^\alpha_{(143)} = e^{\varepsilon \frac{d}{d\lambda}} e^{\varepsilon \frac{d}{d\rho}} x^\alpha(0). \tag{1.101}$$

The fundamental question is: $x^\alpha_{(123)} = x^\alpha_{(143)}$? To prove this let us denote Δ, as follows:

$$\Delta = x^\alpha_{(123)} - x^\alpha_{(143)}. \tag{1.102}$$

Replacing (1.100) and (1.101) into (1.102) we have:

$$\Delta = \left[e^{\varepsilon \frac{d}{d\rho}} e^{\varepsilon \frac{d}{d\lambda}} - e^{\varepsilon \frac{d}{d\lambda}} e^{\varepsilon \frac{d}{d\rho}} \right] x^\alpha(0). \tag{1.103}$$

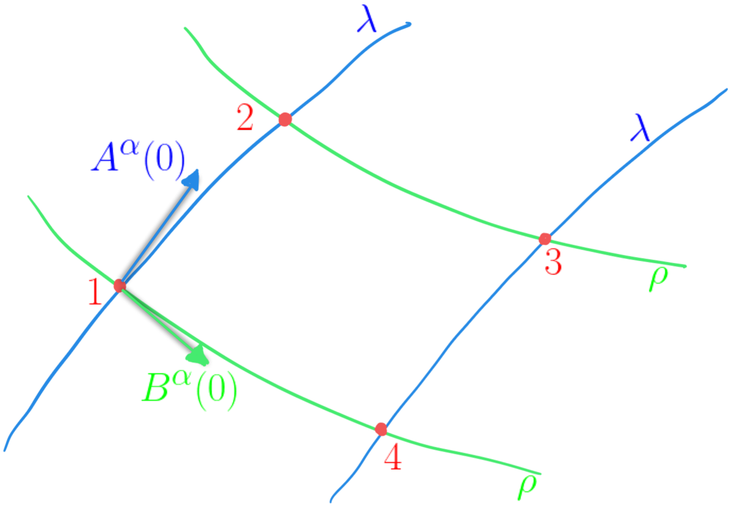

Figure 5 – A^α and B^α and its congruence of curves.

Because $\varepsilon \ll 1$, then we will use the following approximations:

$$e^{\varepsilon \frac{d}{d\lambda}} \approx 1 + \varepsilon \frac{d}{d\lambda}, \tag{1.104}$$

$$e^{\varepsilon \frac{d}{d\rho}} \approx 1 + \varepsilon \frac{d}{d\rho}. \tag{1.105}$$

Replacing (1.104) and (1.105) into (1.103) we have:

$$\Delta = \left[\left(1 + \varepsilon \frac{d}{d\rho}\right)\left(1 + \varepsilon \frac{d}{d\lambda}\right) - \left(1 + \varepsilon \frac{d}{d\lambda}\right)\left(1 + \varepsilon \frac{d}{d\rho}\right)\right] x^\alpha(0). \tag{1.106}$$

Indeed:

$$\Delta = \varepsilon^2 \left[\frac{d}{d\rho}\frac{d}{d\lambda} - \frac{d}{d\lambda}\frac{d}{d\rho}\right] x^\alpha(0) = \varepsilon^2 \left[\frac{d}{d\rho}\frac{dx^\alpha(0)}{d\lambda} - \frac{d}{d\lambda}\frac{dx^\alpha(0)}{d\rho}\right]. \tag{1.107}$$

Replacing (1.98) and (1.99) into (1.107):

$$\Delta = \varepsilon^2 \left[\frac{dA^\alpha(0)}{d\rho} - \frac{dB^\alpha(0)}{d\lambda}\right] = \varepsilon^2 \left[\frac{dx^\beta}{d\rho}\frac{\partial A^\alpha(0)}{\partial x^\beta} - \frac{dx^\beta}{d\lambda}\frac{\partial B^\alpha(0)}{\partial x^\beta}\right]. \tag{1.108}$$

Where $A^\alpha(0)$ and $B^\alpha(0)$ are the vector fields at the point $\lambda = \rho = 0$. Replacing (1.98) and (1.99) into (1.108) we have:

$$\Delta = \varepsilon^2 \left[B^\beta \frac{\partial A^\alpha(0)}{\partial x^\beta} - A^\beta \frac{\partial B^\alpha(0)}{\partial x^\beta}\right] \tag{1.109}$$

If we use the notation used in many books [8] $\frac{\partial}{\partial x^\alpha} \equiv \partial_\alpha$, then:

$$\Delta = \varepsilon^2 (B^\beta \partial_\beta A^\alpha(0) - A^\beta \partial_\beta B^\alpha(0)) = \varepsilon^2 \mathcal{L}_B A^\alpha. \tag{1.110}$$

From the equation (1.110) we can see:

$$\mathcal{L}_B A^\alpha = 0 \rightarrow \Delta = 0. \tag{1.111}$$

In conclusion, if $\Delta = 0$ then, from equation (1.111) $x^\alpha_{(123)} = x^\alpha_{(143)}$.

1.2 General relativity

Already postulating the theory of spatial relativity in 1905, Einstein knew that Newton's gravitation would have to be modified. The main reason for this was that in Newton's theory, the force of gravity propagates between different objects at infinite speed, which contradicts a fundamental principle of relativity: no physical interaction can travel faster than light. It is important to mention that Newton himself never found the existence of this "action at a distance" convincing, but he considered it a necessary hypothesis until a better explanation of the nature of gravity was found. In the decade from 1905 to 1915, Einstein set out to find such an explanation [1] [8] [21] [32].

1.2.1 Principles and postulates of general relativity

The principles that guided Einstein towards the formulation of General Relativity were [8]:

- The principle of covariance: Establishes that the laws of physics must be the same for all observers. In other words, all observers are equivalent. This principle led Einstein to consider that physical laws should be written in tensor form.
- The equivalence principle: It states that all objects fall in the same way in a gravitational field. That is to say, the laws of special relativity apply locally to all inertial observers. Based on this principle, Einstein concluded that the description of gravity should be identified with the geometry of space-time.
- Mach's principle: states that the local inertia of an object must be produced by the total distribution of matter in the Universe. This principle led Einstein to conclude that the geometry of space-time must be altered by the distribution of matter. However, it is important to mention that this principle was not consistently incorporated into general relativity.

The general theory of relativity is a geometric theory of gravitation, its construction is based on a series of postulates, which were developed seeking compatibility with the Newtonian limit and with special relativity. In a more mathematical way, they can be reformulated, as follows

- **Postulate 1:** Space-time is described by the pair (M, g) where M is a 4-dimensional manifold and g is a metric tensor with Lorentzian signature on M.

  This postulate is based on the description of spacetime, where the curvature in the differentiable manifold is determined by the Riemann curvature tensor.

- **Postulate 2:** There exists a symmetric tensor $T_{\alpha\beta} = T_{\beta\alpha} = T_{\alpha\beta}(\varphi)$ that is a function of the matter fields φ and their derivatives such that:

    1. $T_{\alpha\beta} = 0$ over $u \subset M$ if only if $\varphi_i = 0$ for all i over u.
    2. $\nabla_\alpha T^{\alpha\beta} = 0$.

       This postulate is related to the conservation of energy in general relativity.

- **Postulate 3:** The metric on the space-time manifold (M, g) is determined by Einstein's field equations.

1.2.2 The Einstein's field equations

The missing element in Einstein's theory of gravitation is the one that tells us how the geometry of space-time is related to the distribution of matter and energy. This final element is contained in Einstein's field equations. These equations can be derived in various ways, either by seeking a relativistic and consistent generalization of Newton's law of gravitation (the path followed by Einstein) or by deriving them formally from a Lagrangian (the path that Hilbert followed almost simultaneously). In their most compact form, Einstein's equations have the following form [1] [8] [21] [32]:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.112}$$

where $G_{\mu\nu}$ is the Einstein tensor that is related to the Ricci curvature tensor, and $T_{\mu\nu}$ is the energy-momentum tensor. That is, the left-hand side represents the geometry of space-time and the right-hand side represents the distribution of matter and energy. The factor of 8π is simply a normalization needed to get the correct Newtonian limit. It should be noted that there are 10 independent equations in the above tensor equation [1] [8] [21] [32].

The Einstein tensor is defined in terms of the Ricci tensor as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.113}$$

with $R := g^{\mu\nu} R_{\mu\nu}$ as the trace of the Ricci tensor, also called the curvature scalar.

The second part of Einstein's equations (1.112), the energy-momentum tensor, describes the energy density, momentum density, and momentum flux of a field of matter. In effect, T^{00} represents the energy density, T^{0i} represents the momentum density, and T^{ij} represents the flux of momentum i through the surface j [1] [8] [21] [32].

In the case of empty space, we have

$$G_{\mu\nu} = 0, \tag{1.114}$$

or, equivalently,

$$R_{\mu\nu} = 0. \tag{1.115}$$

We have already mentioned that the fact that the Ricci tensor is zero does not imply that the space-time is flat. This is because the gravitational field of an object extends beyond the object itself. Therefore, the curvature of space-time in a vacuum region close to a massive object cannot be zero.

It had been mentioned that the equivalence principle led Einstein to think that gravitation could be identified with the curvature of space-time. Mathematically, this means that the theory of gravity should be a "metric theory", in which gravity manifests itself solely and exclusively through a distortion in the geometry of space-time [1] [8] [21] [32].

1.2.3 Bianchi identities and conservation laws

The Riemann curvature tensor has the following property [1] [8] [21] [32]:

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0. \tag{1.116}$$

The above equation is known as the Bianchi identity. One of its consequences is the fact that the covariant divergence of the Einstein tensor is equal to zero [1] [8] [21] [32]:

$$\nabla_\nu G^{\mu\nu} = 0. \tag{1.117}$$

The Einstein tensor is the only combination that can be obtained from the Ricci tensor that has this property. This is the reason why we do not use the Ricci tensor directly [1] [8] [21] [32]. In this sense, if we use Einstein's equations, we see that the above implies

$$\nabla_\nu T^{\mu\nu} = 0. \tag{1.118}$$

These equations are of fundamental importance since they represent the local conservation laws of energy and momentum [1] [8] [21] [32].

1.2.4 The weak energy condition

The weak energy condition (WEC) states that the local energy density measured by any observer must be non-negative. Let t^μ be a timelike 4-vector, then [1] [8] [21] [30] [32]:

$$T_{\mu\nu} t^\mu t^\nu \geq 0. \tag{1.119}$$

There are other energy conditions such as the Null energy condition (NEC), the Strong energy condition (SEC) and the Dominant energy condition (DEC). To see these energy conditions in more detail, we recommend the reader see [30].

Warp drives violate the WEC and is one of its main problems of a physical nature [2] [18]. However, as we will see later, we will deal with another problem that is also of a physical nature: the horizon problem. That is why we will not go into details regarding the energy conditions.

As we already know, the solutions to the equations (1.113) are too complicated to be treated analytically [1] [8] [21] [30] [32]. A solution to this problem is in the 3+1 formalism, where we can treat certain globally hyperbolic spacetimes, with a low level of symmetry [3]. And as we will see later, warp drive spacetime does not have a high level of symmetry. That is the reason why the 3+1 formalism will be necessary [2] [27].

2 3+1 Formalism

To study the evolution in time of any physical system, the first thing to do is formulate said evolution as an initial value problem, or also called the Cauchy problem. That is, given the appropriate initial conditions, the fundamental equations must be able to predict the future or past evolution of the system [1] [31].

When trying to write Einstein's equations as a Cauchy problem we immediately run into a problem: Einstein's equations are written in such a way that space and time are symmetric. This covariance of the equations is important and very elegant from a theoretical point of view, but it does not allow us to think clearly about the time evolution of the gravitational field [3]. The first thing we must do then, to rewrite Einstein's equations as a Cauchy problem, is to separate space and time clearly [31].

The formulation of general relativity that results from this separation is known as the 3+1 formalism. Arnowitt, Deser, and Misner proposed this formalism, and consequently, this formalism is also called "ADM formalism" [31].

2.1 Frobenius theorem

We already saw in the previous chapter that space-time can be described by means of a 4-dimensional manifold M. This representation can also be decomposed by making a 3-dimensional manifold Σ dependent on a real parameter such as time. This is possible due to the following theorem [29] [31]:

- **Frobenius theorem:** Let $(M, g_{\mu\nu})$ be a globally hyperbolic space-time, then a global function t can be chosen such that every surface of constant t is a Cauchy surface. Therefore, M can be described by Cauchy surfaces and the topology will be $\mathbb{R} \times \Sigma$, where Σ denotes any Cauchy surface.

It is important to mention that 4-dimensional space-time is globally hyperbolic if it admits a space-like sub-manifold that intersects each time curve once and only once. This sub-manifold is called the Cauchy surface. This condition of global hyperbolicity guarantees the good causal behavior of the different solutions of Einstein's equations.

From the previous theorem, we can identify that the surfaces Σ_t, for each fixed $t \in \mathbb{R}$, represent a family of Riemannian sub-spaces (M^3, γ_{ij}) of dimension 3, with an induced metric γ_{ij} on M^3. Physically this time t represents a "universal time". Furthermore, this time t must not necessarily coincide with the proper time of any observer.

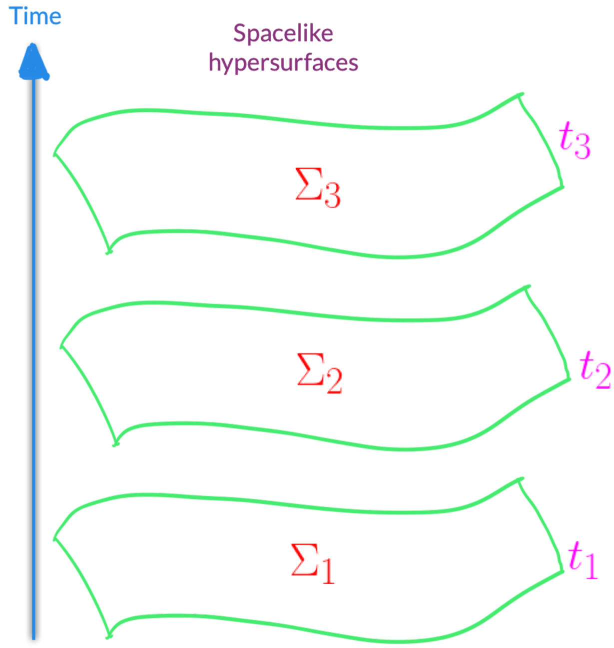

Figure 6 – Spacelike hypersurfaces

Now let us consider a space-time with a metric $g_{\mu\nu}$, which has 10 independent components. Within the ADM formalism, this space-time metric is split into its time components (4) and spatial ones (6). In this way, only these 6 components are calculated from Einstein's field equations. The 4 variables are defined by the "foliation" of space-time in hypersurfaces: one of them is the lapse function α, and the other 3 are the shift functions (shift-vector) β^i, which is defined on the space-like 3-dimensional hypersurfaces with metrics γ_{ij} [3] [29] [31].

2.2 The metric in 3+1 formalism

In order to separate spacetime into spacelike hypersurfaces Σ we must define new variables. These variables are going to replace our metric $g_{\mu\nu}$. These variables are going to be the lapse function α, shift vector β^i and the induced metric γ_{ij} [31].

2.2.1 Lapse function

Let Σ_t be a 3-dimensional hypersurface embedded in a 4-manifold M. Σ_t could be defined by [3] [31]:

$$\Sigma(x^i) = 0. \tag{2.1}$$

Then, the vector field everywhere normal to the surface (2.1) is the following:

$$\xi_\mu = \partial_\mu \Sigma. \tag{2.2}$$

We can define a unit normal vector

$$n_\mu = \frac{\xi_\mu}{\sqrt{|\xi_\lambda \xi^\lambda|}}. \tag{2.3}$$

The function $\alpha : M^4 \to \mathbb{R}$, called the lapse function, is introduced to normalize n^μ, that is:

$$\alpha = \sqrt{|\xi_\lambda \xi^\lambda|}. \tag{2.4}$$

In general, $\alpha = \alpha(t, x^i)$ [31]. We need to know that n^μ is a timelike normalized vector, since $n^\mu n_\mu = -1$. Thus, n^μ is oriented in the sense of increasing t.

The lapse function tells us the variation of proper time when an observer moves between two "neighboring sheets" following the normal trajectory n^μ [31]. The proper time τ between 2 hypersurfaces Σ_t and Σ_{t+dt} that measures an observer moving in a normal direction to them, is related to α in the following way:

$$d\tau = \alpha(t, x^i)dt. \tag{2.5}$$

2.2.2 Induced metric

The induced metric over Σ_t measures the distances within the hypersurface itself, and is defined by [3] [31]:

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{2.6}$$

where $g_{\mu\nu}$ is the metric in M^4, and n_μ is the 4-vector normal to Σ_t.

The induced metric is entirely contained in the three-dimensional hypersurface, therefore $\gamma_{\mu\nu} n^\mu = 0$, thus being a purely spatial quantity. The inverse spatial metric $\gamma^{\mu\nu}$ is obtained by raising the indices of $\gamma_{\mu\nu}$ with $g^{\mu\nu}$:

$$\gamma^{\mu\nu} = g^{\alpha\mu} g^{\beta\nu} \gamma_{\alpha\beta} = g^{\mu\nu} + n^\mu n^\nu. \tag{2.7}$$

Similarly, the projection operator P^μ_ν that sends one space-time vector to another on the hypersurface is defined by:

$$P^\mu_\nu \doteq \gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu. \tag{2.8}$$

2.2.3 Shift vector

An observer can move between two hypersurfaces along a direction that is not necessarily the normal direction n^μ. The time-like vector t^μ that describes an arbitrary trajectory is given by [3] [31]:

$$t^\mu = \alpha n^\mu + N^\mu, \tag{2.9}$$

where N^μ is the shift vector, which is the tangent vector to the hypersurface Σ_t where it is located. Defined N^μ in this way, then $N^\mu n_\mu = 0$. Also α is the lapse function.

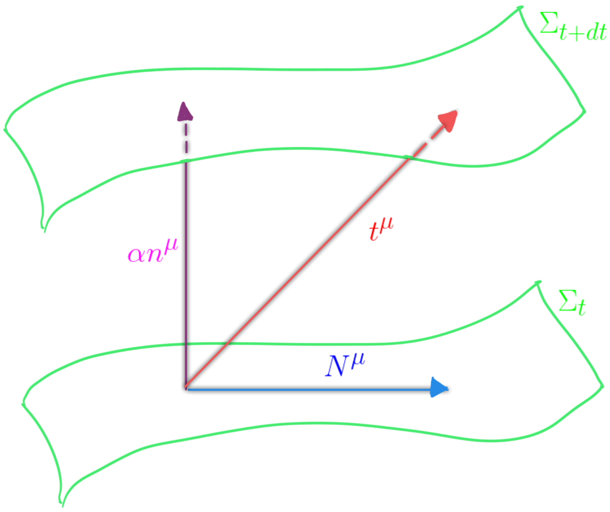

Figure 7 – Decomposition of the timelike vector t^μ

In this way, the 4-vector displacement N^μ can be defined as the projection of t^ν onto Σ_t, in the form:

$$N^\mu = \gamma^\mu_\nu t^\nu. \tag{2.10}$$

Also, N^μ is usually expressed in the following way:

$$N^\mu = (0, \beta^i). \tag{2.11}$$

In this equation, the vector β^i is also called "shift vector" and belongs to the 3-dimensional spacelike hypersurface Σ_t [3]. In general, $\beta^i = \beta^i(t, x^j)$. Also β^i can be interpreted as the relative velocity between the Euler observers and the lines with constant spatial coordinates, as follows

$$x^i_{t+dt} = x^i_t - \beta^i dt \tag{2.12}$$

2.2.4 The metric in terms of the ADM functions

Starting from the relations $n_\mu = (-\alpha, 0, 0, 0)$, $n^\mu n_\mu = -1$ and $N^\mu = -\gamma^\mu_\nu t^\nu$, it is shown that the components of the unit normal vector n^μ to the hypersurfaces are [3] [31]:

$$n^\mu = \left(\frac{1}{\alpha}, \frac{\beta^i}{\alpha}\right). \tag{2.13}$$

Furthermore, from Eqs. (2.6) and (2.9), it can be shown that

$$g^{\mu\nu} = \gamma^{\mu\nu} - \frac{1}{\alpha^2}(t^\mu - N^\mu)(t^\nu - N^\nu). \tag{2.14}$$

Of course $x^0 \equiv t$ and $t^\mu = \delta^\mu_0$. Using these relations together with $\gamma^0_\mu = \gamma^{\mu 0} = 0$, we can write the components of $g^{\mu\nu}$ as:

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & -\frac{\beta^i}{\alpha^2} \\ -\frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix}, \tag{2.15}$$

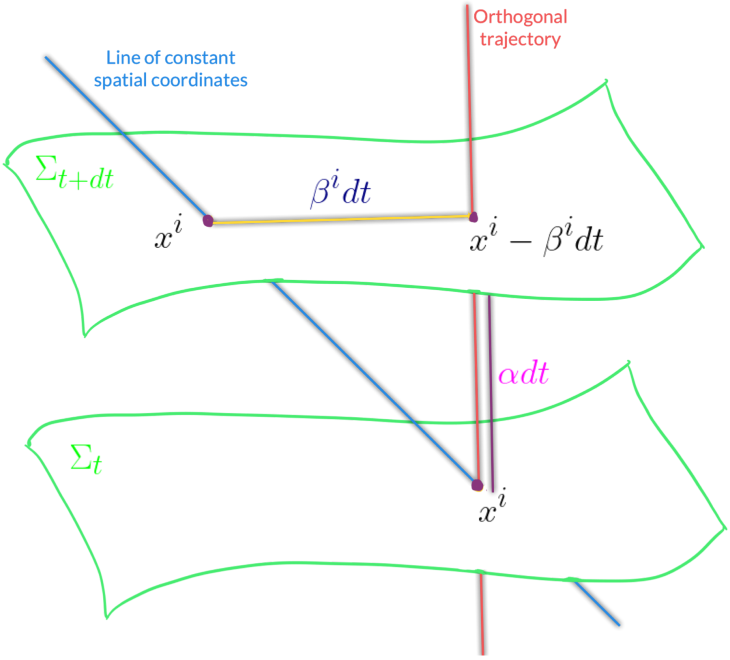

Figure 8 – Shift vector β^i

and the inverse matrix is

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & -\beta^i \\ -\beta^j & \gamma_{ij} \end{bmatrix}. \tag{2.16}$$

Indeed, the line element will be:

$$ds^2 = (-\alpha^2 + \beta^k\beta_k)dt^2 - 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j. \tag{2.17}$$

We can notice that γ_{ij} provides the distances between points of Σ_t with coordinates x^i and $x^i + dx^i$. Therefore, the 3-metric γ_{ij} characterizes the intrinsic geometry of the hypersurface Σ_t.

2.3 Extrinsic curvature tensor

Let n^μ be a vector perpendicular to the spacelike hypersurface Σ_t at point P. Then we will do a parallel transport to this vector n^μ on this hypersurface Σ_t and we will obtain a vector $\bar{n}^\mu$ at point Q. In this context, the extrinsic curvature tensor K_{ij} will measure the variation between vector n^μ and vector $\bar{n}^\mu$ [3] [29] [31]. Naturally, the extrinsic curvature tensor is defined on the manifold M. The formal mathematical definition of the $K_{\alpha\beta}$ is the following [3] [31]:

$$K_{\alpha\beta} = -P\nabla_\alpha n_\beta, \tag{2.18}$$

where P is the "projection operator", defined in equation (2.8). The compact notation above means

$$PT_{\alpha\beta} \equiv P^\mu_\alpha P^\nu_\beta T_{\mu\nu}. \tag{2.19}$$

Immediately, from the definition of the extrinsic curvature tensor $K_{\alpha\beta}$ we can see the following:

$$K_{\alpha\beta} = -P\nabla_\alpha n_\beta = -P\left[\partial_\alpha n_\beta - \Gamma^\rho_{\alpha\beta} n_\rho\right],$$

$$K_{\alpha\beta} = -P\left[\partial_\alpha(\partial_\beta \Sigma) - \Gamma^\rho_{\alpha\beta} n_\rho\right] = -P\left[\partial_\beta(\partial_\alpha \Sigma) - \Gamma^\rho_{\beta\alpha} n_\rho\right] = -P\left[\partial_\beta n_\alpha - \Gamma^\rho_{\beta\alpha} n_\rho\right]. \quad (2.20)$$

Then:

$$K_{\alpha\beta} = -P\nabla_\beta n_\alpha. \quad (2.21)$$

Indeed, $K_{\alpha\beta} = K_{\beta\alpha}$. From this, we can see that the extrinsic curvature tensor $K_{\alpha\beta}$ is symmetric. Now, to use this projection operator P more naturally, let's look at the following example: Let an arbitrary vector v^α be defined in M. We will show that Pv^α (Pv^α represents a vector projected onto the spacelike hypersurface Σ_t) is orthogonal to n^α. Then:

$$(Pv^\alpha)n_\alpha = \left[\delta^\alpha_\beta + n^\alpha n_\beta\right] v^\beta n_\alpha = \delta^\alpha_\beta v^\beta n_\alpha + n^\alpha n_\alpha n_\beta v^\beta. \quad (2.22)$$

We know that $n^\alpha n_\alpha = -1$ (n^α is a timelike vector). Then:

$$(Pv^\alpha)n_\alpha = v^\beta n_\beta + (-1)n_\beta v^\beta = 0. \quad (2.23)$$

Equation (2.23) tells us that Pv^α is a spacelike vector (i.e., Pv^α is on the spacelike hypersurface Σ_t). Now we will see whether the extrinsic curvature tensor $K_{\alpha\beta}$ is orthogonal to n^α. To do this we must calculate

$$n^\alpha K_{\alpha\beta} = n^\alpha\left[-P\nabla_\alpha n_\beta\right] = -n^\alpha P^\gamma_\alpha P^\delta_\beta \nabla_\gamma n_\delta, \quad (2.24)$$

$$n^\alpha K_{\alpha\beta} = -n^\alpha[\delta^\gamma_\alpha + n^\gamma n_\alpha][\delta^\delta_\beta + n^\delta n_\beta][\partial_\gamma n_\delta - \Gamma^\mu_{\gamma\delta} n_\mu]. \quad (2.25)$$

Then:

$$\begin{aligned} n^\alpha K_{\alpha\beta} = &-n^\alpha \delta^\gamma_\alpha \delta^\delta_\beta \partial_\gamma n_\delta - n^\alpha \delta^\gamma_\alpha n^\delta n_\beta \partial_\gamma n_\delta - (n^\alpha n_\alpha) n^\gamma \delta^\delta_\beta \partial_\gamma n_\delta - (n^\alpha n_\alpha) n^\gamma n^\delta n_\beta \partial_\gamma n_\delta + \\ &+ n^\alpha \delta^\gamma_\alpha \delta^\delta_\beta \Gamma^\mu_{\gamma\delta} n_\mu + n^\alpha \delta^\gamma_\alpha n^\delta n_\beta \Gamma^\mu_{\gamma\delta} n_\mu + (n^\alpha n_\alpha) n^\gamma \delta^\delta_\beta \Gamma^\mu_{\gamma\delta} n_\mu + (n^\alpha n_\alpha) n^\gamma n^\delta n_\beta \Gamma^\mu_{\gamma\delta} n_\mu. \quad (2.26) \end{aligned}$$

We need to remember that $n^\alpha n_\alpha = -1$, then, we have:

$$\begin{aligned} n^\alpha K_{\alpha\beta} = &-n^\alpha \partial_\alpha n_\beta - n^\alpha n^\delta n_\beta \partial_\alpha n_\delta + n^\gamma \partial_\gamma n_\beta + n^\gamma n^\delta n_\beta \partial_\gamma n_\delta + n^\alpha \Gamma^\mu_{\alpha\beta} n_\mu + \\ &+ n^\alpha n^\delta n_\beta \Gamma^\mu_{\alpha\delta} n_\mu - n^\gamma \Gamma^\mu_{\gamma\beta} n_\mu - n^\gamma n^\delta n_\beta \Gamma^\mu_{\gamma\delta} n_\mu. \quad (2.27) \end{aligned}$$

Replacing the dumb indexes conveniently, we have that

$$n^\alpha K_{\alpha\beta} = 0. \quad (2.28)$$

The equation (2.28) tells us that the extrinsic curvature tensor $K_{\alpha\beta}$ is purely spatial. This tensor will be defined completely within the spacelike hypersurface Σ_t with metric tensor $\gamma_{ij(t)}$.

Now we are going to express the extrinsic curvature tensor $K_{\alpha\beta}$ in the form of the Lie derivative. To do that, let's take definition (2.19):

$$K_{\alpha\beta} = -P\nabla_\alpha n_\beta = -P^\mu_\alpha P^\nu_\beta \nabla_\mu n_\nu = -[\delta^\mu_\alpha + n^\mu n_\alpha][\delta^\nu_\beta + n^\nu n_\beta]\nabla_\mu n_\nu, \tag{2.29}$$

$$K_{\alpha\beta} = -\delta^\mu_\alpha \delta^\nu_\beta \nabla_\mu n_\nu - \delta^\mu_\alpha n^\nu n_\beta \nabla_\mu n_\nu - n^\mu n_\alpha \delta^\nu_\beta \nabla_\mu n_\nu - n^\mu n_\alpha n^\nu n_\beta \nabla_\mu n_\nu. \tag{2.30}$$

We know the following:

$$\nabla_\mu (n^\nu n_\nu) = 0. \tag{2.31}$$

Also:

$$\nabla_\mu (n^\nu n_\nu) = n^\nu \nabla_\mu n_\nu + n_\nu \nabla_\mu n^\nu = 2n^\nu \nabla_\mu n_\nu \tag{2.32}$$

Thus,

$$n^\nu \nabla_\mu n_\nu = 0. \tag{2.33}$$

Replacing (2.33) into (2.30) then:

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha n^\mu \nabla_\mu n_\beta. \tag{2.34}$$

The expression (2.34) gives us the extrinsic curvature tensor $K_{\alpha\beta}$ for any timelike vector n^α perpendicular to the hypersurface Σ_t.

If we want to study the properties of warp drive spacetime then we must measure physical quantities with respect to an observer in free fall (for those only subject to the gravitational interaction) [31]. Then we are going to identify vector n^α with an observer that follows timelike geodesics (Eulerian observer). So:

$$n^\mu \nabla_\mu n^\rho = \frac{dn^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} n^\mu n^\nu = 0. \tag{2.35}$$

Substituting (2.35) into (2.34), we have the following expression for $K_{\alpha\beta}$:

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta. \tag{2.36}$$

From (2.21) we know that $K_{\alpha\beta}$ is a symmetric tensor, so:

$$K_{\alpha\beta} = -\frac{1}{2}\left[\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right]. \tag{2.37}$$

Also we know:

$$\nabla_\sigma g_{\alpha\beta} = 0. \tag{2.38}$$

From (1.20) and (2.38) we have:

$$K_{\alpha\beta} = -\frac{1}{2}\left[n^\sigma \nabla_\sigma g_{\alpha\beta} + \nabla_\alpha (g_{\sigma\beta} n^\sigma) + \nabla_\beta (g_{\sigma\alpha} n^\sigma)\right], \tag{2.39}$$

$$K_{\alpha\beta} = -\frac{1}{2}\left[n^\sigma \nabla_\sigma g_{\alpha\beta} + g_{\sigma\beta} \nabla_\alpha n^\sigma + g_{\sigma\alpha} \nabla_\beta n^\sigma\right]. \tag{2.40}$$

Using the definition of Lie derivative (1.91) for expression (2.40) we have the following:

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n g_{\alpha\beta}. \tag{2.41}$$

The extrinsic curvature tensor K_{ij} is related to the parallel transport of the normal vector n^μ within a spacelike hypersurface Σ_t. The evolution of the 3-metric γ_{ij} along a direction n^μ gives us the extrinsic curvature [3] [31].

The extrinsic curvature tensor can be expressed in terms of the lapse function α, the shift vector β_i and the induced metric γ_{ij}, as follows [3]

$$K_{ij} = \frac{1}{2\alpha}\left(D_i\beta_j + D_j\beta_i - \frac{\partial\gamma_{ij}}{\partial t}\right), \tag{2.42}$$

where D_i is the covariant derivative corresponding to the metric γ_{ij}.

For the Eulerian observers, we can define the expansion of the volume elements θ in the following way [3]:

$$\theta = \alpha \mathrm{Tr}(K_{ij}), \tag{2.43}$$

where $\mathrm{Tr}(K_{ij})$ is the trace of the extrinsic curvature tensor K_{ij}.

2.4 The Einstein's equations in the 3+1 formalism

Now we must investigate what happens to Einstein's equations (1.112) when we use the variables of the 3+1 formalism.

2.4.1 Gauss-Codazzi relations

Now we need to relate the Einstein tensor to the intrinsic and extrinsic geometry of the Cauchy hypersurfaces. Thus, we need the projections of the Einstein tensor on the hypersurface and its normal vector. Such projections are represented by the so-called Gauss-Codazzi relations. And these relations can be written in terms of the Einstein tensor compactly, as follows [3] [31]

$$G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}\left[{}^{(3)}\mathrm{R} + (\mathrm{Tr}(K_{ij}))^2 - K_{ij}K^{ij}\right], \tag{2.44}$$

and,

$$G_{\mu\nu}n^\mu\gamma^\nu_\alpha = D_\alpha(\mathrm{Tr}(K_{ij})) - D_\mu K^\mu_\alpha, \tag{2.45}$$

where $G_{\mu\nu}$ is the Einstein tensor 4-dimensional. Also n^μ is the normal vector and ${}^{(3)}\mathrm{R}$ is the Ricci scalar of the spacelike Cauchy hypersurface with metric γ_{ij}.

2.4.2 The Einstein's equations

The energy density ρ will be represented by the complete bilateral projection of the energy-momentum tensor $T_{\mu\nu}$ on the normal hypersurface. The complete bilateral projection of $T_{\mu\nu}$ onto the hypersurface is the mechanical stress. The mixed projection of $T_{\mu\nu}$, one on the hypersurface and one on the normal n^{μ} represents the momentum density (or energy flow) of the matter fields. In effect, the energy density ρ, the momentum density $\mathcal{P}_{\sigma}$ and the stress tensor $\mathcal{S}_{\alpha\beta}$ are given by [3] [31]:

$$\rho = T_{\mu\nu} n^{\mu} n^{\nu}, \tag{2.46}$$

$$\mathcal{P}_{\sigma} = -T_{\mu\nu} n^{\mu} \gamma^{\nu}_{\sigma}, \tag{2.47}$$

$$\mathcal{S}_{\alpha\beta} = T_{\mu\nu} \gamma^{\mu}_{\alpha} \gamma^{\nu}_{\beta}. \tag{2.48}$$

In this way, Einstein's field equations are decomposed (using Gauss-Codazzi relations) in the 3+1 formalism into the following equations:

$$^{(3)}\mathrm{R} + (\mathrm{Tr}K_{ij})^2 - K_{ij}K^{ij} = 16\pi\rho, \tag{2.49}$$

$$D_j K^j_i - D_i(\mathrm{Tr}(K_{ij})) = 8\pi \mathcal{P}_i, \tag{2.50}$$

$$(\partial_t - \mathcal{L}_{\beta^i})K_{ij} = -D_i D_j \alpha + \alpha \left\{ {}^{(3)}R_{ij} + \mathrm{Tr}(K_{ij})K_{ij} - 2K_{ik}K^k_j + \right. \\ \left. + 4\pi \left[(\mathrm{Tr}(\mathcal{S}_{ij}) - \rho)\gamma_{ij} - 2\mathcal{S}_{ij} \right] \right\}, \tag{2.51}$$

$$(\partial_t - \mathcal{L}_{\beta^i})\gamma_{ij} = -2\alpha K_{ij}. \tag{2.52}$$

The equations (2.49) and (2.50) are called the Hamiltonian constraint and the momentum constraint, respectively. These Einstein field equations expressed in the 3+1 formalism help us calculate the evolution of gravitational fields in strong gravity without symmetry. However, for our purposes, equation (2.49) will be very important to investigate the nature of the energy density ρ.

3 Warp drive spacetimes

It is well known that the speed of light is the principal problem to want to do interstellar travels. The universe is too vast and the speed of light is too slow. Even if we use the Parker Space Probe to travel to Alpha Centauri (the planetary system nearest to our Earth) we need to travel 65 centuries to reach Alpha Centauri [10].

To solve this problem, wormholes have been proposed to make interstellar travel [30]. In 1993 Alcubierre proposed a new mechanism that would allow superluminal interstellar travel [2]. However, as was the case with wormholes, this new travel mechanism proposed by Alcubierre required negative energy densities [2]. Alcubierre called this mechanism "warp drive".

Later works determined that the amount of negative energy required for an Alcubierre warp drive would be 10 orders of magnitude greater than the mass of the entire observable Universe [11] [24]. However, years later, many metrics were proposed to try to minimize the problem of negative energy. One of the most notable proposals was made by Van Den Broeck [28]. By making a small alteration to the Alcubierre metric, the Van Den Broeck metric could significantly reduce the amount of negative energy (the amount of negative energy was reduced to a scale of few solar masses [28]).

Lobo and Visser demonstrated that any generic Natario warp drive (that we shall define soon) will inevitably have a non-physical nature in the framework of general relativity, so they propose to investigate warp drive-type metrics in alternative theories of gravitation [17].

Nowadays there are many warp drive type metrics. However, in this chapter, we will study some of the most important warp drive metrics that have been proposed to date, within the framework of general relativity.

3.1 Natario's warp drive definition

After Alcubierre published the first article on warp drives [2] it generated great attention from a certain part of the scientific community. As we will see later, the Alcubierre warp drive works with expansion and contraction of space-time around the ship. For some years it was believed that this rate of expansion/contraction was inherent to a spacetime that allowed superluminal travel.

However, some years later Natario stated that this is not the case [22]. First, he generalized the concept of warp drive. After that, he showed a warp drive type spacetime that does not need contraction or expansion of the spacetime surrounding the spacecraft

[22]. Let's now look at the general definition of warp drive proposed by Natario:

- **Natario's warp drive definition:** A warp drive spacetime is a globally hyperbolic spacetime $(M, g_{\mu\nu})$ represented by the differentiable manifold M, where $M = \mathbb{R}^4$. Additionally, the metric tensor $g_{\mu\nu}$ is given by the following line element:

$$ds^2 = -dt^2 + \sum_{i=1}^{3} \left(dx^i - X^i dt\right)^2, \tag{3.1}$$

where X^i is a vector field composed of 3 smooth and bounded arbitrary functions, that is $(X^i) \equiv (X, Y, Z)$. And furthermore, the line element (3.1) is expressed in Cartesian coordinates, that is: $(t, x^i) \equiv (t, x, y, z)$.

Natario's definition tells us that a warp drive spacetime is completely defined by the vector field $\vec{X}$, where:

$$\vec{X} \equiv X^i \frac{\partial}{\partial x^i} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}, \tag{3.2}$$

and, in general,

$$X^i = X^i(t.x.y.z). \tag{3.3}$$

For an observer inside the warp bubble defined with the tetrad basis $\{(e_{\hat{0}})^\mu, (e_{\hat{1}})^\mu, (e_{\hat{2}})^\mu, (e_{\hat{3}})^\mu\}$, we will have the metric $g_{\hat{\alpha}\hat{\beta}}$ (as will be reviewed in the next chapter) and the shift vector $\vec{X}_b$. If our warp drive moves in the $+x$ direction (with respect to an observer located at infinity), this shift vector must satisfy the following condition [22]

- **Inside the warp bubble:**

$$\vec{X}_b = 0. \tag{3.4}$$

- **Outside the warp bubble:**

$$\vec{X}_b = -ve_x, \tag{3.5}$$

where e_x is the unit vector in the $+x$ direction, and v is the speed of the warp bubble with respect to a distant observer.

3.1.1 3+1 formalism

Since the generic warp drive is a globally hyperbolic spacetime, the line element (3.1) can then be expressed in the 3+1 formalism [3] [29]. From (3.1) we have:

$$ds^2 = -dt^2 + (dx - Xdt)^2 + (dy - Ydt)^2 + (dz - Zdt)^2. \tag{3.6}$$

Indeed:

$$ds^2 = (-1 + X^2 + Y^2 + Z^2)dt^2 - 2(Xdx + Ydy + Zdz)\,dt + \left(dx^2 + dy^2 + dz^2\right). \tag{3.7}$$

From equation (3.7), we can see that:

$$\beta_i \beta^i = X^2 + Y^2 + Z^2, \tag{3.8}$$

$$\gamma_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2, \tag{3.9}$$

$$\beta_i dx^i dt = (X dx + Y dy + Z dz) dt, \tag{3.10}$$

From the equations (3.7) - (3.10) and (2.17), we can reach the following conclusion:

$$\alpha = 1, \tag{3.11}$$

$$\beta^i = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z}, \tag{3.12}$$

$$\gamma_{ij} = \delta_{ij}. \tag{3.13}$$

From the equation (3.11), replacing the lapse function α into (2.5) we have:

$$d\tau = dt. \tag{3.14}$$

From (3.14) we see that the proper time τ measured by an observer moving in the direction normal to two spacelike hypersurfaces Σ_t and Σ_{t+dt} has the same value as the time coordinate t.

Also, from equation (3.13), we can see that spacelike hypersurfaces are 3-dimensional Euclidean spaces with metric δ_{ij}. As will be seen later, using these hypersurfaces will greatly simplify mathematical calculations.

We should also note from (3.11) and (3.12) that the 4-velocity of the Eulerian observers defined in (2.13) for a general warp drive spacetime is given by:

$$n_\mu = (-1, 0, 0, 0) = -dt, \tag{3.15}$$

$$n^\mu = (1, \beta^i) = (1, X^i) = \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial x^i}. \tag{3.16}$$

3.1.2 Is the Eulerian observer in free fall?

We will now calculate the geodesics for a general warp drive spacetime. These geodesics were calculated by Natario. It is true that the equation (1.64) could be used. However, an easier way to calculate geodesics is to take a given Lagrangian and use the Euler Lagrange equations [22]. We will see an analogous mathematical treatment later, in the calculation of the geodesics for the Alcubierre metric. From the equations (1.59) and (3.1) we have:

$$-d\tau^2 = -dt^2 + \sum_{i=1}^{3} (dx^i - X^i dt)^2, \tag{3.17}$$

$$-1 = -\left(\frac{dt}{d\tau}\right)^2 + \sum_{i=1}^{3} \left(\frac{dx^i}{d\tau} - X^i \frac{dt}{d\tau}\right)^2. \tag{3.18}$$

We will denote the following:

$$\dot{t} \equiv \frac{dt}{d\tau}, \tag{3.19}$$

$$\dot{x}^i \equiv \frac{dx^i}{d\tau}. \tag{3.20}$$

Replacing (3.19) and (3.20) into (3.18) we have:

$$-1 = -\dot{t}^2 + \sum_{i=1}^{3} \left(\dot{x}^i - X^i \dot{t}\right)^2. \tag{3.21}$$

Now we are going to define the following Lagrangian:

$$L = -\dot{t}^2 + \sum_{i=1}^{3} \left(\dot{x}^i - X^i \dot{t}\right)^2. \tag{3.22}$$

From (3.21) we notice that the Lagrangian (3.22) is always constant. Applying then the Principle of Least Action (see Chapter 1), we will have the following Euler-Lagrange equations:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0, \tag{3.23}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0. \tag{3.24}$$

Replacing (3.22) into (3.23) and (3.24) we have the following equations of the geodesics for a general spacetime warp drive:

$$\frac{d}{d\tau}\left[-\dot{t} - X^i(\dot{x}^i - X^i \dot{t})\right] + \dot{t}\frac{\partial X^i}{\partial t}(\dot{x}^i - X^i \dot{t}) = 0, \tag{3.25}$$

$$\frac{d}{d\tau}(\dot{x}^i - X^i \dot{t}) + \dot{t}\frac{\partial X^i}{\partial x^j}(\dot{x}^i - X^i \dot{t}) = 0. \tag{3.26}$$

If we establish the following values:

$$\dot{t} = 1, \tag{3.27}$$

$$\dot{x}^i = X^i. \tag{3.28}$$

We see that the values given by (3.27) and (3.28) satisfy the geodesic equations (3.25) and (3.26). Therefore, the 4-velocity is given by:

$$n^\mu = (\dot{t}, \dot{x}^i) = (1, X^i). \tag{3.29}$$

The expression (3.29) corresponds to the 4-velocity of an Eulerian observer. In conclusion, we have shown that Eulerian observers are in free fall in general warp drive spacetime.

3.1.3 Extrinsic curvature tensor and expansion coefficient

Let's calculate the extrinsic curvature tensor K_{ij} for a general warp drive spacetime. Replacing (3.11), (3.12) and (3.13) in the equation (2.42) we have:

$$K_{ij} = \frac{1}{2}\left[D_i X^j + D_j X^i - \frac{\partial \delta_{ij}}{\partial t}\right], \tag{3.30}$$

$$K_{ij} = \frac{1}{2}\left[D_i X^j + D_j X^i\right]. \tag{3.31}$$

Since the spacelike hypersurfaces are 3-Euclidean ($\gamma_{ij} = \delta_{ij}$), then the covariant derivative D_i defined on the hypersurface with metric δ_{ij} will be:

$$D_i \to \partial_i. \tag{3.32}$$

Replacing (3.32) into (3.31) we have:

$$K_{ij} = \frac{1}{2}\left[\partial_i X^j + \partial_j X^i\right]. \tag{3.33}$$

Now, let's calculate the expansion θ for a general warp drive spacetime. Replacing (3.11) and (3.33) into (2.43), we have:

$$\theta = Tr(K_{ij}) = \partial_i X^i. \tag{3.34}$$

Therefore:

$$\theta = \vec{\nabla} \cdot \vec{X}. \tag{3.35}$$

From (3.35) we see that the expansion θ for a general warp drive spacetime is the divergence of the vector field $\vec{X}$.

3.2 Alcubierre metric

The Alcubierre metric is the most studied warp drive metric. For example, Santos-Pereira, Abreu, and Ribeiro studied the Alcubierre metric using Einstein's field equations with cosmological constant and a perfect fluid [25]. Carneiro, Ulhoa, Maluf, and da Rocha-Neto address the problem of negative energy in the Alcubierre metric using a modified theory of gravity, in this case, the Teleparallel Equivalent of General Relativity [6]. Barceló et al. analyzed the stability of Alcubierre-type warp drives (warp drives with a more generic Alcubierre shift-vector) due to the presence of quantum matter [4]. An analysis of a hypothetical interaction between an Alcubierre warp bubble and external particles was done by Mc Monigal, Lewis and O' Byrne [20].

The Alcubierre metric is the first theoretical way to attempt to do superluminal interstellar travels [2]. This metric itself is not a solution for Einstein's field equations in "a normal sense". To solve Einstein's field equations we define certain momentum-energy tensors. After that, we need to solve 10 very difficult non-linear partial differential equations to obtain the metric tensor.

Instead, Alcubierre gives a certain metric tensor. With that metric tensor is very easy to calculate the Einstein tensor and, consequently, the energy-momentum tensor. The Alcubierre strategy was very useful to construct, using the physical intuition, a metric that allows superluminal interstellar travel. In that sense, if we want to travel from star A to star B, the space-time behind our spaceship must be expanded, and in the same way, the space-time in front of our spaceship must be contracted. The Alcubierre metric is given by the following line element [2]:

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left[dz - v(t)f(r(t))dt\right]^2, \tag{3.36}$$

In matrix form:

$$g_{\mu\nu} = \begin{bmatrix} -1+v^2f^2 & 0 & 0 & -vf \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -vf & 0 & 0 & 1 \end{bmatrix}. \tag{3.37}$$

The metric (3.36) represents the spacetime where the warp bubble moves in the $+z$ direction at a speed $v \equiv v(t)$. The spacecraft is at each instant of time in the z-axis position equal to $z_0(t)$. Indeed:

$$v(t) \equiv \frac{dz_0(t)}{dt}. \tag{3.38}$$

Also $r(t)$ is defined by:

$$r(t) = \sqrt{x^2 + y^2 + (z - z_0(t))^2}, \tag{3.39}$$

where $r(t)$ is the spatial distance at time t to the center of the warp bubble. Here too, the radius of the warp bubble will be R and the thickness of the bubble walls will be ε [27]. The center of the warp bubble is where the spacecraft is located. The function $f \equiv f(r(t))$ is called the "form function". In general,

$$\lim_{r \to 0} f(r) = 1, \tag{3.40}$$

The limit (3.40) means that $f(r) = 1$ inside the warp bubble. That is, inside the warp bubble space-time is flat. On the other hand:

$$\lim_{r \to \infty} f(r) = 0. \tag{3.41}$$

The limit (3.41) means that $f(r) = 0$ outside the warp bubble. That is, in the outer region space-time will be flat (as expected). Also, the form function f must drop sharply from the inner region of the warp bubble to the outer region. The abrupt change of f must occur on the walls of the warp bubble. From $f = 1$ to $f = 0$. For that, we must choose a function f appropriate to the requirements we mentioned. Alcubierre made his own choice of form function f [2], and it is as follows:

$$f(r) = \frac{\tanh\left(\frac{r+R}{\varepsilon}\right) - \tanh\left(\frac{r-R}{\varepsilon}\right)}{2\tanh\left(\frac{R}{\varepsilon}\right)}, \tag{3.42}$$

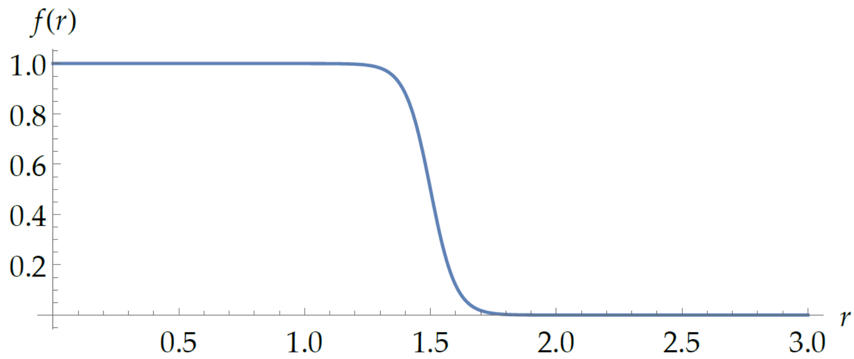

Figure 9 – Alcubierre form function [27].

This form function has the following limit with respect to ε:

$$\lim_{\varepsilon\to 0} f(r) = \begin{cases} 1; & 0 < r \leq R, \\ 0; & R < r < \infty. \end{cases} \tag{3.43}$$

As we mentioned above, the interesting thing about the Alcubierre metric is that it allows trips with a speed $v > 1$ (that is, superluminal trips!).

3.2.1 Is there time dilation?

It is known from special relativity that if a body moves at speeds close to the speed of light, the well-known phenomenon of time dilation will occur. We will also see later in chapter 5 that time dilation will occur in general relativity. A natural question would then be: Does time dilation occur in the Alcubierre metric? To answer this question, let's first look at the trajectory of a spacecraft moving along the z axis in the $+z$ direction [27].

$$(x, y, z) = (0, 0, z_0(t)). \tag{3.44}$$

We know that at the center of the warp bubble $f(0) = 1$. Furthermore, since the spacecraft moves only in the direction of the z axis, then:

$$dx = dy = 0. \tag{3.45}$$

Furthermore, as $z = z_0(t)$ at each instant of time t:

$$d(z) = d(z_0(t)). \tag{3.46}$$

Replacing (3.45) and (3.46) into (3.36) we have:

$$ds^2 = -dt^2 + (dz - v(t)dt)^2. \tag{3.47}$$

Then:

$$ds^2 = -dt^2 + \left(dz_0(t) - \frac{dz_0(t)}{dt}dt\right)^2. \tag{3.48}$$

Therefore:

$$ds^2 = -dt^2. \tag{3.49}$$

From the equation (3.49) we can see that the spacecraft follows a timelike path inside the warp bubble, since $ds^2 < 0$. This is true even for $v > 1$ (since there is no restriction on the magnitude of v). We also know that proper time τ is defined in the following way:

$$ds^2 = -d\tau^2. \tag{3.50}$$

From the equations (3.49) and (3.50) we have:

$$dt^2 = d\tau^2. \tag{3.51}$$

The equation (3.51) indicates that the proper time τ (time interval inside the spacecraft) and the coordinate time t (time measured by a distant observer) are equal. In effect, there is no time dilation.

3.2.2 Light cones

In order to analyze the inclination of light cones we must analyze the path of light in certain directions. Now let's look at the path of light in the z direction. Indeed:

$$dx = dy = 0. \tag{3.52}$$

Also, since we are analyzing the light, then:

$$ds^2 = 0. \tag{3.53}$$

Replacing (3.52) and (3.53) into (3.36) he have:

$$\frac{dz}{dt} = \pm 1 + v(t)f(r(t)). \tag{3.54}$$

The equation (3.54) tells us the following: Outside the warp bubble we have:

$$f \approx 0 \rightarrow \frac{dz}{dt} \approx \pm 1. \tag{3.55}$$

From (3.55) we can see that it represents the light cones of a Minkowski spacetime (as expected). On the other hand, inside the warp bubble we have:

$$f \approx 1 \rightarrow \frac{dz}{dt} \approx \pm 1 + v(t). \tag{3.56}$$

Then, the light cones inside the warp bubble "globally" will have a tilt proportional to $v(t)$. The light cones inside and outside the warp bubble will be illustrated in the figure (10).

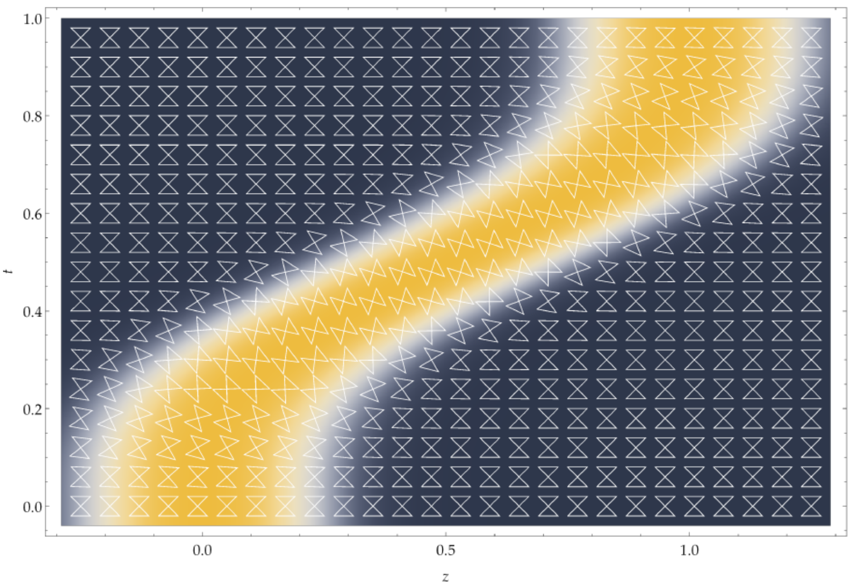

Figure 10 – Space-time diagram for an Alcubierre warp bubble [27]

Figure (10) represents a space-time diagram for a warp bubble (in yellow) with $\varepsilon = 0,1$, $R = 0,25$ and $v(t) = 2\sin^2(\pi t)$. At $t = 0$ the warp bubble is at rest and accelerates to its maximum speed $v = 2$ at $t = 0,5$. After that, the warp bubble decelerates to rest at $t = 1$. As we see from the figure (10), the light cones have slopes $\pm 1 + vf$.

From equations (3.55) and (3.56) we see that, although the spacecraft travels faster than light globally, locally the spacecraft is inside its light cones (as expected for a massive object). The explanation for this fact is as follows: the warp bubble is not a massive object. The warp bubble is spacetime itself curved. Therefore, the warp bubble must not necessarily obey the equations of geodesics. Thus, although globally $v(t) > 1$ the path taken by the warp bubble is spacelike, this does not violate the locality principle of special relativity.

3.2.3 Timelike geodesics

We could calculate the geodesics for the Alcubierre metric using equation (1.64). However, another easier way to do this calculation would be to find an extreme (maximum) of proper time τ. Then:

$$-d\tau^2 = ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{3.57}$$

$$d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}. \tag{3.58}$$

Now we are going to introduce the parameter λ, so that $x^\mu = x^\mu(\lambda)$, then:

$$d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda. \tag{3.59}$$

If we denote $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$ and integrate the equation (3.59), we have:

$$\tau = \int \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\lambda. \tag{3.60}$$

We need to remember:

$$v = v(t), \tag{3.61}$$

$$f = f(r(t)). \tag{3.62}$$

From the Alcubierre metric (3.36), we have:

$$\tau = \int \sqrt{(-1 + v^2 f^2)\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2vf\dot{t}\dot{z}}\, d\lambda. \tag{3.63}$$

In other form:

$$\tau = \int \sqrt{-\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \left(\dot{z} - vf\dot{t}\right)^2} d\lambda. \tag{3.64}$$

Since this square root is a function that increases monotonically (that is, L increases as $r(t)$ increases), we can define the following Lagrangian:

$$L = \frac{1}{2}\left[-\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \left(\dot{z} - vf\dot{t}\right)^2\right]. \tag{3.65}$$

Now we are going to parameterize the geodesic using proper time τ. In this case, we have:

$$x^\mu = x^\mu(\tau). \tag{3.66}$$

And also, let's redefine:

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}. \tag{3.67}$$

Equation (3.67) must represent a timelike 4-velocity. Indeed;

$$\dot{x}_\mu \dot{x}^\mu = -1, \tag{3.68}$$

where (3.67) represents a 4-velocity timelike. Now, in order to maximize this Lagrangian, we will use the Euler-Lagrange equations:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0. \tag{3.69}$$

Replacing (3.65) into (3.69), we will have 4 equations:

$$\frac{d}{d\tau}\left[-\dot{t} - vf(\dot{z} - vf\dot{t})\right] + \frac{\partial(vf)}{\partial t}\dot{t}(\dot{z} - vf\dot{t}) = 0, \tag{3.70}$$

$$\frac{d}{d\tau}(\dot{z} - vf\dot{t}) + \frac{\partial(vf)}{\partial z}\dot{t}(\dot{z} - vf\dot{t}) = 0, \tag{3.71}$$

$$\frac{d\dot{x}}{d\tau} + \frac{\partial(vf)}{\partial x}\dot{t}(\dot{z} - vf\dot{t}) = 0, \tag{3.72}$$

$$\frac{d\dot{y}}{d\tau} + \frac{\partial(vf)}{\partial y}\dot{t}(\dot{z} - vf\dot{t}) = 0. \tag{3.73}$$

In general we need to integrate the 4 equations to obtain the geodesics $\dot{t} = \dot{t}(\tau)$, $\dot{x} = \dot{x}(\tau)$, $\dot{y} = \dot{y}(\tau)$ and $\dot{z} = \dot{z}(\tau)$. And this is generally very complicated. Geodesics are usually found using numerical methods (as we will see later for null geodesics for the Alcubierre metric [7]). However, in this case, by simple inspection, we can find a very simple solution. If we have:

$$\dot{z} - vf\dot{t} = 0. \tag{3.74}$$

Replacing (3.74) into (3.70) - (3.73), we have: $\frac{d\dot{t}}{d\tau} = 0$, $\frac{d\dot{x}}{d\tau} = 0$ and $\frac{d\dot{y}}{d\tau} = 0$. Also, from (3.74) we have:

$$\dot{x} = 0, \tag{3.75}$$

$$\dot{y} = 0. \tag{3.76}$$

We also see that from (3.74), $\dot{t}$ cannot be zero. Also, we know that $\dot{t}$ must be a constant. Then, we will set the value of $\dot{t}$ as follows:

$$\dot{t} = 1. \tag{3.77}$$

Indeed, from (3.74) and (3.77) also we have:

$$\dot{z} = vf. \tag{3.78}$$

Therefore, our 4-velocity $\dot{x}^\mu = (\dot{t}, \dot{x}, \dot{y}, \dot{z})$ will be as follows

$$\dot{x}^\mu = (1, 0, 0, vf). \tag{3.79}$$

Observers with the 4-velocity given by (3.79) are called Eulerian observers. These observers follow timelike geodesics. However, despite this, its apparent speed $v(t)$ relative to a distant observer may be greater than the speed of light. Now, we going to calculate $\dot{x}_\mu$, then:

$$\dot{x}_\mu = g_{\mu\nu}\dot{x}^\nu. \tag{3.80}$$

Then:

$$\dot{x}_0 = g_{00}\dot{x}^0 + g_{01}\dot{x}^1 + g_{02}\dot{x}^2 + g_{03}\dot{x}^3. \tag{3.81}$$

Replacing the metric (3.37) and the 4-velocity (3.79), then:

$$\dot{x}_0 = (-1 + v^2 f^2) - (vf)(vf) = -1. \tag{3.82}$$

In the same way, we can calculate $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$. Therefore:

$$\dot{x}_\mu = (-1, 0, 0, 0). \tag{3.83}$$

The covector $\dot{x}_\mu$ is normal to the spacelike hypersurface with metric $\gamma_{ij} = \delta_{ij}$. Also from (3.79) and (3.83) we can see that:

$$\dot{x}_\mu \dot{x}^\mu = -1. \tag{3.84}$$

Equation (3.84) indicates that the 4-velocity $\dot{x}^\mu$ is timelike and is also normalized.

3.2.4 3+1 formalism

As we have mentioned above, to know more complex properties of the Alcubierre space-time (energy density ρ, expansion θ) it is necessary to express the metric in the 3+1 formalism. Again, for simplicity we will denote $v = v(t)$ and $f = f(r(t))$. Indeed, from the equation (3.36) we have:

$$ds^2 = (-1 + v^2 f^2)dt^2 - 2vf dzdt + dx^2 + dy^2 + dz^2. \tag{3.85}$$

Also, from the equation (3.85) it is easy to show that α, β^i and γ_{ij} will have the following values for the Alcubierre metric:

$$\alpha = 1, \tag{3.86}$$

$$\beta^i = (0, 0, vf), \tag{3.87}$$

$$\gamma_{ij} = \delta_{ij}. \tag{3.88}$$

From (3.86) we can see that the lapse function α corresponds to an Eulerian observer. Also, from the equation (3.88) we can see that the induced metric will correspond to the 3-dimensional flat space.

3.2.5 Expansion and contraction of space

Our purpose now is to know how spacetime contracts or expands in Alcubierre spacetime. For this, let us first calculate the trace of the extrinsic curvature tensor K_{ij} given in (2.42) for the Alcubierre metric. So:

$$K_{xx} = \frac{1}{2}\left(\partial_x\beta_x + \partial_x\beta_x - \frac{\partial\gamma_{xx}}{\partial t}\right) = 0, \tag{3.89}$$

$$K_{yy} = \frac{1}{2}\left(\partial_y\beta_y + \partial_y\beta_y - \frac{\partial\gamma_{yy}}{\partial t}\right) = 0. \tag{3.90}$$

From equations (3.89) and (3.90) we can notice that we have replaced the covariant derivatives D_i with the partial derivatives ∂_i. This is because the spacelike hypersurfaces are flat, that is, $\gamma_{ij} = \delta_{ij}$. Now we are going to calculate K_{zz}, then:

$$K_{zz} = \frac{1}{2}\left(\partial_z\beta_z + \partial_z\beta_z - \frac{\partial\gamma_{zz}}{\partial t}\right), \tag{3.91}$$

$$K_{zz} = \frac{1}{2}\left(\partial_z(vf) + \partial_z(vf) - \frac{\partial\delta_{zz}}{\partial t}\right), \tag{3.92}$$

$$K_{zz} = \frac{\partial(vf)}{\partial z} = v\frac{\partial f}{\partial z} = v\frac{\partial r}{\partial z}\frac{\partial f}{\partial r}. \tag{3.93}$$

Now, we need to remember that $r = r(t)$. And from (3.93) we have:

$$K_{zz} = v\frac{\partial}{\partial z}\left[x^2 + y^2 + (z - z_0(t))^2\right]^{1/2}\frac{\partial f}{\partial r}, \tag{3.94}$$

$$K_{zz} = \frac{v}{r}\left(z - z_0(t)\right)\frac{\partial f}{\partial r}. \tag{3.95}$$

Finally, we replace the equations (3.11), (3.89), (3.90) and (3.95) in (2.43), and we obtain the expansion θ:

$$\theta = \frac{v}{r}\left(z - z_0(t)\right)\frac{\partial f}{\partial r}. \tag{3.96}$$

The equation (3.96) can be graphed on a computer. If parameters $v = 2$, $R = 1,5$, and $\varepsilon = 0,1$ are entered, we can see the expansion θ. Then, we can see a spacecraft moving in the direction of the positive z axis, in the figure (11).

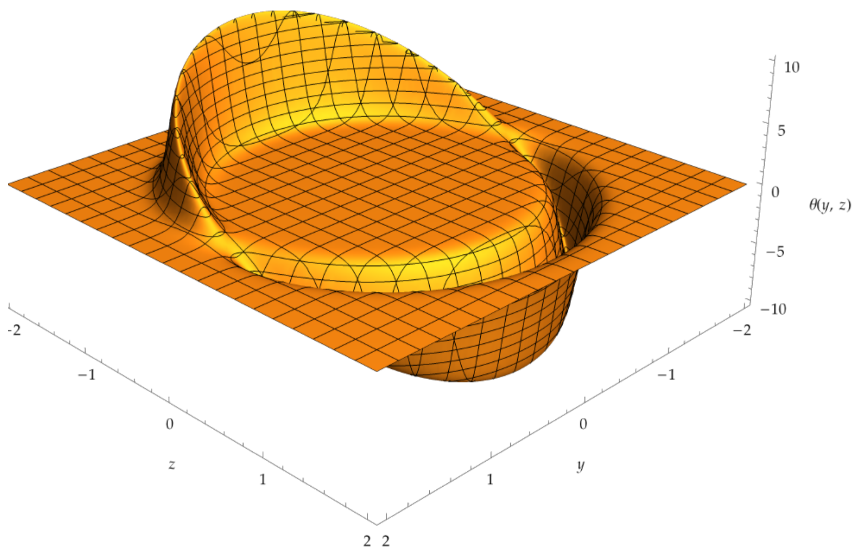

Figure 11 – Plot of the expansion $\theta(y, z)$ of the Alcubierre warp drive [27].

We know that $f \approx 1$ inside the bubble. On the other hand, $f \approx 0$ outside the bubble. In both cases, from the figure (11) we can see that $\theta \approx 0$ in both regions (inside and outside the warp bubble). Only on the walls of warp bubble $\theta \neq 0$. Also, as we saw previously, f decreases rapidly from inside the warp bubble outward, from 1 to 0. In effect:

$$\frac{\partial f}{\partial r} < 0. \tag{3.97}$$

So, from (3.96) and (3.97) we can deduce that:

$$z > z_0(t) \to \theta < 0, \tag{3.98}$$

$$z < z_0(t) \to \theta > 0. \tag{3.99}$$

Physically (3.98) and (3.99) tell us the following: For values $z > z_0(t)$ (ahead of the spaceship), then the spacetime on the walls of the warp bubble contract. And similarly, for values of $z < z_0(t)$ (behind the spaceship), the spacetime on the walls of the warp bubble expands. This behavior can be seen in the graph of expansion θ.

3.2.6 Negative energy density

Let n^{μ} be the 4-velocity (timelike) of an Eulerian observer. If n^{μ} is normalized, then, $n^{\mu}n_{\mu} = -1$. Now, from the Einstein field equations (1.112) and the equation (2.46) we have:

$$\rho = T_{\mu\nu}n^{\mu}n^{\nu} = \frac{1}{8\pi}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)n^{\mu}n^{\nu}. \tag{3.100}$$

Using the equation (2.49) (expression of the energy density ρ in the 3+1 formalism) we have:

$$\rho = \frac{1}{16\pi}\left[(K_{xx} + K_{yy} + K_{zz})^2 - K_{ij}K^{ij}\right]. \tag{3.101}$$

In the equation (3.101) ${}^{(3)}\mathrm{R} = 0$ because $\gamma_{ij} = \delta_{ij}$ (that is, we have flat spacelike hypersurfaces). Also, from (3.89) and (3.90), we know that $K_{xx} = K_{yy} = 0$. Then:

$$\rho = \frac{1}{16\pi}\left[(K_{zz})^2 - \{K_{zz}K^{zz} + 2K_{xy}K^{xy} + 2K_{xz}K^{xz} + 2K_{yz}K^{yz}\}\right]. \tag{3.102}$$

We also know that the contravariant extrinsic curvature tensor K^{ij} is given by:

$$K^{ij} = \delta^{ip}\delta^{jp}K_{pq} = K_{ij}. \tag{3.103}$$

Then, replacing (3.103) into (3.102) we have:

$$\rho = \frac{1}{16\pi}\left[(K_{zz})^2 - \left\{(K_{zz})^2 + 2(K_{xy})^2 + 2(K_{xz})^2 + 2(K_{yz})^2\right\}\right], \tag{3.104}$$

$$\rho = -\frac{1}{8\pi}\left[(K_{xy})^2 + (K_{xz})^2 + (K_{yz})^2\right]. \tag{3.105}$$

Applying the definition of extrinsic curvature tensor K_{ij} from (2.42) in (3.105) we have:

$$\rho = -\frac{1}{8\pi}\left[\frac{1}{4}(\partial_x\beta_y + \partial_y\beta_x)^2 + \frac{1}{4}(\partial_x\beta_z + \partial_z\beta_x)^2 + \frac{1}{4}(\partial_y\beta_z + \partial_z\beta_y)^2\right]. \tag{3.106}$$

We need to remember that in the Alcubierre metric, from the expression (3.87), the shift vector β^i is given by the following components: $\beta_x = \beta_y = 0$ and $\beta_z = vf$. Indeed, replacing (3.87) into (3.106) we have:

$$\rho = -\frac{1}{32\pi}\left[(\partial_x\beta_z)^2 + (\partial_y\beta_z)^2\right]. \tag{3.107}$$

Replacing (3.87) into (3.107) we have:

$$\rho = -\frac{1}{32\pi}\left[\left(\frac{\partial(vf)}{\partial x}\right)^2 + \left(\frac{\partial(vf)}{\partial y}\right)^2\right]. \tag{3.108}$$

Indeed:

$$\rho = -\frac{v^2}{32\pi}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \leq 0. \tag{3.109}$$

From equation (3.109) we can see the energy density ρ will always have a non-positive value. Also, ρ will be zero if and only if $v = 0$ and/or $f = 0$. Therefore, physically the expression (3.109) tells us that the weak energy condition (WEC) is violated for the Alcubierre spacetime (with respect to an Eulerian observer).

Also, since we know that f is a function of $r(t)$, so we replace (3.39) into (3.109) and we have:

$$\rho = -\frac{v^2}{32\pi}\left[\left(\frac{\partial r}{\partial x}\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial r}{\partial y}\frac{\partial f}{\partial r}\right)^2\right], \tag{3.110}$$

$$\rho = -\frac{v^2}{32\pi}\left[\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2+(z-z_0(t))^2}\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial}{\partial y}\sqrt{x^2+y^2+(z-z_0(t))^2}\frac{\partial f}{\partial r}\right)^2\right], \tag{3.111}$$

$$\rho = -\frac{v^2}{32\pi}\left[\frac{x^2}{r^2}\left(\frac{\partial f}{\partial r}\right)^2 + \frac{y^2}{r^2}\left(\frac{\partial f}{\partial r}\right)^2\right]. \tag{3.112}$$

Indeed:

$$\rho = -\frac{v^2}{32\pi}\frac{(x^2+y^2)}{r^2}\left(\frac{df}{dr}\right)^2. \tag{3.113}$$

From equation (3.113) we can see that the dependence of the derivative $\frac{df}{dr}$ physically means that the WEC violation is negligible inside and outside the warp bubble. The violation of the WEC will be noticeable only on the walls of the warp bubble. This is illustrated in the graph (12).

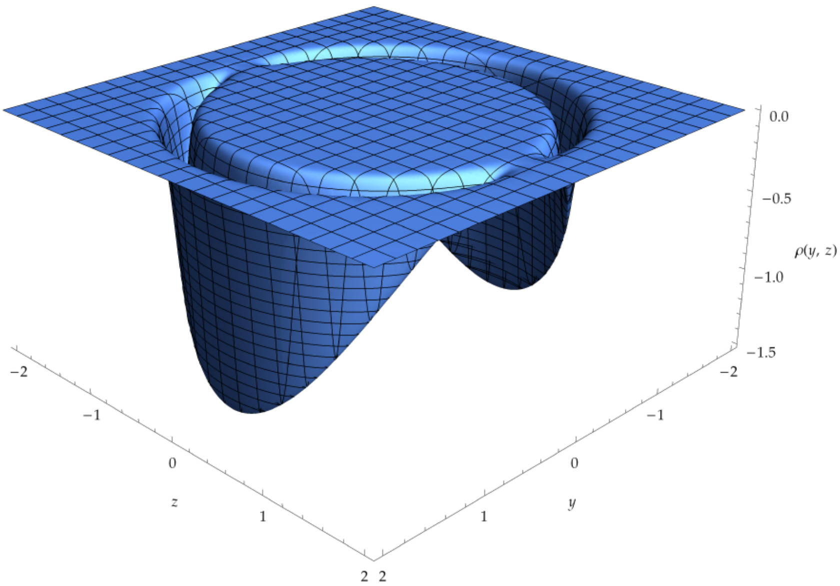

Figure 12 – Plot of the density $\rho(y,z)$ of the Alcubierre warp drive [27].

Figure (12) represents the energy density distribution ρ around a warp bubble with $v = 2$, $R = 1,5$, and $\varepsilon = 0,1$. We are considering that this warp bubble moves in the direction of the positive z axis.

Now the natural question would be: how much energy E is needed for an Alcubierre warp drive? So far, we have calculated ρ (measured by an Eulerian observer). Now with this result, we must integrate all spacelike hypersurfaces (which in this case are Euclidean spaces). Indeed:

$$E = \int \rho d^3x. \tag{3.114}$$

We will use spherical coordinates, and in this case:

$$d^3x = r^2 \sin(\theta) dr d\theta d\phi. \tag{3.115}$$

Also x and y in spherical coordinate are: $x = r\sin(\theta)\cos(\phi)$ and $y = r\sin(\theta)\sin(\phi)$. Then:

$$x^2 + y^2 = r^2 \sin^2(\theta). \tag{3.116}$$

Then, replacing (3.113), (3.115) and (3.116) into (3.114):

$$E = -\frac{v^2}{32\pi} \int \frac{(r^2 \sin^2(\theta))}{r^2} \left(\frac{df}{dr}\right)^2 r^2 \sin(\theta) dr d\theta d\phi, \tag{3.117}$$

$$E = -\frac{v^2}{32\pi} \int_0^\infty \left(\frac{df}{dr}\right)^2 r^2 dr \int_0^\pi \sin^3(\theta) d\theta \int_0^{2\pi} d\phi, \tag{3.118}$$

We know that: $\int_0^{2\pi} d\phi = 2\pi$ and $\int_0^\pi \sin^3(\theta) d\theta = 4/3$. Then finally:

$$E = -\frac{v^2}{12} \int_0^\infty \left(\frac{df}{dr}\right)^2 r^2 dr. \tag{3.119}$$

The value of the integral (3.119) was estimated [11] [27] and its approximate result is the following:

$$E = -\frac{v^2 R^2}{\varepsilon}. \tag{3.120}$$

From equation (3.120) we can see that we need large amounts of negative energy E to have high apparent warp bubble speeds v. Also, large amounts of negative energy are also needed for a larger warp bubble. We notice that the negative energy E and the thickness of the warp bubble ε are inversely proportional. Finally, it is important to note that, even for apparent subluminal speeds $v < 1$, an Alcubierre warp drive will always require negative energy.

A quite natural question would be: Can the form function chosen by Alcubierre be changed by another form function, in such a way that the amounts of negative energy can be reduced? Answering this question, Pfenning and Ford [11] introduced the following piecewise continuous form function:

$$f_{(r)} = \begin{cases} 1 & r \leq R - \frac{\varepsilon}{2}, \\ \frac{1}{2} + \frac{R-r}{\varepsilon} & R - \frac{\varepsilon}{2} < r < R + \frac{\varepsilon}{2}, \\ 0 & r \geq R + \frac{\varepsilon}{2}, \end{cases} \tag{3.121}$$

where R is the radius of the warp bubble and ε is the wall thickness of the warp bubble. Replacing (3.121) into (3.119) we have:

$$E_{PF} = -\frac{v^2}{12} \int_{R-\frac{\varepsilon}{2}}^{R+\frac{\varepsilon}{2}} \frac{1}{\varepsilon^2} r^2 dr = -\frac{v^2}{12} \left(\frac{R^2}{\varepsilon} + \frac{\varepsilon}{12}\right), \tag{3.122}$$

where E_{PF} represents the energy calculated using the form function introduced by Pfenning and Ford [11]. It is important to note that also in (3.122) v is assumed constant. Then, for superluminal speeds $v > 1$ and for a thin warp bubble wall, from equations (3.120) and (3.122) we can see:

$$E_{PF} < E. \tag{3.123}$$

However, we can see that, although the form function introduced by Pfenning and Ford somewhat reduces the amounts of negative energy for a warp bubble, this reduction does not seem to be very important [11].

Since we are dealing with negative energy, then quantum fields are going to be necessary (to take advantage of phenomena such as the Casimir effect, for example). On the other hand, Ford and Roman established the relationship between quantum fields and quantum energy inequalities [12]. Taking advantage of this result, Pfenning and Ford estimated the order of magnitude of the energy needed for an Alcubierre warp drive [11]. Considering a warp bubble of radius $R = 100m$ traveling with an apparent speed $v = 1$ (in geometric units), the amount of energy would have the absurd value of:

$$E \approx -10^{63}kg. \tag{3.124}$$

This amount of energy is 10 orders of magnitude larger than the mass of the entire observable universe (which is approximately $10^{53}kg$ [24]). This result gives the Alcubierre warp drive a non-physical nature [11].

3.3 Natario without expansion metric

It was believed that the expansion/contraction rate of spacetime would allow the superluminal characteristic of spacetime warp drives. However, Natario demonstrated that this is not a necessary condition [22]. The warp drive with zero expansion proposed by Natario will move with superluminal speed with $\theta = 0$. Using isomorphisms in 3-Euclidean space (for more details see [22]) Natario proposed the following vector field $\vec{X}$ in spherical coordinates:

$$\vec{X} = X^r e_r + X^\theta e_\theta + X^\phi e_\phi, \tag{3.125}$$

$$\vec{X} = -2v_s f\cos(\theta)e_r + v_s(2f + rf')\sin(\theta)e_\theta, \tag{3.126}$$

Namely:

$$X^r = -2v_s f\cos(\theta), \tag{3.127}$$

$$X^\theta = v_s(2f + rf')\sin(\theta), \tag{3.128}$$

$$X^\phi = 0, \tag{3.129}$$

where $v_s = v_s(t)$ is the speed of the spaceship. Also e_r and e_θ are unit coordinate vectors in spherical coordinates. The function $f = f(r)$ is the form function (a bounded and soft function). with the following properties:

$$r = \infty \rightarrow f_{(r)} = \frac{1}{2}, \tag{3.130}$$

$$r = 0 \rightarrow f_{(r)} = 0. \tag{3.131}$$

Also:

$$f' \equiv \frac{df}{dr}. \tag{3.132}$$

If $r = \infty$, replacing (3.127) - (3.130) and (3.132) into (3.126), then we have:

$$\vec{X}_{(r=\infty)} = -v_s \cos(\theta) e_r + v_s \sin(\theta) e_\theta. \tag{3.133}$$

In Cartesian coordinates (with unit vectors e_x, e_y and e_z) we have:

$$\begin{aligned}\vec{X}_{(r=\infty)} = &-v_s \cos(\theta) \left[\sin(\theta)\cos(\phi)e_x + \sin(\theta)\sin(\phi)e_y + \cos(\theta)e_z\right] + \\ &+ v_s \sin(\theta)\left[\cos(\theta)\cos(\phi)e_x + \cos(\theta)\sin(\phi)e_y - \sin(\theta)e_z\right].\end{aligned} \tag{3.134}$$

Indeed:

$$\vec{X}_{(r=\infty)} = -v_s e_z. \tag{3.135}$$

From (3.135) we can see that the warp bubble will move at the speed of v_s. Now let's look at the other case when $r = 0$. Then:

$$\vec{X}_{(r=0)} = 0. \tag{3.136}$$

From the expression (3.136) we can see that in the vicinity of the center of the warp bubble, the spacetime will be Minkowski.

3.3.1 Zero expansion case

Now we are going to calculate the expansion θ in this spacetime. The expression (2.42) that gives us the extrinsic curvature tensor K_{ij} is in Cartesian coordinates. However, our vector field $\vec{X}$ is in spherical coordinates. Transforming the expression (2.42) into spherical coordinates (for more details see [22]) we have the following components of the extrinsic curvature tensor K_{ij}:

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s f' \cos(\theta), \tag{3.137}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial \theta} + \frac{X^r}{r} = v_s f' \cos(\theta), \tag{3.138}$$

$$K_{\phi\phi} = \frac{1}{r\sin(\theta)}\frac{\partial X^\phi}{\partial \phi} + \frac{X^r}{r} + \frac{X^\theta \cot(\theta)}{r} = v_s f' \cos(\theta). \tag{3.139}$$

Replacing (3.137), (3.138) and (3.139) into (2.43) we have:

$$\theta = K_{rr} + K_{\theta\theta} + K_{\phi\phi}, \tag{3.140}$$

$$\theta = 0. \tag{3.141}$$

The expression (3.141) shows that a general superluminal spacetime warp drive can have $\theta = 0$. To better understand this, let's look at a small example. On the front area of the warp bubble wall, namely $\cos(\theta) > 0$ and $f' \neq 0$, analyzing the expressions (3.137) - (3.139) we have the following:

$$f' > 0 \rightarrow K_{rr} < 0, \tag{3.142}$$

$$f' > 0 \rightarrow K_{\theta\theta} > 0, \tag{3.143}$$

$$f' > 0 \rightarrow K_{\phi\phi} > 0. \tag{3.144}$$

The expressions (3.142), (3.143) and (3.144) tell us the following: On the front area of the warp bubble wall, the contraction of spacetime is in a radial direction. However, this contraction is perfectly balanced by the expansion of spacetime in directions θ and ϕ. In effect, we have: $K_{\theta\theta} + K_{\phi\phi} = -K_{rr}$.

3.3.2 Negative energy density

To calculate the energy density ρ it is necessary to calculate the other components of the extrinsic curvature tensor K_{ij}. Then we have:

$$K_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^{\theta}}{r}\right) + \frac{1}{r}\frac{\partial X^{r}}{\partial\theta}\right] = v_s\sin(\theta)\left(f' + \frac{rf''}{2}\right), \tag{3.145}$$

$$K_{r\phi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{X^{\phi}}{r}\right) + \frac{1}{r\sin(\theta)}\frac{\partial X^{r}}{\partial\phi}\right] = 0, \tag{3.146}$$

$$K_{\theta\phi} = \frac{1}{2}\left[\frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\left(\frac{X^{\phi}}{\sin(\theta)}\right) + \frac{1}{r\sin(\theta)}\frac{\partial X^{\theta}}{\partial\phi}\right] = 0, \tag{3.147}$$

Now, we use the equation (2.49) to calculate the energy density ρ. Since we have $\theta = 0$, then we have the following:

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij}. \tag{3.148}$$

Using the equation (2.42) in spherical coordinates we have:

$$\rho = -\frac{1}{16\pi}\left[(K_{rr})^2 + (K_{\theta\theta})^2 + (K_{\phi\phi})^2 + 2(K_{r\theta})^2 + 2(K_{r\phi})^2 + 2(K_{\theta\phi})^2\right]. \tag{3.149}$$

Replacing (3.137) - (3.139) and (3.145) - (3.147) into (3.149), then:

$$\rho = -\frac{1}{16\pi}\left[6\left(v_s f'\cos(\theta)\right)^2 + 2v_s^2\sin^2(\theta)\left(f' + \frac{rf''}{2}\right)^2\right]. \tag{3.150}$$

Indeed, finally:

$$\rho = -\frac{v_s^2}{8\pi}\left[3(f')^2\cos^2(\theta) + \left(f' + \frac{rf''}{2}\right)^2\sin^2(\theta)\right]. \tag{3.151}$$

From equations (3.148) and (3.151) we can see that a warp drive without expansion will always require negative energy density ρ (measured with respect to an Eulerian observer).

4 The horizon problem

Warp drive spacetimes not only has the problem of negative energy. They also have an additional problem, the so-called horizon problem. Specifically, when we deal with superluminal warp drives, horizons will form. That is, if we have a superluminal warp drive, the interior of the warp drive will be causally isolated from the outside. This feature means that observers within the warp drive cannot control the warp bubble at will [18].

Hiscock showed that in an Alcubierre warp drive in 2+1 dimensions the momentum energy tensor diverges (for more details see [14]). Hiscock showed that this divergence is associated with the formation of a horizon [14]. Analyzing the Alcubierre metric in 1+1 dimensions, Krasnikov also found the horizon problem in the Alcubierre metric [15]. Because of this, Krasnikov proposed a new superluminal travel mechanism, different from the warp drive, which was later called the "Krasnikov tube" [9].

In this chapter, we will show the necessary elements to understand the horizon problem. In addition to that, we will show how to determine horizons. Calculating null geodesics, we will first look at the horizon problem for the general case (Natario warp drive). Then we will do an analogous procedure for the case of the Alcubierre metric. In addition, we will see a new problem related to the horizon problem [7] [22]: The blueshift problem.

4.1 The infinite redshift surface

First, before thinking about the horizon, we must know how to identify surfaces that represent a horizon, with others that are not (which is due to an inappropriate choice of coordinates). For this, we are going to study the redshift of the light emitted from a source at rest in a given point x_s^μ of space-time with metric $g_{\mu\nu}$.

The proper time that passes at the source (where the light beam leaves) is τ_s. The proper time that passes at the point of arrival (where the light beam arrives) is τ_a [1] [21]. Then:

$$d\tau_s = \sqrt{g_{00(x_s^\mu)}}dt, \tag{4.1}$$

$$d\tau_a = \sqrt{g_{00(x_a^\mu)}}dt, \tag{4.2}$$

where x_s^μ represents the event when the light beam leaves and x_a^μ represents the event when the light beam arrives. In the weak gravitational field:

$$\sqrt{g_{00(x_s^\mu)}}dt \approx \sqrt{\left(1 + \frac{2\phi_s}{c^2}\right)}dt, \tag{4.3}$$

$$\sqrt{g_{00(x_a^\mu)}}dt \approx \sqrt{\left(1+\frac{2\phi_a}{c^2}\right)}dt, \tag{4.4}$$

where ϕ_s represents the gravitational field of the starting point of the light beam. And ϕ_a represents the gravitational field of the point of arrival of the light beam.

Now consider n waves with frequency ν_s that are emitted from x_s^μ in the time interval $\Delta\tau_s$, then:

$$n = \nu_s \Delta\tau_s. \tag{4.5}$$

Now, in x_a^μ the n waves are going to arrive. These n waves will arrive with another frequency ν_a in a time interval $\Delta\tau_a$:

$$n = \nu_a \Delta\tau_a. \tag{4.6}$$

From (4.5) and (4.6):

$$\nu_a = \nu_s \frac{\Delta\tau_s}{\Delta\tau_a}. \tag{4.7}$$

Replacing (4.1) and (4.2) into (4.7), we finally have:

$$\nu_a = \nu_s \sqrt{\frac{g_{00(x_s^\mu)}}{g_{00(x_a^\mu)}}}. \tag{4.8}$$

The infinite red shift surfaces $\mathcal{I}$ are given when $g_{00(x_s^\mu)} = 0$. From equation (4.8), then:

$$\nu_a = 0. \tag{4.9}$$

In conclusion, if we want to search for infinite redshift surfaces $\mathcal{I}$ we simply have to do the following:

$$g_{00} = 0. \tag{4.10}$$

From (4.10) we can find a certain surface $\mathcal{I}_{(x^i)} = constant$ that will represent the infinite redshift surface (or hypersurface) of that spacetime in a certain coordinate system [1] [21].

4.1.1 Example: The Schwarzschild metric

The Schwarzschild metric has the following element line (with coordinates t, r, θ and ϕ):

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2m}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.11}$$

From the metric (4.11) g_{00} is:

$$g_{00} = -\left(1-\frac{2m}{r}\right). \tag{4.12}$$

From (4.10) and (4.12) we can see that the infinite redshift surface $\mathcal{I}_{(x^i)}$ for the Schwarzschild metric will be:

$$r = 2m. \tag{4.13}$$

The surface given by (4.13) represents a spherical surface with radius $r = 2m$. This surface is widely known in the literature as the event horizon of a Schwarzschild black hole [8][13][31].

4.1.2 Example: The Kerr metric

The Kerr metric has the following element line in the coordinates t, ρ, θ, and ϕ [1] [21].

$$ds^2 = -\left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)dt^2 + \left(\frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho}\right)d\rho^2 + \left(\rho^2 + a^2\cos^2\theta\right)d\theta^2 + \\ + \left[\left(\rho^2 + a^2\right)\sin^2\theta + \frac{2m\rho a^2\sin^4\theta}{\rho^2 + a^2\cos^2\theta}\right]d\phi^2 + 2\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}dtd\phi. \quad (4.14)$$

From the metric (4.14) g_{00} is:

$$g_{00} = -\left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right). \quad (4.15)$$

From (4.10) and (4.15) we can see that the infinite redshift surface $\mathcal{I}_{(x^i)} = constant$ for the Kerr metric will be:

$$\rho = m \pm \sqrt{m^2 - a^2\cos^2\theta}. \quad (4.16)$$

The equation (4.16) represents the infinite redshift surface for the Kerr metric in the coordinates t, ρ, θ and ϕ. We must also assume that $|a| < m$. This last condition will make the surface well defined, that is, ρ is real. In this case, we have 2 surfaces:

$$\rho_{\pm} = m \pm \sqrt{m^2 - a^2\cos^2\theta}. \quad (4.17)$$

We can also see from (4.17) that in the limit:

$$a \to 0. \quad (4.18)$$

The equations (4.17) will be:

$$\rho^{+}_{a\to 0} = 2m, \quad (4.19)$$

$$\rho^{-}_{a\to 0} = 0. \quad (4.20)$$

By including the condition (4.18) in the surfaces (4.17), the infinite redshift surfaces are reduced to $\rho^{+}_{a\to 0}$ and $\rho^{-}_{a\to 0}$. Here $\rho^{+}_{a\to 0}$ represents the event horizon of a Schwarzschild black hole and $\rho^{-}_{a\to 0}$ represents the intrinsic singularity in the Schwarzschild black hole [1] [21].

4.2 One-way membrane

In this section, we will show one of the first treatments that were done to the one-way membranes $\mathcal{N}$. It does not have a very sophisticated mathematical treatment, but it was very useful for studying various classical space-times. The one-way membranes will be studied using the concept of the so-called null hypersurface (following the mathematical treatment in [1]). Consider a smooth hypersurface $\mathcal{S}$ defined by the equation:

$$\mathcal{U}_{(x^\mu)} = constant. \quad (4.21)$$

The vector n_α will represent the vector normal to the hypersurface, then:

$$n_\alpha = \partial_\mu \mathcal{U}. \tag{4.22}$$

Also, we know:

$$d\mathcal{U} = n_\alpha dx^\alpha = 0. \tag{4.23}$$

We know that any manifold is locally flat (Minkowski). In that case, at any point, we have the following line element:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{4.24}$$

Without loss of generality, we can define n^α as follows:

$$n^\alpha = (n^0, n^1, 0, 0). \tag{4.25}$$

Then:

$$n^\alpha n_\alpha = -\left(n^0\right)^2 + \left(n^1\right)^2. \tag{4.26}$$

Let t^α be the tangent vector to n_α, then:

$$n_\alpha t^\alpha = 0. \tag{4.27}$$

Then:

$$-n^0 t^0 + n^1 t^1 = 0. \tag{4.28}$$

Consequently:

$$\frac{t^0}{t^1} = \frac{n^1}{n^0}. \tag{4.29}$$

From (4.29), t^α should have the following form:

$$t^\alpha = \lambda(n^1, n^0, a, b), \tag{4.30}$$

where λ, a and b are arbitrary. From (4.30), then, for $t^\alpha t_\alpha$ we have the following:

$$t^\alpha t_\alpha = \lambda^2 \left[-\left(n^1\right)^2 + \left(n^0\right)^2 + a^2 + b^2\right]. \tag{4.31}$$

Replacing (4.26) in (4.31), we have:

$$t^\alpha t_\alpha = \lambda^2 \left[-n^\alpha n_\alpha + \left(a^2 + b^2\right)\right]. \tag{4.32}$$

This simple relation (4.32) between the norms of n_α and t^α leads to a beautiful geometric result. Thus, analyzing (4.32) we will have 3 cases:

- **Case I:** If n^α is timelike, then $n^\alpha n_\alpha < 0$. Therefore $t^\alpha t_\alpha > 0$, so t^α is spacelike.

- **Case II:** If n^α is null, then $n^\alpha n_\alpha = 0$. Therefore $t^\alpha t_\alpha \geq 0$.

  In particular, we are going to be interested in the particular case where: $a = b = 0$. If $a = b = 0$ so t^α is null. Then, there exists a tangent vector t^α to $\mathcal{S}$, which together with its multiples remain on the local light cone at point P.

- **Case III:** If n^α is spacelike, then $n^\alpha n_\alpha > 0$. Therefore $t^\alpha t_\alpha$ could be positive, negative, or zero.

  In particular, we have $t^\alpha t_\alpha = 0$ on the circle defined by $a^2 + b^2 = n^\alpha n_\alpha > 0$. In effect, there exists a "family" of tangent vectors t^α to $\mathcal{S}$, which also remain on the local light cone centered at P.

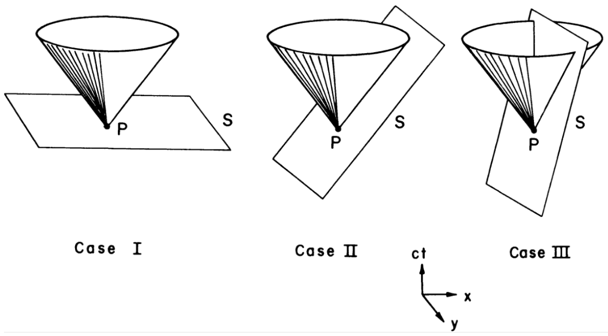

Figure 13 – The three possible relationships between a hypersurface $\mathcal{S}$ with the local light cone. [1]

Figure (13) represents the three types of relationship between the hypersurface $\mathcal{S}$ and the local light cone, with the z coordinate suppressed. Now we see the physical interpretation of (13). We know that light has trajectories on the light cone of the future. We also know that massive bodies have trajectories within the light cone. Based on this, let's look at the three cases:

- **Case I:** We have n^α timelike. Therefore, the hypersurface $\mathcal{S}$ will be timelike too. And from (13) we can see that physical objects can traverse a timelike hypersurface in only one direction.

- **Case III:** We have n^α spacelike. Therefore, the hypersurface $\mathcal{S}$ will be spacelike too. And from (13) we can see that physical objects can traverse a spacelike hypersurface in any direction.

- **Case II:** We have n^α null. Therefore, the hypersurface $\mathcal{S}$ will be null too. And from (13) we can see that this is the critical case, where the unidirectional behavior starts. This is the reason why we are going to interpret the null hypersurface as a "one-way membrane".

We can have an easy example for Minkowski spacetime. The hypersurface $t = constant$ is timelike. In this case, physical objects can traverse this hypersurface only in one direction. On the other hand, hypersurface $x = constant$ is spacelike. In this case, physical objects can traverse this hypersurface in any direction. The null hypersurface is given by $ct - x = 0$, and indeed, this hypersurface is a one-way membrane. We can continue investigating one-way membranes in other types of more complex spacetime. We will do that later.

4.2.1 Example: Schwarzschild metric

We saw from the metric (4.11) that it has spherical symmetry. For that reason, a spherical surface with $r = constant$ has the following normal vector n_α:

$$n_\alpha = (0, 1, 0, 0). \tag{4.33}$$

Now we must calculate the contravariant tensor $g^{\alpha\beta}$. After that, we must use the following:

$$n^\alpha = g^{\alpha\beta} n_\beta. \tag{4.34}$$

Using the last result, we can calculate:

$$n^\alpha n_\alpha = 1 - \frac{2m}{r}. \tag{4.35}$$

Now, let's impose that n_α is null. indeed:

$$n^\alpha n_\alpha = 0. \tag{4.36}$$

From (4.35) and (4.36) we have the following:

$$r = 2m. \tag{4.37}$$

We can see that equation (4.37) represents the one-way membrane $\mathcal{N}$ for the Schwarzschild metric in these coordinates. We must note that in this case, from equation (4.13), the one-way membrane $\mathcal{N}$ and the infinite redshift hypersurface $\mathcal{I}$ are the same [1] [21].

4.2.2 Example: Kerr metric

Now let's look at null hypersurfaces in the Kerr metric. We had found $\mathcal{I}$ for the Kerr metric, and obtained two hypersurfaces: $\rho_+^{\mathcal{I}}$ and $\rho_-^{\mathcal{I}}$. From the previous section, we might think that in the Kerr metric, the infinite redshift surfaces are also one-way membranes. But as we will see later this will not be the case. For that, let's see, if $\rho_+^{\mathcal{I}}$ is a one-way membrane. From (4.17) we have:

$$\varphi \equiv \rho - m - \sqrt{m^2 - a^2 \cos^2\theta} = 0, \tag{4.38}$$

where φ represents the hypersurface. If we define n_α as the normal vector to the hypersurface φ. Then:

$$n_\alpha = \left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial\rho}, \frac{\partial\varphi}{\partial\theta}, \frac{\partial\varphi}{\partial\phi}\right). \tag{4.39}$$

Replacing (4.39) with (4.16), we have:

$$n_\alpha = \left(0, 1, -\frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{m^2 - a^2\cos^2(\theta)}}, 0\right). \tag{4.40}$$

Now, from the Kerr metric (4.14), we can calculate $g^{\alpha\beta}$. Using (1.21) we can calculate $n^\alpha n_\alpha$:

$$n^\alpha n_\alpha = \frac{\rho^2 + a^2 - 2m\rho + \frac{a^4\cos^2\theta\sin^2\theta}{m^2 - a^2\cos^2\theta}}{\rho^2 + a^2\cos^2\theta}. \tag{4.41}$$

From equation (4.41) we can see that $n^\alpha n_\alpha$ will always be positive, that is, $\rho_+^{\mathcal{I}}$ is a spacelike hypersurface. Consequently, the infinite redshift hypersurface $\rho_+^{\mathcal{I}}$ is not a one-way membrane.

Our task now is to search for an axially symmetric, time-independent null hypersurface. Let this hypersurface $\mathcal{U}$ be given in the following way [1]:

$$\mathcal{U}_{(\rho,\theta)} = 0. \tag{4.42}$$

Then, n_α will be defined by:

$$n_\alpha = \left(0, \frac{\partial\mathcal{U}}{\partial\rho}, \frac{\partial\mathcal{U}}{\partial\theta}, 0\right). \tag{4.43}$$

Because we are imposing that n_α is null, then $n^\alpha n_\alpha = 0$. In effect, from (4.36) and (4.43) we have:

$$\left(\rho^2 - 2m\rho + a^2\right)\left(\frac{\partial\mathcal{U}}{\partial\rho}\right)^2 + \left(\frac{\partial\mathcal{U}}{\partial\theta}\right)^2 = 0. \tag{4.44}$$

The differential equation (4.44) will be solved using the following [1]:

$$\mathcal{U}_{(\rho,\theta)} = \mathcal{R}_{(\rho)}\Theta_{(\theta)}. \tag{4.45}$$

Replacing (4.44) with (4.45) we find:

$$-\left(\rho^2 - 2m\rho + a^2\right)\frac{1}{\mathcal{R}^2}\left(\frac{\partial\mathcal{R}}{\partial\rho}\right)^2 = \frac{1}{\Theta^2}\left(\frac{\partial\Theta}{\partial\theta}\right)^2. \tag{4.46}$$

From the equation (4.46) we see that the left side is a function only of ρ, and the right side is a function only of θ. Furthermore, we can see that both terms must be equal to a positive constant, which we will call γ. Then, from the left side we have:

$$\frac{\partial\Theta}{\partial\theta} = \sqrt{\gamma}\Theta. \tag{4.47}$$

The solution of the equation (4.47) is very simple and is given by:

$$\Theta = Me^{\sqrt{\gamma}\theta}. \tag{4.48}$$

Where M is an arbitrary constant. However, we can see that the solution of equation (4.48) is not periodic with respect to θ. And that implies that for Θ to be real:

$$\Theta = \text{constant} \rightarrow \gamma = 0. \tag{4.49}$$

Also, by replacing (4.46) with (4.49) we have:

$$\frac{1}{\mathcal{R}^2}\left(\frac{\partial \mathcal{R}}{\partial \rho}\right)^2 (\rho^2 - 2m\rho + a^2) = 0. \tag{4.50}$$

From (4.50) we can see that:

$$\frac{\partial \mathcal{R}}{\partial \rho} \neq 0. \tag{4.51}$$

Then, from (4.50) and (4.51) we finally can see that:

$$\rho^2 - 2m\rho + a^2 = 0. \tag{4.52}$$

Therefore:

$$\rho_+^{\mathcal{N}} = m + \sqrt{m^2 - a^2}, \tag{4.53}$$

$$\rho_-^{\mathcal{N}} = m - \sqrt{m^2 - a^2}. \tag{4.54}$$

The surfaces $\rho_+^{\mathcal{N}}$ and $\rho_-^{\mathcal{N}}$ are one-way membranes $\mathcal{N}$ of the Kerr metric. Also, the surfaces (4.53) and (4.54) are well defined if only if $|a| < m$. It is also important to mention that at the Schwarzschild limit:

$$a \rightarrow 0. \tag{4.55}$$

With the approximation (4.55), the surfaces (4.53) and (4.54) are reduced to the Schwarzschild surfaces $\rho = 2m$ and $\rho = 0$, respectively.

4.3 The horizon

In the previous section, we dealt with infinite redshift surfaces $\mathcal{I}$. It was mentioned that these surfaces do not necessarily have physical meaning [1] [21] [8]. Also, there are other types of surfaces that have physical meaning. These surfaces are called one-way membranes $\mathcal{N}$. For example, the so-called "event horizon" in a Schwarzschild black hole (a spherical surface where information, if it enters, can no longer leave) had already been mentioned.

The horizons were studied in various ways. Its mathematical treatment has improved significantly with the works of Hawking, Penrose, among others [31]. Currently, horizons are investigated using the causal structure of the spacetime [13]. This involves the use of sophisticated mathematical techniques such as differential geometry and topology [13]. Also, the concept of horizon can vary from one context to another. In addition, there are many types of horizons (absolute horizon, apparent horizon, Cauchy horizon, among others). For a more detailed study of these different types of horizons we recommend the reader review [13] [26] [30].

As we will see later, the horizon generated in a superluminal warp drive is very different from the event horizon of a Schwarzschild black hole. One of the main differences is that the event horizon of a Schwarzschild black hole contains a singularity [13]. However, in a warp drive spacetime there are no singularities. Another difference is that a Schwarzschild black hole's event horizon is closed (compact) [13]. While a horizon in a superluminal warp drive is non-compact [22]. As we will see later, when we deal with horizons in superluminal warp drives we are going to refer exclusively to one-way membranes $\mathcal{N}$.

4.4 The tetrad and the observers

As we have seen previously, spacetime is given by a pair $(M, g_{\alpha\beta})$ where $g_{\alpha\beta}$ is the metric tensor. Also, in that spacetime, we can make measurements. However, the natural question will be: who is the observer who makes these measurements? We could represent that observer as a 4-vector (as already seen in Chapter 2). However, given a metric tensor $g_{\alpha\beta}$, there is an observer located at infinity (Minkowski spacetime) who implicitly makes any measurement in that spacetime. This observer located at infinity is represented by a "tetrad". This tetrad is a set of four 4-vectors that are defined as follows [21] [32]:

$$(e_0)^\mu = (1, 0, 0, 0), \tag{4.56}$$

$$(e_1)^\mu = (0, 1, 0, 0), \tag{4.57}$$

$$(e_2)^\mu = (0, 0, 1, 0), \tag{4.58}$$

$$(e_3)^\mu = (0, 0, 0, 1). \tag{4.59}$$

The four 4-vectors (4.56) - (4.59) form the global Lorentz reference frame. Also, those 4-vectors are called the basis vectors of the global Lorentz reference frame [21]. Indeed:

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu \equiv (e_\mu)^\alpha (e_\nu)_\alpha. \tag{4.60}$$

At any point of a manifold M, an arbitrary tetrad can be defined locally on an infinitesimal interval of time, in the following way:

$$\{(h_0)^\mu, (h_1)^\mu, (h_2)^\mu, (h_3)^\mu\} \tag{4.61}$$

Such arbitrary tetrad will fulfill the following properties [32]:

$$(h_0)^\mu (h_0)_\mu = -1 = n_{00}, \tag{4.62}$$

$$(h_1)^\mu (h_1)_\mu = +1 = n_{11}, \tag{4.63}$$

$$(h_2)^\mu (h_2)_\mu = +1 = n_{22}, \tag{4.64}$$

$$(h_3)^\mu (h_3)_\mu = +1 = n_{33}. \tag{4.65}$$

Also:

$$(h_\alpha)^\mu (h_\beta)_\mu = \delta_{\alpha\beta}. \tag{4.66}$$

Indeed:

$$(h_\alpha)^\mu (h_\beta)_\mu = n_{\alpha\beta}, \tag{4.67}$$

where $n_{\alpha\beta}$ is the Minkowski metric tensor. Physically this is related to the equivalence principle (spacetime is locally flat) [21] [32].

The most natural way to associate a tetrad with an observer is to assign the basis vector $(e_0)^\mu$ with the 4-velocity u^μ of the observer. This physically implies that the observer would be at rest with respect to the reference frame given by the tetrad [21]. Indeed:

$$(e_0)^\mu = u^\mu. \tag{4.68}$$

A well-known example is in relation to an observer moving with 4-velocity u^μ and with constant acceleration g in Minkowski spacetime (for details see [8] [21] [32]). The tetrads related to that accelerated observer will be the following:

$$(e_{0'})^\mu = u^\mu = (\cosh(g\tau), \sinh(g\tau), 0, 0), \tag{4.69}$$

$$(e_{1'})^\mu = \frac{1}{g} a^\mu = (\sinh(g\tau), \cosh(g\tau), 0, 0), \tag{4.70}$$

$$(e_2)^\mu = (0, 0, 1, 0), \tag{4.71}$$

$$(e_3)^\mu = (0, 0, 0, 1), \tag{4.72}$$

where a^μ is the 4-vector acceleration and τ is the proper time of the accelerated observer.

Now let's define the metric $g_{\hat{\alpha}\hat{\beta}}$ defined in the tetrad $\{(e_{\hat{\alpha}})^\mu\}$ (related to the observer inside the warp bubble) as follows:

$$g_{\hat{\alpha}\hat{\beta}} = \vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} \equiv (e_{\hat{\alpha}})^\mu (e_{\hat{\beta}})_\mu. \tag{4.73}$$

Then:

$$g_{\hat{\alpha}\hat{\beta}} = (e_{\hat{\alpha}})^\mu \left[g_{\mu\nu} (e_{\hat{\beta}})^\nu \right]. \tag{4.74}$$

Indeed:

$$g_{\hat{\alpha}\hat{\beta}} = g_{\mu\nu} (e_{\hat{\alpha}})^\mu (e_{\hat{\beta}})^\nu. \tag{4.75}$$

4.5 Generic Natario metric

Natario not only perceived that the expansion/contraction rate of spacetime is not necessary to generate a warp drive. He also noted that any warp drive (generic warp drive) would always have horizons [22]. This means that events inside the warp bubble cannot influence events outside the warp bubble. This would be a problem because an observer inside the spacecraft could not control it at will. Hiscock, Krasnikov, and Low had already noticed this anomalous feature in the Alcubierre warp drive [14] [15] [19]. However, Natario demonstrated that the horizons would present for a superluminal general warp drive [22] (according to the definition of warp drive that he proposed).

We will now first present the tetrads related to an observer within a generic warp drive. Next, we will analyze the horizons and their characteristics.

4.5.1 Observer inside the warp bubble and its tetrad

Now let's consider an observer who is located inside the warp bubble of a generic Natario warp drive. We will consider that the warp bubble moves along the x axis with constant speed v. The line element (3.1) is related to the metric tensor $g_{\alpha\beta}$ (relative to a distant observer). On the other hand, the tetrad related to the observer inside the warp bubble is as follows:

$$(e_{\hat{0}})^\mu = u^\mu = (1, v, 0, 0), \tag{4.76}$$

$$(e_{\hat{1}})^\mu = (0, 1, 0, 0) = (e_1)^\mu, \tag{4.77}$$

$$(e_{\hat{2}})^\mu = (0, 0, 1, 0) = (e_2)^\mu, \tag{4.78}$$

$$(e_{\hat{3}})^\mu = (0, 0, 0, 1) = (e_3)^\mu. \tag{4.79}$$

Using (4.75) we will calculate the metric tensor $g_{\hat{\alpha}\hat{\beta}}$ related to the observer that is inside the warp bubble. Let's start with component $g_{\hat{0}\hat{0}}$. Then:

$$g_{\hat{0}\hat{0}} = g_{\mu\nu}(e_{\hat{0}})^\mu (e_{\hat{0}})^\nu. \tag{4.80}$$

Then:

$$\begin{aligned} g_{\hat{0}\hat{0}} = g_{00}(e_{\hat{0}})^0(e_{\hat{0}})^0 + g_{11}(e_{\hat{0}})^1(e_{\hat{0}})^1 + g_{22}(e_{\hat{0}})^2(e_{\hat{0}})^2 + g_{33}(e_{\hat{0}})^3(e_{\hat{0}})^3 + 2g_{01}(e_{\hat{0}})^0(e_{\hat{0}})^1 + \\ + 2g_{02}(e_{\hat{0}})^0(e_{\hat{0}})^2 + 2g_{03}(e_{\hat{0}})^0(e_{\hat{0}})^3 + 2g_{12}(e_{\hat{0}})^1(e_{\hat{0}})^2 + 2g_{13}(e_{\hat{0}})^1(e_{\hat{0}})^3 + 2g_{23}(e_{\hat{0}})^2(e_{\hat{0}})^3. \end{aligned} \tag{4.81}$$

Replacing (4.76) - (4.79) into (4.81) we have:

$$g_{\hat{0}\hat{0}} = g_{00}(e_{\hat{0}})^0(e_{\hat{0}})^0 + g_{11}(e_{\hat{0}})^1(e_{\hat{0}})^1 + 2g_{01}(e_{\hat{0}})^0(e_{\hat{0}})^1. \tag{4.82}$$

Indeed:

$$g_{\hat{0}\hat{0}} = (-1 + X^2 + Y^2 + Z^2) + v^2 + 2(-X)v. \tag{4.83}$$

Finally:

$$g_{\hat{0}\hat{0}} = -1 + (X - v)^2 + Y^2 + Z^2. \tag{4.84}$$

Now let's calculate $g_{\hat{0}\hat{1}}$. Then:

$$g_{\hat{0}\hat{1}} = g_{\mu\nu}(e_{\hat{0}})^\mu (e_{\hat{1}})^\nu. \tag{4.85}$$

Indeed:

$$g_{\hat{0}\hat{1}} = g_{00}(e_{\hat{0}})^0(e_{\hat{1}})^0 + g_{11}(e_{\hat{0}})^1(e_{\hat{1}})^1 + g_{22}(e_{\hat{0}})^2(e_{\hat{1}})^2 + g_{33}(e_{\hat{0}})^3(e_{\hat{1}})^3 + g_{01}(e_{\hat{0}})^0(e_{\hat{1}})^1 + \\ + g_{10}(e_{\hat{0}})^1(e_{\hat{1}})^0 + g_{02}(e_{\hat{0}})^0(e_{\hat{1}})^2 + g_{20}(e_{\hat{0}})^2(e_{\hat{1}})^0 + g_{03}(e_{\hat{0}})^0(e_{\hat{1}})^3 + g_{30}(e_{\hat{0}})^3(e_{\hat{1}})^0 + \\ + g_{12}(e_{\hat{0}})^1(e_{\hat{1}})^2 + g_{21}(e_{\hat{0}})^2(e_{\hat{1}})^1 + g_{13}(e_{\hat{0}})^1(e_{\hat{1}})^3 + g_{31}(e_{\hat{0}})^3(e_{\hat{1}})^1 + g_{23}(e_{\hat{0}})^2(e_{\hat{1}})^3 + \\ + g_{32}(e_{\hat{0}})^3(e_{\hat{1}})^2. \quad (4.86)$$

Replacing (4.76) - (4.79) into (4.86) we have:

$$g_{\hat{0}\hat{1}} = g^{11}(e_{\hat{0}})^1(e_{\hat{1}})^1 + g_{01}(e_{\hat{0}})^0(e_{\hat{1}})^1. \quad (4.87)$$

Because, as we can see:

$$g_{10}(e_{\hat{0}})^1(e_{\hat{1}})^0 = 0. \quad (4.88)$$

Finally, we have:

$$g_{\hat{0}\hat{1}} = v - X. \quad (4.89)$$

In the same way, we can calculate the following components of $g_{\hat{\alpha}\hat{\beta}}$:

$$g_{\hat{0}\hat{2}} = -Y. \quad (4.90)$$

$$g_{\hat{0}\hat{3}} = -Z. \quad (4.91)$$

$$g_{\hat{1}\hat{1}} = g_{\hat{2}\hat{2}} = g_{\hat{3}\hat{3}} = 1. \quad (4.92)$$

Also, the other components of the metric tensor will be zero. Finally, the metric tensor $g_{\hat{\alpha}\hat{\beta}}$ related to the tetrad of an observer inside the warp bubble will be expressed by the following line element:

$$ds^2 = \left[-1 + (X-v)^2 + Y^2 + Z^2\right] d\hat{t}^2 + 2\left[(v-X)d\hat{x} - Y d\hat{y} - Z d\hat{z}\right] d\hat{t} + \\ + \left[d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2\right], \quad (4.93)$$

where $\hat{t}$, $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the coordinates related to the tetrad of the observer inside the warp bubble. In matrix form, $g_{\hat{\alpha}\hat{\beta}}$ has the form:

$$g_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} [-1 + (X-v)^2 + Y^2 + Z^2] & (v-X) & -Y & -Z \\ (v-X) & 1 & 0 & 0 \\ -Y & 0 & 1 & 0 \\ -Z & 0 & 0 & 1 \end{bmatrix}. \quad (4.94)$$

The metric (4.93) can take the following form:

$$ds^2 = -d\hat{t}^2 + \left[(X-v)^2 d\hat{t}^2 - 2(X-v)d\hat{x}d\hat{t} + d\hat{x}^2\right] + \left[Y^2 d\hat{t}^2 - 2Y d\hat{y}d\hat{t} + d\hat{y}^2\right] + \\ + \left[Z^2 d\hat{t}^2 - 2Z d\hat{z}d\hat{t} + d\hat{z}^2\right]. \quad (4.95)$$

Finally:

$$ds^2 = -d\hat{t}^2 + \left[d\hat{x} - (X-v)d\hat{t}\right]^2 + \left[d\hat{y} - Y d\hat{t}\right]^2 + \left[d\hat{z} - Z d\hat{t}\right]^2. \quad (4.96)$$

4.5.2 Angular deflection

Let's take into account the null geodesics in a generic Natario warp drive with respect to a distant observer. So:

$$ds^2 = 0. \tag{4.97}$$

Replacing (3.1) with (4.97) we have:

$$dt^2 = \sum_{i=1}^{3} \left(dx^i - X^i dt\right)^2. \tag{4.98}$$

Then:

$$1 = \sum_{i=1}^{3} \left(\frac{dx^i}{dt} - X^i\right)^2. \tag{4.99}$$

In other words:

$$1 = \left(\frac{dx}{dt} - X\right)^2 + \left(\frac{dy}{dt} - Y\right)^2 + \left(\frac{dz}{dt} - Z\right)^2. \tag{4.100}$$

Then:

$$1 = \sqrt{\left(\frac{dx}{dt} - X\right)^2 + \left(\frac{dy}{dt} - Y\right)^2 + \left(\frac{dz}{dt} - Z\right)^2}. \tag{4.101}$$

$$1 = \sqrt{\sum_{i=1}^{3} \left(\frac{dx^i}{dt} - X^i\right)^2}. \tag{4.102}$$

Indeed:

$$\left|\frac{d\vec{x}}{dt} - \vec{X}\right| = 1. \tag{4.103}$$

If we define $\vec{n}$ as a unit vector, that is:

$$|\vec{n}| = 1. \tag{4.104}$$

Then, from (4.103) and (4.104) we have:

$$\frac{d\vec{x}}{dt} = \vec{n} + \vec{X}. \tag{4.105}$$

Now let us consider the null geodesics with respect to an observer inside the warp bubble. From the metric (4.96) and from (4.97) we have:

$$d\hat{t}^2 = \left[d\hat{x} - (X - v)d\hat{t}\right]^2 + \left[d\hat{y} - Y d\hat{t}\right]^2 + \left[d\hat{z} - Z d\hat{t}\right]^2. \tag{4.106}$$

Then:

$$1 = \left[\frac{d\hat{x}}{d\hat{t}} - (X - v)\right]^2 + \left[\frac{d\hat{y}}{d\hat{t}} - Y\right]^2 + \left[\frac{d\hat{z}}{d\hat{t}} - Z\right]^2. \tag{4.107}$$

Operating analogously to the case of the distant observer, we have:

$$1 = \sqrt{\sum_{i=1}^{3} \left(\frac{d\hat{x}^i}{d\hat{t}} - X_b^i\right)^2}. \tag{4.108}$$

Therefore:

$$\left|\frac{d\hat{\vec{x}}}{d\hat{t}} - \vec{X}_b\right| = 1. \tag{4.109}$$

Using (4.108) in equation (4.109) we have:

$$\frac{d\hat{\vec{x}}}{d\hat{t}} = \vec{n} + \vec{X}_b, \tag{4.110}$$

where:

$$\beta^i = \vec{X}_b = (X - v)e_x + Ye_y + Ze_z, \tag{4.111}$$

where $\vec{X}_b$ is the shift vector of metric $g_{\hat{\alpha}\hat{\beta}}$. Also $d\hat{\vec{x}}/d\hat{t}$ represents the trajectory of the null geodesics with respect to the observer inside the warp bubble.

Now we will see the angular deflection with respect to the observer who is inside the warp bubble. Also, for simplicity, we will take into account that the thickness of the warp bubble wall approaches zero (to avoid the effect of aberration in null geodesics [22]). It is clear that if the wall of the warp bubble has a certain non-zero thickness, then there will always be an aberration in the null geodesics [22]. Indeed, we will have two cases. Inside the bubble and outside the warp bubble. Then, outside the warp bubble we have:

$$\vec{X}_b \neq 0 \rightarrow \frac{d\hat{\vec{x}}}{d\hat{t}} = \vec{n} + \vec{X}_b. \tag{4.112}$$

Also, inside the warp bubble we have:

$$\vec{X}_b = 0 \rightarrow \frac{d\hat{\vec{x}}}{d\hat{t}} = \vec{n}. \tag{4.113}$$

Using (4.105) we can do an analogous procedure obtaining expressions analogous to the equations (4.112) and (4.113). This is clearly shown in the figure (14) [22].

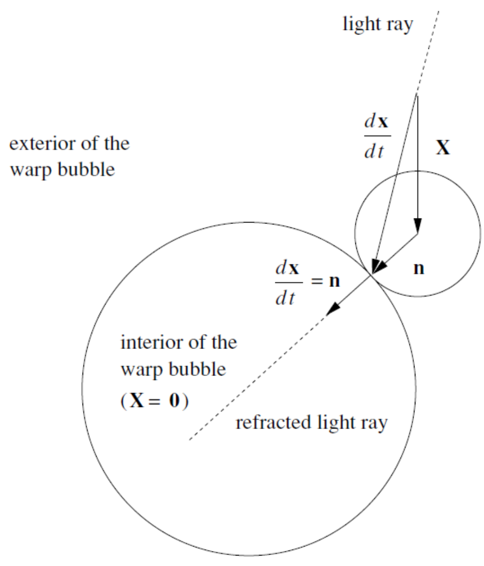

Figure 14 – Angular deflection [22].

4.5.3 Horizons

From the metric (4.96) that is related to the observer that is inside the superluminal warp bubble, we have the shift vector $\vec{X}_b$. Then:

$$\left|\vec{X}_b\right| = \sqrt{(X-v)^2 + Y^2 + Z^2}. \tag{4.114}$$

From (3.4), inside the warp bubble we always have:

$$\vec{X}_b = 0. \tag{4.115}$$

In the same way, from (3.5) outside the warp bubble we have:

$$X = Y = Z = 0. \tag{4.116}$$

Then:

$$\vec{X}_b = -v e_x. \tag{4.117}$$

On the walls of the warp bubble, we have:

$$0 < \left|\vec{X}_b\right| < v. \tag{4.118}$$

So, inside the wall of the warp bubble (as Low demonstrated [19]) there will be a point x_c where we have the following:

$$\left|\vec{X}_b\right| = 1. \tag{4.119}$$

Equation (4.118) shows us that at point x_c, $\left|\vec{X}_b\right|$ will be equal to the speed of light $c = 1$ (using geometric units). Now, let's define the following [22]:

$$\sin(\alpha) = \frac{1}{\left|\vec{X}_b\right|}. \tag{4.120}$$

The relation (4.120) represents the horizon $\mathcal{H}$ and tells us the relationship between the speed of light and the shift vector. From this relation, we deduce that always:

$$\left|\vec{X}_b\right| \geq 1. \tag{4.121}$$

The inequality (4.121) means that the horizon $\mathcal{H}$ will be formed exclusively in a superluminal warp drive (at speeds less than light we will not have a horizon).

Figure (15) illustrates the superluminal motion of the warp drive in the $-x$ axis [22]. The curve represents the horizon $\mathcal{H}$. Furthermore, in the upper right part of the figure (15) a spherical wavefront of light is represented. In this case, this wavefront is generated within the wall of the warp bubble. For example, at the point (within the walls of the warp bubble [19]) where the horizon $\mathcal{H}$ intersects the x axis we have the following:

$$\alpha = 90^\circ \rightarrow \left|\vec{X}_b\right| = 1. \tag{4.122}$$

In the same way, for points far from the warp bubble we have:

$$\alpha = \alpha_{\infty} \rightarrow \left|\vec{X}_b\right| = v. \tag{4.123}$$

From the expressions (4.122) and (4.123) we can deduce the following:

$$\alpha_{\infty} \leq \alpha \leq 90^{\circ}. \tag{4.124}$$

Taking into account the evident cylindrical symmetry, from the figure (15) we can see that the relationship (4.120) gives us the horizon $\mathcal{H}$. At this horizon, information cannot escape from the warp bubble to the outside [22]. We can also see from figure (15) that this horizon $\mathcal{H}$ is represented by a conical region and is in front of the warp bubble.

In a way analogous to horizon $\mathcal{H}$, we will also have the so-called ”visibility horizon” $\mathcal{H}_v$. This visibility horizon $\mathcal{H}_v$ delimits the region of space where light can not reach the warp bubble [22]. The visibility horizon $\mathcal{H}_v$ is represented in the figure (16). Here also it is also assumed that the warp bubble travels in the $-x$ direction. We can also see from figure (16) that the visibility horizon $\mathcal{H}_v$ is represented by a conical region and is behind the warp bubble.

The calculation of the visibility horizon $\mathcal{H}_v$ is analogous to the horizon $\mathcal{H}$ (see figure (16)). Furthermore, it is important to mention that the horizons $\mathcal{H}$ and $\mathcal{H}_v$ (as will later be demonstrated for the 2-D Alcubierre warp drive) are one-way membranes $\mathcal{N}$.

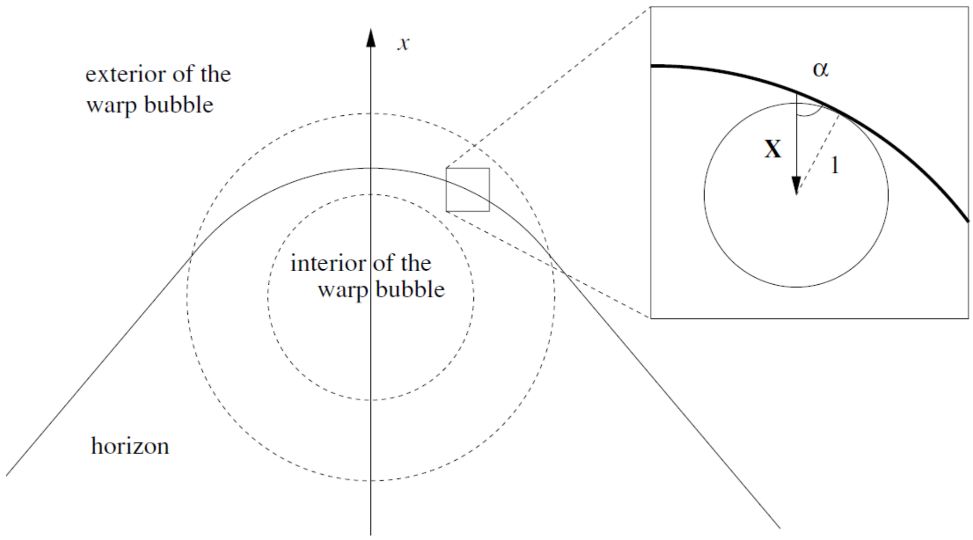

Figure 15 – Horizon in generic warp drive [22].

4.5.4 The blueshift problem

Suppose our warp bubble is moving in the $+x$ direction (relative to a distant observer). As we have seen above, photons starting from directions that are within the visibility horizon $\mathcal{H}_v$ cannot reach the warp bubble. And therefore the photons traveling

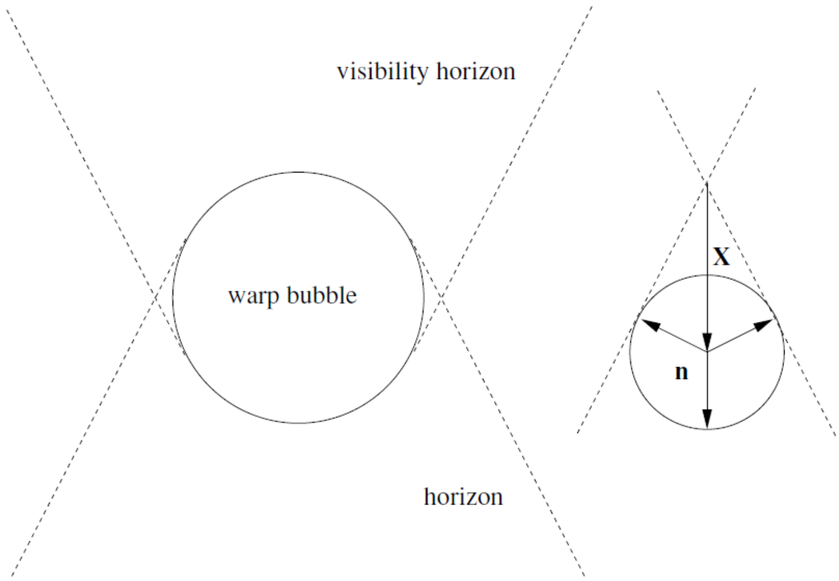

Figure 16 – Two horizons in generic warp drive [22].

in the opposite direction to the warp bubble will be blue-shifted. That is, blueshift will occur for these photons that reach the warp bubble [22]. So if we assume the following:

$$\frac{\partial \vec{X}_b}{\partial t} = \sum_{i=1}^{3} \frac{\partial X_b^i}{\partial t} = \sum_{i=1}^{3} \frac{\partial X_b^i}{\partial \hat{t}} = 0. \tag{4.125}$$

To replace (4.125) in null geodesics, we will apply a procedure analogous to (3.22). In this case for the metric $g_{\hat{\alpha}\hat{\beta}}$ (with respect to an observer who is inside the warp drive) we have:

$$\frac{d}{d\lambda}\left[-\dot{\hat{t}} - X_b^i(\dot{\hat{x}}^i - X_b^i\dot{\hat{t}})\right] - \frac{\partial X_b^i}{\partial \hat{t}}\dot{\hat{t}}\left[\dot{\hat{x}}^i - X_b^i\dot{\hat{t}}\right] = 0, \tag{4.126}$$

where:

$$\dot{\hat{t}} = \frac{d\hat{t}}{d\lambda}. \tag{4.127}$$

And:

$$\dot{\hat{x}}^i = \frac{d\hat{x}^i}{d\lambda}, \tag{4.128}$$

where λ is the affine parameter. Replacing (4.125) in (4.126) we have:

$$\frac{d}{d\lambda}\left[-\dot{\hat{t}} - X_b^i\left\{\dot{\hat{x}}^i - X_b^i\dot{\hat{t}}\right\}\right] = 0. \tag{4.129}$$

Indeed:

$$\frac{d}{d\lambda}\left[-\dot{t} - X_b^i\left\{\frac{d\hat{x}^i}{d\lambda} - X_b^i\dot{t}\right\}\right] = 0. \tag{4.130}$$

Therefore:

$$\dot{t} + X_b^i\left\{\frac{d\hat{x}^i}{d\lambda} - X_b^i\dot{t}\right\} = A. \tag{4.131}$$

Then:

$$\dot{t} + \vec{X}_b \cdot \left\{\frac{d\hat{\vec{x}}}{d\lambda} - \vec{X}_b\dot{t}\right\} = A, \tag{4.132}$$

where A is a constant. Replacing (4.112) in (4.132) then:

$$\dot{t} + \vec{X}_b \cdot \left\{ \frac{d\hat{\vec{x}}}{d\lambda} + \dot{t} \left[\vec{n} - \frac{d\hat{\vec{x}}}{dt} \right] \right\} = A. \tag{4.133}$$

Then:

$$\dot{t} + \vec{X}_b \cdot \left\{ \frac{d\hat{\vec{x}}}{d\lambda} + \dot{t}\vec{n} - \frac{dt}{d\lambda} \frac{d\hat{\vec{x}}}{dt} \right\} = A. \tag{4.134}$$

Indeed:

$$\dot{t} \left[1 + \vec{X}_b \cdot \vec{n} \right] = A. \tag{4.135}$$

We need to remember:

$$\dot{t} = \dot{\hat{t}} = E, \tag{4.136}$$

where E is the photon energy measured by a distant observer outside the warp bubble. Also, from (4.135) and (4.136) we have inside the warp bubble the following:

$$\vec{X}_b = 0 \rightarrow E_0 = A, \tag{4.137}$$

where E_0 is the photon energy measured by an observer inside the warp bubble. Replacing (4.136) and (4.137) into (4.135) we have:

$$E \left[1 + \vec{X}_b \cdot \vec{n} \right] = E_0. \tag{4.138}$$

The equation (4.138) tells us that the photons that enter the warp bubble can be blueshifted. In general, as can be seen in equation (4.138), the photons entering the warp bubble also can be redshifted (as we will see later). Let us analyze two cases in detail. The first case is where the photon enters the warp bubble at 90° (with respect to the direction of the $+x$ axis). Since $\vec{n}$ indicates the direction of the photon inside the warp bubble we will have the following:

$$\vec{X}_b \cdot \vec{n} = 0 \rightarrow E = E_0. \tag{4.139}$$

The expression (4.139) tells us that if a photon enters at an angle 90° then this photon will not be blueshifted. The photon energy measured by a distant observer outside the warp drive and the photon energy measured by an observer inside the warp bubble will be exactly the same. The second case that we will see is when the photon enters with an angle of 180° with respect to the $+x$ axis (photon that moves in the opposite direction to the motion of the warp drive). In this case, we need to remember (3.5) and we would have the following:

$$\vec{X}_b \cdot \vec{n} = v \rightarrow E(1 + v) = E_0, \tag{4.140}$$

where v is the speed of the warp bubble. From equation (4.140) we can see that the photon is blueshifted, by a factor of $1 + v$.

As we can see, this blue shift may be an additional problem that the crew of our hypothetical warp drive must take into account. With a sufficiently large speed v, the high energy photons would be a danger!

4.6 The Alcubierre metric

Above we have analyzed the angular deflection of light and the horizons in a generic Natario warp drive. In that generic warp drive it was assumed that the thickness of the warp bubble walls tends to zero [22]. However, in this section we will take into account the thickness of the warp bubble wall. And specifically we will study the angular deflection of light and the blueshift in the Alcubierre warp drive.

4.6.1 Observer inside the warp bubble and its tetrad

In our case, we are interested in associating a tetrad with an observer who is inside the warp drive bubble. As seen above, the spacetime inside the warp bubble is Minkowski. By considering that the warp bubble moves with constant speed v along the z axis, the tetrad associated with that observer will be the following [7]:

$$(e_{\hat{0}})^\mu = u^\mu = (1, 0, 0, v), \tag{4.141}$$

$$(e_{\hat{1}})^\mu = (0, 1, 0, 0) = (e_1)^\mu, \tag{4.142}$$

$$(e_{\hat{2}})^\mu = (0, 0, 1, 0) = (e_2)^\mu, \tag{4.143}$$

$$(e_{\hat{3}})^\mu = (0, 0, 0, 1) = (e_3)^\mu. \tag{4.144}$$

Also:

$$(e_{\hat{0}})^\mu = (e_0)^\mu + v(e_3)^\mu. \tag{4.145}$$

We are going to do a mathematical treatment analogous to the case of Natario's generic warp drive. Now we are going to calculate the components of the metric tensor $g_{\hat{\alpha}\hat{\beta}}$. Let's start with the component $g_{\hat{0}\hat{0}}$. Then:

$$g_{\hat{0}\hat{0}} = g_{00}(e_{\hat{0}})^0(e_{\hat{0}})^0 + g_{11}(e_{\hat{0}})^1(e_{\hat{0}})^1 + g_{22}(e_{\hat{0}})^2(e_{\hat{0}})^2 + g_{33}(e_{\hat{0}})^3(e_{\hat{0}})^3 + 2g_{01}(e_{\hat{0}})^0(e_{\hat{0}})^1 + \\ + 2g_{02}(e_{\hat{0}})^0(e_{\hat{0}})^2 + 2g_{03}(e_{\hat{0}})^0(e_{\hat{0}})^3 + 2g_{12}(e_{\hat{0}})^1(e_{\hat{0}})^2 + 2g_{13}(e_{\hat{0}})^1(e_{\hat{0}})^3 + 2g_{23}(e_{\hat{0}})^2(e_{\hat{0}})^3. \tag{4.146}$$

Replacing the metric (3.36) and the relations (4.141) - (4.144) into (4.146) we have:

$$g_{\hat{0}\hat{0}} = g_{00}(e_{\hat{0}})^0(e_{\hat{0}})^0 + 2g_{03}(e_{\hat{0}})^0(e_{\hat{0}})^3 + g_{33}(e_{\hat{0}})^3(e_{\hat{0}})^3. \tag{4.147}$$

Finally:

$$g_{\hat{0}\hat{0}} = (-1 + v^2 f^2) + 2(-vf)v + v^2, \tag{4.148}$$

$$g_{\hat{0}\hat{0}} = -1 + v^2 (1 - f)^2. \tag{4.149}$$

Now let's calculate $g_{\hat{0}\hat{3}}$. Then:

$$g_{\hat{0}\hat{3}} = g_{00}(e_{\hat{0}})^0(e_{\hat{3}})^0 + g_{11}(e_{\hat{0}})^1(e_{\hat{3}})^1 + g_{22}(e_{\hat{0}})^2(e_{\hat{3}})^2 + g_{33}(e_{\hat{0}})^3(e_{\hat{3}})^3 + g_{01}(e_{\hat{0}})^0(e_{\hat{3}})^1 + \\ + g_{10}(e_{\hat{0}})^1(e_{\hat{3}})^0 + g_{02}(e_{\hat{0}})^0(e_{\hat{3}})^2 + g_{20}(e_{\hat{0}})^2(e_{\hat{3}})^0 + g_{03}(e_{\hat{0}})^0(e_{\hat{3}})^3 + g_{30}(e_{\hat{0}})^3(e_{\hat{3}})^0 + \\ + g_{12}(e_{\hat{0}})^1(e_{\hat{3}})^2 + g_{21}(e_{\hat{0}})^2(e_{\hat{3}})^1 + g_{13}(e_{\hat{0}})^1(e_{\hat{3}})^3 + g_{31}(e_{\hat{0}})^3(e_{\hat{3}})^1 + g_{23}(e_{\hat{0}})^2(e_{\hat{3}})^3 + \\ + g_{32}(e_{\hat{0}})^3(e_{\hat{3}})^2. \tag{4.150}$$

Replacing the metric (3.36) and the relations (4.141) - (4.144) into (4.150) we have:

$$g_{\hat{0}\hat{3}} = g_{33}(e_{\hat{0}})^3(e_{\hat{3}})^3 + g_{03}(e_{\hat{0}})^0(e_{\hat{3}})^3. \tag{4.151}$$

Because, as we can see:

$$g_{30}(e_{\hat{0}})^3(e_{\hat{3}})^0 = 0. \tag{4.152}$$

Finally:

$$g_{\hat{0}\hat{3}} = v + (-vf) = v(1-f). \tag{4.153}$$

In the same way, we can calculate the following components of $g_{\hat{\alpha}\hat{\beta}}$:

$$g_{\hat{1}\hat{1}} = g_{\hat{2}\hat{2}} = g_{\hat{3}\hat{3}} = 1. \tag{4.154}$$

Also, the other components of the metric tensor will be zero. Finally, the metric tensor $g_{\hat{\alpha}\hat{\beta}}$ related to the tetrad of an observer inside the warp bubble will be expressed by the following line element:

$$ds^2 = \left[-1 + v^2(1-f)^2\right] d\hat{t}^2 + 2(1-f)v\,d\hat{z}d\hat{t} + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2. \tag{4.155}$$

Where $\hat{t}$, $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the coordinates related to the tetrad of the observer inside the Alcubierre warp bubble. In matrix form, $g_{\hat{\alpha}\hat{\beta}}$ has the form:

$$g_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} -1 + v^2(1-f)^2 & 0 & 0 & (1-f)v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (1-f)v & 0 & 0 & 1 \end{bmatrix}. \tag{4.156}$$

There is another way to find $g_{\hat{\alpha}\hat{\beta}}$. The coordinates $\hat{t}$, $\hat{x}$, $\hat{y}$ and $\hat{z}$ that are related for an observer at rest inside the warp bubble can be related to the coordinates t, x, y and z of a very distant external observer who is at rest in the following way:

$$\hat{t} = t, \qquad \hat{x} = x, \qquad \hat{y} = y, \qquad \hat{z} = z - z_0, \tag{4.157}$$

where z_0 indicates the position of the warp bubble.

4.6.2 Null geodesics

Now let's calculate the null geodesics for the Alcubierre warp drive. Using the geodesic equation (1.64) for the Alcubierre metric (3.36) we have:

$$\frac{dp^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu} p^\mu p^\nu = 0. \tag{4.158}$$

If the warp bubble moves in the $+x$ axis (taking into account the condition of movement used by Clark, Hiscock and Larson [7]), we can see that our spacetime has cylindrical symmetry [7]. Additionally, our p^α will be:

$$p^\alpha = (p^t, p^x, p^y, p^z). \tag{4.159}$$

Also, since we are dealing with null geodesics, then:

$$p^{\alpha}p_{\alpha} = 0. \tag{4.160}$$

As mentioned above, due to the cylindrical symmetry of spacetime we can have the following:

$$p^{z} = 0. \tag{4.161}$$

If the warp bubble moves in the $+x$ direction (analogous to Alcubierre's warp drive moving in the $+z$ direction). Then the metric will be:

$$g_{\mu\nu} = \begin{bmatrix} (-1+v^2f^2) & -vf & 0 & 0 \\ -vf & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.162}$$

Then:

$$g^{\mu\nu} = \begin{bmatrix} -1 & -vf & 0 & 0 \\ -vf & (1-v^2f^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.163}$$

Furthermore, for simplicity, we will consider the speed of the warp bubble constant and equal to v. Then, replacing (4.159) and (4.161) into (4.158) we have three null geodesic equations:

$$\frac{dp^t}{d\lambda} + \Gamma^t_{tt}(p^t)^2 + \Gamma^t_{xx}(p^x)^2 + 2\Gamma^t_{tx}p^tp^x + 2\Gamma^t_{ty}p^tp^y + 2\Gamma^t_{xy}p^xp^y = 0, \tag{4.164}$$

$$\frac{dp^x}{d\lambda} + \Gamma^x_{tt}(p^t)^2 + \Gamma^x_{xx}(p^x)^2 + 2\Gamma^x_{tx}p^tp^x + 2\Gamma^x_{ty}p^tp^y + 2\Gamma^x_{xy}p^xp^y = 0, \tag{4.165}$$

$$\frac{dp^y}{d\lambda} + \Gamma^y_{tt}(p^t)^2 + 2\Gamma^y_{tx}p^tp^x = 0. \tag{4.166}$$

From the equations (4.164) - (4.166) it is easy to show that the other Christoffel symbols are equal to zero (the Christoffel symbols that are not in the geodesic equations). Now, using (1.56) let's calculate the non-zero Christoffel symbols. Then:

$$\Gamma^t_{tt} = \frac{1}{2}g^{t\rho}\left[\partial_t g_{\rho t} + \partial_t g_{\rho t} - \partial_\rho g_{tt}\right]. \tag{4.167}$$

Then;

$$\Gamma^t_{tt} = \frac{1}{2}g^{tt}\left[\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt}\right] + \frac{1}{2}g^{tx}\left[\partial_t g_{xt} + \partial_t g_{xt} - \partial_x g_{tt}\right]. \tag{4.168}$$

Finally:

$$\Gamma^t_{tt} = v^3 f^2 \partial_x f. \tag{4.169}$$

In an analogous way we can calculate the other non-zero Christoffel symbols for the geodesic equation (4.164). So:

$$\Gamma^t_{xx} = v\partial_x f, \qquad \Gamma^t_{tx} = -v^2 f\partial_x f, \qquad \Gamma^t_{ty} = -\frac{1}{2}v^2 f\partial_y f, \qquad \Gamma^t_{xy} = \frac{v}{2}\partial_y f \tag{4.170}$$

Also, for the geodesic equation (4.165), we have:

$$\Gamma^x_{tt} = -v\partial_t f + v^2 f(v^2 f^2 - 1)\partial_x f, \qquad \Gamma^x_{xx} = v^2 f \partial_x f, \tag{4.171}$$

$$\Gamma^x_{tx} = -v^3 f^2 \partial_x f, \qquad \Gamma^x_{ty} = -\frac{v}{2}(v^2 f^2 + 1)\partial_y f, \qquad \Gamma^x_{xy} = \frac{1}{2} v^2 f \partial_y f, \tag{4.172}$$

Finally, for the geodesic equation (4.166), we have:

$$\Gamma^y_{tt} = -v^2 f \partial_y f, \qquad \Gamma^y_{tx} = \frac{v}{2}\partial_y f. \tag{4.173}$$

If we replace the Christoffel symbols in the geodesic equations, we will obtain three very complicated differential equations. Therefore, Clark et al. [7] solved these equations using numerical methods. An interesting feature of Clark et al.'s results is that they calculated the geodesics that would be observed by an observer inside the warp bubble. To do this, two observers must be considered: one inside the warp bubble and another distant observer who is outside the warp bubble.

We could compute null geodesics with respect to a distant observer (an astronomer, for example). However, this would not be very practical. The angular size that a warp drive would generate in the sky would be too small [7]. In addition, the superluminal movement of the warp drive would also generate another problem for astronomical observation. However, we could do something more interesting: compute the null geodesics as seen by an observer inside the warp bubble. To do this, we must consider the initial conditions of the null geodesics inside the warp bubble. Let us consider the components of 4-momentum for an observer inside the warp bubble as follows:

$$\hat{p}^x = \cos(\theta_0), \tag{4.174}$$

$$\hat{p}^y = \sin(\theta_0). \tag{4.175}$$

Now we must consider the tetrad of the observer inside the warp bubble and the tetrad of the observer outside the warp bubble (as we saw above). From (4.157) we have:

$$\hat{p}^y = p^y. \tag{4.176}$$

Also from (4.157) we have:

$$\hat{x} = x - vt. \tag{4.177}$$

Then:

$$\frac{d\hat{x}}{d\lambda} = \frac{dx}{d\lambda} - v\frac{dt}{d\lambda}. \tag{4.178}$$

Indeed:

$$\hat{p}^x = p^x - vp^t. \tag{4.179}$$

Replacing (4.179) and (4.176), into (4.174) and (4.175), respectively, we have:

$$p^x = \cos(\theta_0) + vp^t, \tag{4.180}$$

$$p^y = \sin(\theta_0). \tag{4.181}$$

Furthermore, we will also choose as the initial condition the energy of the photon equal to unity, then:

$$p^t = 1. \tag{4.182}$$

Indeed, equations (4.180), (4.181), and (4.182) represent the initial conditions of the null geodesics leaving from the center of the warp bubble.

Now, knowing p^t, p^x and p^y we can calculate the null geodesics with the following well-known definitions:

$$\frac{dt}{d\lambda} - p^t = 0, \tag{4.183}$$

$$\frac{dx}{d\lambda} - p^x = 0, \tag{4.184}$$

$$\frac{dy}{d\lambda} - p^y = 0. \tag{4.185}$$

With all this procedure we can calculate the null geodesics for an Alcubierre warp drive. As mentioned above, those null geodesics are leaving the center of the warp bubble. If we want to calculate the null geodesics that enter from the outside to the center of the warp bubble, we would simply have to reverse the time from t to $-t$ in the numerical integrations [7]. Another important detail to mention is that the numerical integrations were done from $r = 0$ to $r = 100r_b$, where r_b is the radius of the warp bubble [7].

4.6.3 Angular deflection

For a distant outside observer, the 4-momentum components of the null geodesics will be related to an angle θ_∞ that is formed with respect to the x axis in the following way:

$$\tan(\theta_\infty) = \frac{p^y}{p^x}. \tag{4.186}$$

For an observer inside the warp bubble, we will have something similar. However, due to the obvious aberration of light, the angle θ will be different from that measured by the external observer. We will call this angle θ_0. Then:

$$\tan(\theta_0) = \frac{\hat{p}^y}{\hat{p}^x}. \tag{4.187}$$

Substituting (4.176) and (4.179) into (4.187) we have finally:

$$\tan(\theta_0) = \frac{p^y}{p^x - vp^t}. \tag{4.188}$$

With the equations (4.186) and (4.188) the graph (17) can be made [7]. On the horizontal axis, we have the angle measured by an observer located at infinity θ_∞. On the vertical axis, we have the angle measured by the observer located in the center of the warp bubble θ_0.

It is true that the geodesic equations are very complicated and require numerical methods for their solution. However, there are some special cases where we can find analytical solutions. First, let's consider a null geodesic originating at $r = 0$ and $\theta_0 = 90°$ from the direction of travel. Then from (4.180) - (4.182) our initial conditions will be the following:

$$p_0^t = 1, \qquad p_0^x = v, \qquad p_0^y = 1, \tag{4.189}$$

Let's analyze the equation (4.166). The Christoffel symbols of that equation given in (4.173) will be related as follows:

$$\Gamma^y_{tt} = -2vf\Gamma^y_{tx}. \tag{4.190}$$

Replacing (4.190) in (4.166):

$$\frac{dp^y}{d\lambda} + 2\Gamma^y_{tx}p^t\left\{p^x - vfp^t\right\} = 0. \tag{4.191}$$

From (4.191) we have:

$$p^x = vfp^t \rightarrow \frac{dp^y}{d\lambda} = 0. \tag{4.192}$$

Let's analyze the equation (4.164). The Christoffel symbols of that equation given in (4.169) and (4.170) will be related as follows:

$$\Gamma^t_{tt} = -vf\Gamma^t_{tx}, \qquad \Gamma^t_{tx} = -vf\Gamma^t_{xx}, \qquad \Gamma^t_{ty} = -vf\Gamma^t_{xy}. \tag{4.193}$$

Replacing (4.193) in (4.164):

$$\frac{dp^t}{d\lambda} + \left[p^x - vfp^t\right]\left\{\Gamma^t_{tx}p^t + \Gamma^t_{xx}p^x + 2\Gamma^t_{xy}p^y\right\} = 0. \tag{4.194}$$

Then:

$$\frac{dp^t}{d\lambda} + \left[p^x - vfp^t\right]\left\{\Gamma^t_{xx}\left(p^x - vf\right) + 2\Gamma^t_{xy}p^y\right\} = 0. \tag{4.195}$$

If we establish the solution:

$$p^t = 1, \tag{4.196}$$

we can see that equation (4.192) and equation (4.196) satisfy equation (4.195). Then:

$$p^x = vf. \tag{4.197}$$

And, from (4.192) we have:

$$p^y = p_0^y = 1. \tag{4.198}$$

Therefore, with the initial conditions (4.189) the solution will be trivial for the equations (4.164) - (4.166). As we have shown, those solutions are the following:

$$p^t = 1, \qquad p^x = vf, \qquad p^y = 1. \tag{4.199}$$

Replacing (4.199) into (4.186) we have:

$$\tan(\theta_\infty) = \frac{1}{vf}. \tag{4.200}$$

To a distant observer outside the warp bubble, we must remember (3.43). So:

$$\theta_\infty = 90^\circ. \tag{4.201}$$

Equations (4.199) and (4.201) physically tell us that photons with $\theta_0 = 90^\circ$ will have neither blueshift nor angular deflection, respectively.

It is true that for angles $\theta_0 \neq 90^\circ$ the solutions for p^t, p^x and p^y are much more complicated. As we mentioned above Clark et al. solved that problem with numerical methods and their results were presented in figure (17). In figure (17) we see a plot of the angular deflection of the photons entering the warp bubble. The direction of movement of the warp bubble will be given by $\theta_\infty = \theta_0 = 0^\circ$ [7].

The "filled" curves represent warp speeds v of 0, 1, 2, 5, 10, and 100, from top to bottom and in the left half of the graph (17). On the other hand, the "dotted" curves also in figure (17) represent the special relativity light aberration phenomenon for a ship traveling at speeds of 0.5, 0.9, and 0.99, from top to bottom. We can see that for $\theta_\infty < 90^\circ$ the light sources seen by an observer inside the warp bubble are more clustered towards the direction of warp drive motion. As we can see, this phenomenon is similar to that of special relativity [7].

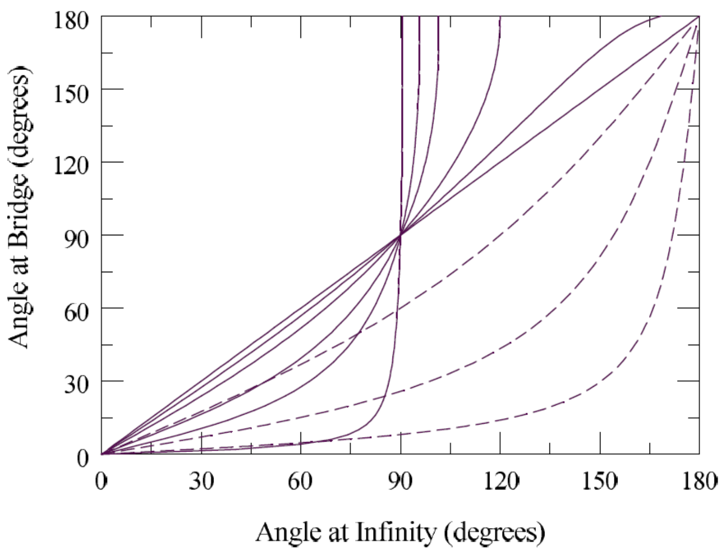

Figure 17 – Angular deflection in Alcubierre warp drive [7].

However, also from the figure (17), we can see that for $\theta_\infty > 90^\circ$ there is a big difference between our warp drive and the ship traveling in a Minkowski spacetime. Figure (17) shows that there is a region where photons will not reach the warp bubble. This is related to the visibility horizon $\mathcal{H}_v$ already mentioned above. Also, from figure (17) it is easy to see that $\theta_\infty = \theta_0 = 90^\circ$. We have already obtained this result analytically above.

We can also see from figure (17) and (18) that as the speed of the warp bubble v increases, the half-angle of the visibility horizon $\mathcal{H}_v$ also increases [7]. The limit value of this half-angle (as we can see) is $90°$. Also from figure (18) we can see that the visibility horizon $\mathcal{H}_v$ will be formed if and only if $v \geq 1$.

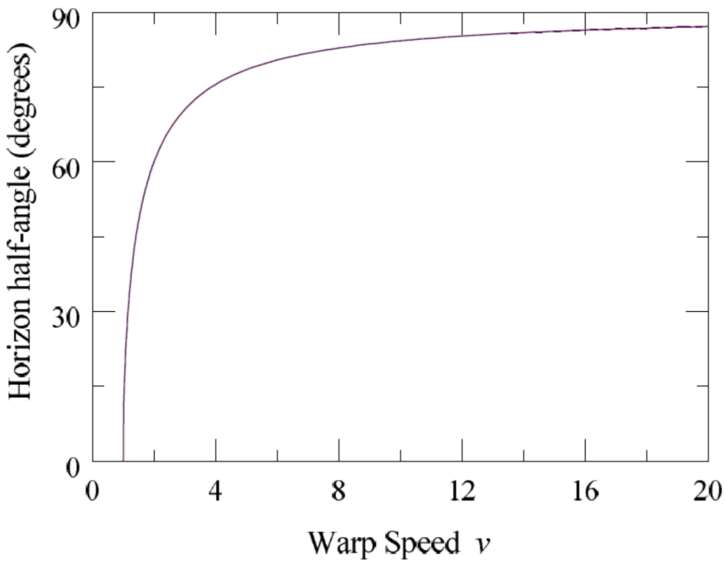

Figure 18 – Visibility horizont in Alcubierre warp drive [7].

4.6.4 Horizon

Now, let's consider that the Alcubierre warp drive moves in the +z direction. In this case, the Alcubierre metric takes the following form:

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left[dz - v(t)f(x,y,z-z_0(t))dt\right]^2, \tag{4.202}$$

where $z_0(t)$ represents the position in the z coordinate at each instant of time t of the spacecraft. Furthermore, $v(t)$ represents the speed of the spacecraft. Also, for simplicity, we will denote $v(t) \equiv v$ and $f(x,y,z-z_0(t)) \equiv f$. Now, we want to know what happens when we use the reference frame of an observer located inside the spacecraft. To do this, we will use the transformation (4.157) [18]:

$$\hat{z} = z - z_0(t). \tag{4.203}$$

Replacing the transformation (4.203) with the metric (3.36) we have the following:

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left[d\hat{z} + (1-f)vdt\right]^2. \tag{4.204}$$

Indeed:

$$ds^2 = \left[-1 + v^2(1-f)^2\right]dt^2 + 2(1-f)vd\hat{z}dt + dx^2 + dy^2 + dz'^2. \tag{4.205}$$

Now, let's look at the trajectory of an emitted photon along the z-axis and from the center of the warp drive bubble outwards. In this case:

$$ds^2 = dx = dy = 0. \tag{4.206}$$

Replacing (4.206) into the metric (4.205), we have:

$$\frac{d\hat{z}}{dt} = 1 - (1 - f)v. \tag{4.207}$$

We know that function f takes values from 0 to 1, where $f = 1$ corresponds when the photon is in the center of the spacecraft. Indeed, the speed of light at the center of the spacecraft will be:

$$\frac{d\hat{z}}{dt} = 1. \tag{4.208}$$

However, from equation (4.207) we can see that, for some critical value $\hat{z}_c$ where $\hat{z} = \hat{z}_c$, the value of f will be:

$$f = 1 - \frac{1}{v}. \tag{4.209}$$

Replacing (4.209) into the equation (4.207), we have:

$$\frac{d\hat{z}}{dt} = 0. \tag{4.210}$$

Physically, the equation (4.210) means the following: When photons leave the center of the spacecraft and arrive at point $\hat{z} = \hat{z}_c$, these photons will remain at relative rest with respect to an observer located in the center of the spacecraft. In this way, the photons are "frozen" and are carried along with the warp bubble. Indeed, the emitted photons never reach the outer region of the warp bubble [18]. Consequently, we have found an horizon $\mathcal{H}$ for our warp drive. This behavior is very similar to an event horizon of a Schwarzschild black hole [18].

This means that the spaceship crew (who are inside the warp bubble) cannot control the warp bubble at will. So, to avoid this problem, the warp bubble must be created and driven by some observer, whose light cone must be in the direction of the warp drive's movement, and must contain its entire trajectory [9].

To study if there is an horizon $\mathcal{H}$ in the Alcubierre metric, we can analyze the 2-D Alcubierre metric [15]. Only considering the z coordinate (direction in which the spacecraft is moving), the metric (3.36) reduces to the following:

$$ds^2 = -(1 - v^2 f^2)dt^2 - 2vf dzdt + dz^2. \tag{4.211}$$

To simplify the calculations, consider the warp bubble velocity constant, indeed:

$$v \equiv v(t) = v_b. \tag{4.212}$$

Furthermore, considering the equation (3.39) in this metric we will have:

$$r_s(t) \equiv r = \left[(z - v_b t)^2\right]^{1/2}. \tag{4.213}$$

If we consider $z > v_b t$, then $r = z - v_b t$. Indeed, we have:

$$dz = dr + v_b dt. \tag{4.214}$$

We can see that the transform (4.214) is the two-dimensional equivalent of the transform (4.203). That is, by taking the r coordinates, we will also have the space-time seen by an observer who is inside the warp bubble. Then, using (4.214), the metric (4.211) will be:

$$ds^2 = -A(r)\left[dt - \frac{v_b(1 - f(r))dr}{A(r)}\right]^2 + \frac{dr^2}{A(r)}, \tag{4.215}$$

where $A(r)$ is the following:

$$A(r) = 1 - v_b^2\left[1 - f(r)\right]^2. \tag{4.216}$$

Now, if we denote τ by the following:

$$d\tau = dt - \frac{v_b(1 - f(r))dr}{A(r)}. \tag{4.217}$$

Replacing (4.217) with (4.215), we have the metric:

$$ds^2 = -A(r)d\tau^2 + \frac{dr^2}{A(r)}. \tag{4.218}$$

From the latest results, we can have the following conclusion: We know that $f \to 1$ as we approach the center of the warp bubble, that is, $r \to 0$. Now, from the equation (4.216) then:

$$A(r) \to 1. \tag{4.219}$$

Replacing (4.219) into the metric (4.218) we have:

$$ds^2 = -d\tau^2 + dr^2. \tag{4.220}$$

The metric (4.220) tells us that inside the warp bubble, the space-time is Minskowki. This characteristic had already been calculated using 3+1 formalism by Alcubierre [2].

Now let's see the behavior of the metric (4.218) for any value of r. In order to analyze the speed v_b we must remember the following:

$$0 \leq f(r) \leq 1. \tag{4.221}$$

If $v_b < 1$ (implying a speed less than that of light), then $A(r) \neq 0$ for any value of r. On the other hand, if $v_b > 1$ (which implies a superluminal speed) we will have a singularity at point r_c, where we have the following:

$$f(r_c) = 1 - \frac{1}{v_b} \to A(r_c) = 0. \tag{4.222}$$

From (4.222) we can see that at $r = r_c$ we have an horizon $\mathcal{H}$. Finally, from the metric (4.218) we can see that τ is interpreted as the proper time measured by an observer who is traveling inside the spacecraft [9].

Figure (18) also shows the horizon half-angle as a function of the speed v of the warp bubble. In figure (18) we can see that as the speed of the warp bubble v increases, the value of the horizon half-angle also increases. The maximum value of the horizon half-angle is 90°. Furthermore, as we have already mentioned, figure (18) shows that the horizon $\mathcal{H}$ is formed only if the warp bubble has a speed equal to or greater than the speed of light. In conclusion, the half-angle describes the behavior of the horizon $\mathcal{H}$ and also the visibility horizon $\mathcal{H}_v$.

4.6.5 One-way membranes $\mathcal{N}$

We can use the same algorithms used above to find one-way membranes $\mathcal{N}$. From the 2-dimensional Alcubierre metric (4.218), we have:

$$g_{\mu\nu} = \begin{bmatrix} -A(r) & 0 \\ 0 & \frac{1}{A(r)} \end{bmatrix}. \tag{4.223}$$

Therefore:

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{A(r)} & 0 \\ 0 & A(r) \end{bmatrix}. \tag{4.224}$$

Now, let's take the following vector n_μ:

$$n_\mu = (0, 1). \tag{4.225}$$

Now, let's calculate n^μ. Also, we know that $n^\mu = g^{\mu\nu} n_\nu$, so:

$$n^0 = g^{00} n_0 + g^{01} n_1 = 0, \tag{4.226}$$

$$n^1 = g^{10} n_0 + g^{11} n_1 = A(r). \tag{4.227}$$

From the expressions (4.225) - (4.227), and remembering that n_ν is null and therefore $n_\mu n^\mu = 0$, we have:

$$A(r) = 0. \tag{4.228}$$

From this last expression, it follows that (4.228) not only represents an infinite redshift surface $\mathcal{I}$. Also represents a one-way membrane $\mathcal{N}$. Now, let's use the metric (4.215) defined in other coordinates. We have:

$$ds^2 = -\left[1 - v^2(1-f)^2\right] dt^2 + 2(1-f)v\,dr\,dt + dr^2. \tag{4.229}$$

Then:

$$g_{\mu\nu} = \begin{bmatrix} -(1 - v^2(1-f)^2) & v(1-f) \\ v(1-f) & 1 \end{bmatrix}. \tag{4.230}$$

Therefore:

$$g^{\mu\nu} = \begin{bmatrix} -1 & v(1-f) \\ v(1-f) & (1-v^2(1-f)^2) \end{bmatrix}. \tag{4.231}$$

We will take the vector n_μ given in (4.225), and using (4.230) and (4.231) we have:

$$n^0 = v(1-f), \tag{4.232}$$

$$n^1 = 1 - v^2(1-f)^2. \tag{4.233}$$

Applying the condition that $n_\mu n^\mu = 0$, then:

$$1 - v^2(1-f)^2 = 0. \tag{4.234}$$

And we must remember from (4.216) that $A(r) \doteq 1 - v^2(1-f)^2$. Indeed $A(r) = 0$. Therefore, it is shown that equation (4.234) is a one-way membrane $\mathcal{N}$.

4.6.6 The blueshift problem

Now let's look at the case when the photon forms an angle of $\theta_0 = 180°$. That is, a photon that travels in the opposite direction to warp drive. The initial conditions in this case will be:

$$p_0^t = 1, \qquad p_0^x = -1 + v, \qquad p_0^y = 0. \tag{4.235}$$

If we replace the initial conditions (4.235) in equation (4.191) we have:

$$2\Gamma^y_{tx}\left\{-1 + v - vf\right\} = 0 \rightarrow \Gamma^y_{tx} = 0. \tag{4.236}$$

Replacing (4.236) into (4.191):

$$\frac{dp^y}{d\lambda} = 0 \rightarrow p^y = p_0^y = 0. \tag{4.237}$$

In this case, we would only have two equations to solve. Replacing (4.237) into (4.195):

$$\frac{dp^t}{d\lambda} + \left(p^x - vfp^t\right)\left(p^x - vf\right)\Gamma^t_{xx} = 0. \tag{4.238}$$

Also, replacing (4.237) in (4.165) we have:

$$\frac{dp^x}{d\lambda} + \Gamma^x_{tt}\left(p^t\right)^2 + \Gamma^x_{xx}\left(p^x\right)^2 + 2\Gamma^x_{tx}p^t p^x = 0. \tag{4.239}$$

So, from the equations (4.238) and (4.239) we will have the following:

$$\frac{dp^t}{d\lambda} + v\partial_x f\left(p^t\right)^2 = 0, \tag{4.240}$$

$$\frac{dp^x}{d\lambda} + v^2\partial_x f\left(p^t\right)^2 = 0. \tag{4.241}$$

Multiplying equation (4.240) by v and subtracting it with equation (4.241) we will have the following:

$$\frac{d}{d\lambda}\left[p^x - vp^t\right] = 0 \to p^x - vp^t = A, \tag{4.242}$$

where A is a constant. Now, we need to remember that for null geodesics:

$$p^\alpha p_\alpha = 0. \tag{4.243}$$

Finally, replacing (4.235) and (4.242) in (4.243) we have the following:

$$p^t = \frac{E_\infty(v+1)}{v(1-f)+1}, \tag{4.244}$$

$$p^x = \frac{E_\infty(v+1)(vf-1)}{v(1-f)+1}, \tag{4.245}$$

where E_∞ represents the energy of the photon measured by an observer at infinity. In the center of the warp bubble ($f = 1$) we have the following:

$$\frac{E_0}{E_\infty} = 1 + v, \tag{4.246}$$

where E_0 represents the photon energy measured by an observer inside the warp bubble. In equation (4.246) we can see that the photon that impacts in the opposite direction to the movement of the warp drive has a blueshift.

For the case of a photon traveling in the same direction of the warp bubble $\theta_\infty = 0°$, the analysis is analogous to the previous case. Indeed we have the following:

$$p^t = \frac{E_\infty(v-1)}{v(1-f)-1}, \tag{4.247}$$

$$p^x = \frac{E_\infty(v-1)(vf+1)}{v(1-f)-1}. \tag{4.248}$$

In the center of warp bubble $f = 1$ we have the following:

$$\frac{E_0}{E_\infty} = 1 - v. \tag{4.249}$$

In equation (4.249) we can see that for $v > 1$ the photon will never impact the warp bubble. That photon is within the visibility horizon $\mathcal{H}_v$.

Figure (19) shows the redshift and blueshift of the photons that are perceived by an observer inside the warp bubble. It is important to mention that all the photons shown in figure (19) will come from the outer region of the visibility horizon $\mathcal{H}_v$.

Indeed, from figure (19) angle θ_0 is represented on the horizontal axis. This is the angle perceived by an observer traveling within the warp drive. The vertical axis represents the ratio E_0/E_∞. If we have $E_0/E_\infty > 1$ then the photon will have blueshift. If we have $E_0/E_\infty < 1$ then the photon will have redshift.

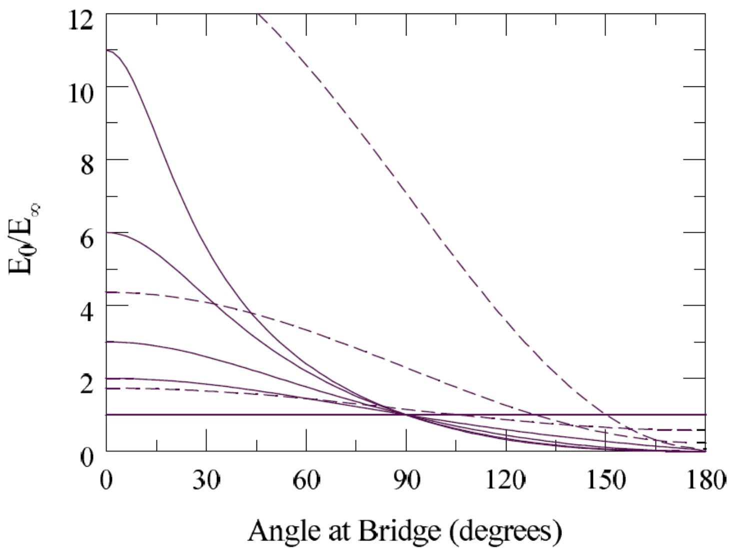

Figure 19 – The blueshift and redshift of the fotons [7].

In figure (19) we can also see the three special cases that we have analyzed analytically: $\theta_0 = 90°$, $\theta_0 = 0°$ and $\theta_0 = 180°$. As we have shown above, from figure (19) for $\theta_0 = 90°$ there is no blueshift or redshift. Also, for $\theta_0 = 0°$ there is blueshift. On the other hand, for $\theta_0 = 180°$ there is infinite redshift. In general for $\theta_0 < 90°$ the photons will have a blueshift. And for $\theta_0 > 90°$ the photons will have a redshift.

The filled curves in figure (19) represent the speeds of a warp bubble of 0, 1, 2, 5, and 10, from bottom to top and at the left edge of the graph. The dotted curves represent the Doppler effect of special relativity for a ship traveling at speeds of 0.99, 0.9, and 0.5, from top to bottom left of the graph [7].

Indeed, if we analyze figures (17) and (19) we reach the following conclusion. An observer traveling in a warp bubble will observe photons coming from absolutely all directions. This traveler will not be able to see the dark region formed by the visibility horizon $\mathcal{H}_v$. He will detect photons that "apparently" come from behind the warp bubble. However, these photons come from the outer region of the visibility horizon $\mathcal{H}_v$ (regions close to $\mathcal{H}_v$). And by aberration those photons will appear behind the warp bubble [7]. Also, these photons that appear behind of the warp bubble, as shown in figure (19), will be redshifted, where for an angle of $\theta_0 \to 180°$ we will have an infinite redshift.

To finish our discussion we are going to comment on how dangerous it would be to travel in a warp drive. Photons that impact at $\theta_0 < 90°$ can be very dangerous for the crew. Clark et al. estimated that for an Alcubierre warp drive traveling at a constant speed of $v = 200$, the cosmic microwave background (CMB) photons would be extremely blueshifted [7]. The energies at which the photons would reach the center of the warp bubble would be comparable to the photons emanated by the solar photosphere!

5 Fell-Heisenberg metric

It is well known that negative density energy is a great problem for the realistic construction of a superluminal Warp Drive [11]. Even Olum claimed to prove that all superluminal travel requires negative energy density in general relativity [23]. However, in recent years several papers indicated that they have demonstrated warp drive type solutions with positive energy [5] [10] [16].

Fell and Heisenberg presented a special class of metrics with a geometric interpretation of Eulerian energy, in order to solve the negative energy problem [10]. However, apparently this problem is not solved, since it only contemplates a certain type of observers (Eulerian observers). That is, in order to not violate the WEC, it is necessary to analyze all observers (not only Eulerian observers). And this does not happen with the Fell-Heisenberg metric [24].

However, although the Fell-Heisenberg metric does not solve the problem of negative energy, we consider that the geometric interpretation they give to the energy density ρ is too original. As we will see later, this metric has many advantages that could be used in future research.

5.1 Definition

To further investigate the positivity of ρ for Eulerian observers, the shift vector β^i will be decomposed using the Helmholtz decomposition. In this case β^i will be assumed to be independent of time, so:

$$\beta^i \equiv \vec{\beta}(x,y,z) = \vec{\nabla}\phi(x,y,z) + \vec{\omega}(x,y,z), \tag{5.1}$$

where $\phi(x,y,z)$ is an escalar field and $\vec{\omega}(x,y,z)$ is a solenoidal field, namely $\vec{\nabla}\cdot\vec{\omega} = 0$. Also:

$$\vec{\nabla}\phi(x,y,z) = (\partial_x\phi(x,y,z), \partial_y\phi(x,y,z), \partial_z\phi(x,y,z)), \tag{5.2}$$

$$\vec{\omega}(x,y,z) = (\omega_x(x,y,z), \omega_y(x,y,z), \omega_z(x,y,z)). \tag{5.3}$$

As we will see later, the Fell-Heisenberg metric is a subclass of the Natario metric. The introduction of the geometric interpretation to Eulerian energy through the shift vector $\vec{\beta}$ (which is a function of $\phi(x,y,z)$ and $\vec{\omega}(x,y,z)$) gives us new paths to find new Warp Drive metrics [10].

- **Fell-Heisenberg's metric:** A Warp Drive is a globally hyperbolic spacetime that has the following metric:

$$ds^2 = -dt^2 + \{dx + (\partial_x\phi + \omega_x)dt\}^2 + \{dy + (\partial_y\phi + \omega_y)dt\}^2 + \\ + \{dz + (\partial_z\phi + \omega_z)dt\}^2 , \quad (5.4)$$

where $\vec{\omega}$ is the purely rotational part and $\vec{\nabla}\phi$ is the purely irrotational part of $\vec{\beta}$.

After seeing this definition, one could validly think: Why was it necessary to decompose $\vec{\beta}$ into $\vec{\nabla}\phi + \vec{\omega}$? To answer this question let's take an arbitrary shift vector:

$$\vec{\beta} = (\beta_x, \beta_y, \beta_z). \quad (5.5)$$

Now let's consider $\alpha = 1$ and $\gamma_{ij} = \delta_{ij}$ Introducing the lapse function and the induced metric mentioned above into the Hamiltonian constraint (2.49) we have:

$$16\pi\rho = 2(\partial_x\beta_x\partial_y\beta_y + \partial_x\beta_x\partial_z\beta_z + \partial_y\beta_y\partial_z\beta_z) - \\ - \frac{1}{2}\left\{(\partial_z\beta_y + \partial_y\beta_z)^2 + (\partial_y\beta_x + \partial_x\beta_y)^2 + (\partial_z\beta_x + \partial_x\beta_z)^2\right\}. \quad (5.6)$$

From equation (5.6) we can see that the second term is purely negative. However, the first term $2(\partial_x\beta_x\partial_y\beta_y + \partial_x\beta_x\partial_z\beta_z + \partial_y\beta_y\partial_z\beta_z)$ has an indeterminate sign. In this case, the equation (5.6) cannot give us more information about the positivity of ρ.

In order to further investigate the equation (5.6), the decomposition of the shift vector $\vec{\beta}$ was done as shown in (5.1). Of course, there is no single way to decompose the shift vector [10]. However, that was the Ansatz chosen by Fell and Heisenberg. Now, considering the shift vector expressed in terms of ϕ and $\vec{\omega}$, and substituting it into equation (2.49), we have a much more general and very complicated expression:

$$16\pi\rho = 2(h_1 + h_2 + h_3) - 2\langle\mathcal{J}, \mathcal{H}\rangle_F - \langle\mathcal{J}, \mathcal{J}\rangle_F + \frac{1}{2}\left|\vec{\nabla}\times\vec{\omega}\right|^2, \quad (5.7)$$

where, $\mathcal{J}(x, y, z)$ is the Jacobian matrix of $\vec{\omega}(x, y, z)$ and is defined as follows:

$$\mathcal{J}(x, y, z) = \begin{bmatrix} \partial_x\omega_x & \partial_y\omega_x & \partial_z\omega_x \\ \partial_x\omega_y & \partial_y\omega_y & \partial_z\omega_y \\ \partial_x\omega_z & \partial_y\omega_z & \partial_z\omega_z \end{bmatrix}. \quad (5.8)$$

Also, $\mathcal{H}(x, y, z)$ is the Hessian matrix of $\phi(x, y, z)$ and is defined as follows:

$$\mathcal{H}(x, y, z) = \begin{bmatrix} \partial_x^2\phi & \partial_x\partial_y\phi & \partial_x\partial_z\phi \\ \partial_y\partial_x\phi & \partial_y^2\phi & \partial_y\partial_z\phi \\ \partial_z\partial_x\phi & \partial_z\partial_y\phi & \partial_z^2\phi \end{bmatrix}. \quad (5.9)$$

From the equation (5.7) h_1, h_2 and h_3 are the second order principal minors of $\mathcal{H}(x, y, z)$ and are defined:

$$h_1(x, y, z) = (\partial_y^2\phi)(\partial_z^2\phi) - (\partial_y\partial_z\phi)^2, \quad (5.10)$$

$$h_2(x,y,z) = (\partial_x^2\phi)(\partial_z^2\phi) - (\partial_x\partial_z\phi)^2, \tag{5.11}$$

$$h_3(x,y,z) = (\partial_x^2\phi)(\partial_y^2\phi) - (\partial_x\partial_y\phi)^2. \tag{5.12}$$

Also $\vec{\nabla} \times \vec{\omega}$ is the rotational of the solenoidal field. It is important to emphasize that $\langle \cdot, \cdot \rangle_F$ is the Frobenius product defined as:

$$\langle A, B \rangle_F = \sum_{ij} A_{ij} B_{ij}. \tag{5.13}$$

5.2 Geometric interpretation of the Eulerian energy

Now let's look at the geometric interpretation of energy density introduced by Fell and Heisenberg. To simplify the calculations, we will start by analyzing the purely irrotational sector of $\vec{\beta}$, that is, $\vec{\nabla}\phi$. Indeed:

$$\vec{\beta} = \vec{\nabla}\phi. \tag{5.14}$$

Replacing equation (5.14) with equation (5.7) we will have the following very simple expression for ρ:

$$8\pi\rho = h_1 + h_2 + h_3. \tag{5.15}$$

If density ρ is positive (only for Eulerian observers), then, it must be established:

$$h_1 + h_2 + h_3 \geq 0. \tag{5.16}$$

It is important to mention that the second-order principal minors h_1, h_2 and h_3 describe the curvature (convexity or concavity) of the 2-dimensional subspaces $\phi(y,z)$, $\phi(x,z)$ and $\phi(x,y)$, respectively. With this, we will define the geometric interpretation of Eulerian energy as follows:

- **Geometric interpretation of the Eulerian energy:** For $\vec{\beta} = \vec{\nabla}\phi(x,y,z)$, the convexity of the 2-dimensional subspaces of the scalar field $\phi(x,y,z)$, that is, the convexity of $\phi(y,z)$, $\phi(x,z)$ and $\phi(x,y)$ determines the positivity of the Eulerian energy density ρ.

It is evident that this geometric interpretation of ρ is based on the equation (5.15). It is also important to mention that the convexity of the scalar function $\phi(x,y,z)$ is not directly related to the positivity of ρ.

Also, one could explore the geometric interpretation of the purely rotational sector of $\vec{\beta}$, that is, $\vec{\omega}$. However, according to equation (5.7), it would be very complicated. So, that problem still remains open to future research.

In conclusion, by decomposing the shift vector $\vec{\beta}$ in the Helmholtz decomposition we can more clearly investigate the nature of ρ (only for Eulerian observers). Furthermore, by introducing ϕ and $\vec{\omega}$, the Fell-Heisenberg metric has some versatility (as we will see later in the examples): According to the choice of ϕ and $\vec{\omega}$ we can find metrics, with arbitrarily different properties.

5.2.1 Example: Alcubierre metric

Let's consider a shift vector $\vec{\beta}$ with a single component, that is:

$$\vec{\beta} = (\beta_x, 0, 0). \tag{5.17}$$

We replace this shift-vector (5.17) in the Hamiltonian constraint (2.49) and it gives us the following:

$$16\pi\rho = -\frac{1}{2}\left[(\partial_y\beta_x)^2 + (\partial_z\beta_x)^2\right]. \tag{5.18}$$

From equation (5.18) it is clear why the Alcubierre metric needs negative ρ (measured by an Eulerian observer). Also, from the same equation, we reach the conclusion that we must at least introduce two components to the shift vector $\vec{\beta}$. So, ρ could not be negative.

We also note that the Alcubierre metric could be a specific Fell-Heisenberg metric with the following values of ϕ and $\vec{\omega}$:

$$(\partial_x\phi + \omega_x, \partial_y\phi + \omega_y, \partial_z\phi + \omega_z) = (-v_s(t)f(r(t)), 0, 0). \tag{5.19}$$

From (5.19) we have 3 partial differential equations along with $\partial_x\omega_x + \partial_y\omega_y + \partial_z\omega_z = 0$ to determine: $\omega_x, \omega_y, \omega_z$ and ϕ. Then, from these equations, also we have:

$$\partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi + \partial_x\left[v_s(t)f(r(t))\right] = 0. \tag{5.20}$$

By solving (5.20) we could calculate ϕ (it is not known if this equation has a solution). Then, from ϕ would be possible to calculate $\omega_x, \omega_y, \omega_z$. Indeed, if the partial differential equation (5.20) has a solution, the Alcubierre metric would be a Fell-Heisenberg-type metric.

5.2.2 Example: Natario without expansion metric

We know from Natario's metric that:

$$Tr(K_{ij}) = 0. \tag{5.21}$$

We also know that the shift vector of the Natario metric (3.126) is purely solenoidal, that is, it has the form: $\vec{\beta} = \vec{\omega}$, with $\vec{\nabla} \cdot \vec{\omega} = 0$. Replacing this shift vector into the Hamiltonian constraint (2.49), then:

$$\rho = -\frac{1}{16\pi}K_{ij}K^{ij} \leq 0. \tag{5.22}$$

From equation (5.22) we can also see why Natario spacetime also needs negative energy densities ρ (measured by an Eulerian observer).

We also note that Natario's metric (3.126) is a specific Fell-Heisenberg metric with the following values of ϕ and $\vec{\omega}$:

$$\vec{\omega} = -2v_s f\cos(\theta)e_r + v_s\left(2f + rf'\right)\sin(\theta)e_\theta \tag{5.23}$$

$$\vec{\nabla}\phi = 0 \tag{5.24}$$

5.3 Some examples

As seen above, the scalar field $\phi(x, y, z)$ defines completely the warp drive spacetime (in a spacetime with $\vec{\omega}(x, y, z) = 0$). Below we will show some examples of warp drives with $\vec{\beta} = \vec{\nabla}\phi(x, y, z)$. The first warp drive spacetime will be defined by a piecewise ϕ. The second warp drive spacetime will be defined by a C^2-differentiable ϕ.

Fell and Heisenberg proposed as an example two types of scalar fields (as we mentioned above), with properties different from each other, however, these scalar fields are generated by Eulerian densities of positive energy [10].

5.3.1 Example I

Let $\vec{\beta} = \vec{\nabla}\phi^I(x, y, z)$ be a purely irrotational shift vector. $\phi^I(x, y, z)$ is a scalar field defined as follows:

$$\phi^I(x, y, z) = -z + \begin{cases} -\left(\sqrt{x^2+y^2}\right)^6 - z^2; & -\left(\sqrt{x^2+y^2}\right)^6 - z^2 > z, \\ \left(\sqrt{x^2+y^2}\right)^6 + z^2; & \left(\sqrt{x^2+y^2}\right)^6 + z^2 < z, \\ z; & \text{otherwise.} \end{cases} \tag{5.25}$$

The scalar field given by (5.25) will generate a warp drive with positive Eulerian energy density and superluminal. From (5.25) we can see that the scalar field has cylindrical symmetry (it is possible to change coordinates from Cartesian to cylindrical to simplify some calculations). The plot of this scalar field $\phi(x, 0, z)$ (with $y = 0$) is given by figure (20).

We also see that the scalar field is not C^1-differentiable. The most notable consequence of this will occur when we graph the energy density $\rho(x, 0, z)$. To see this characteristic, first, let's substitute the scalar field into (5.10) - (5.12) to get h_1, h_2 and h_3. Then we substitute these last terms into equation (5.15) to get the energy density ρ.

The energy density plot $\rho(x, 0, z)$ (with $y = 0$) is given by figure (21). We clearly see discontinuities in the energy density distribution $\rho(x, 0, z)$ because the scalar field is given by a piecewise function. As we can see in Figure (21) the Eulerian energy density is positive. However, discontinuities can present problems from a physical point of view. If these discontinuities are very small (Planck scale), then the problem would be even

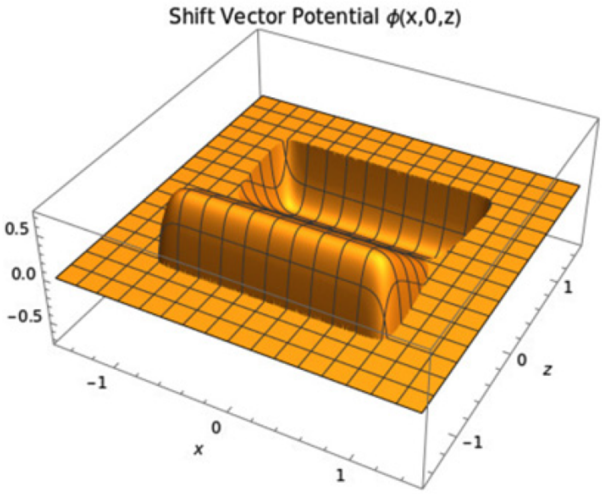

Figure 20 – Plot of scalar field $\phi(x, 0, z)$ [10]

greater. Small discontinuities cannot be treated within the context of general relativity. A quantum theory of gravity would be needed to deal with them [10].

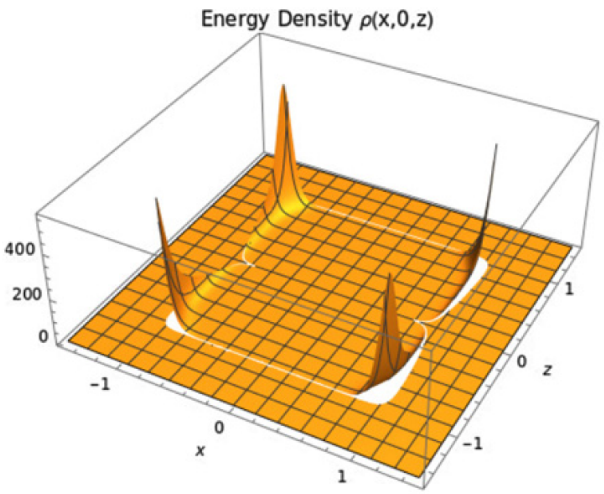

Figure 21 – Plot of energy density $\rho(x, 0, z)$ [10]

In the figure (22) we can see the plot of the energy density $\rho(x, y, z)$. From the figure, it is clear that the energy will be distributed in two toroids perpendicular to the z axis. One of the toroids is located behind and the other is located in front of the spacecraft. Furthermore, in the region where the spacecraft is located, the energy density $\rho(x, y, z)$ is zero (minimum). The energy density $\rho(x, y, z)$ will have its maximum values at the edges of the toroids (as seen in Figure (22)).

The expansion $\theta(x, y, z)$ is represented in the figure (23). From there, we can see that the magnitude of the expansion $|\theta|$ is maximum on the outer face of the toroids. It is also minimal (zero) in the region surrounding the spacecraft. Also, if z_T represents the position of a torus, we can see the following:

$$z_T > 0 \rightarrow \theta(x, y, z) > 0, \tag{5.26}$$

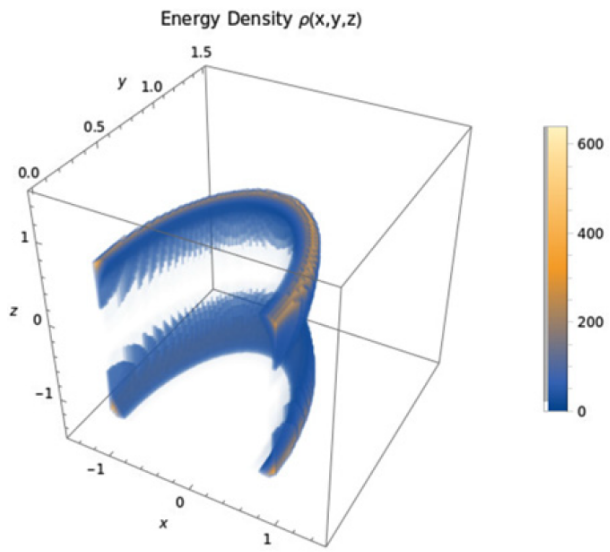

Figure 22 – Plot of energy density $\rho(x, y, z)$ [10]

Also:

$$z_T < 0 \rightarrow \theta(x, y, z) < 0. \quad (5.27)$$

The expression (5.26) tells us that the expansion of spacetime will be in the positive z region. Analogously, the expression (5.27) tells us that the contraction of spacetime will be in the negative z region.

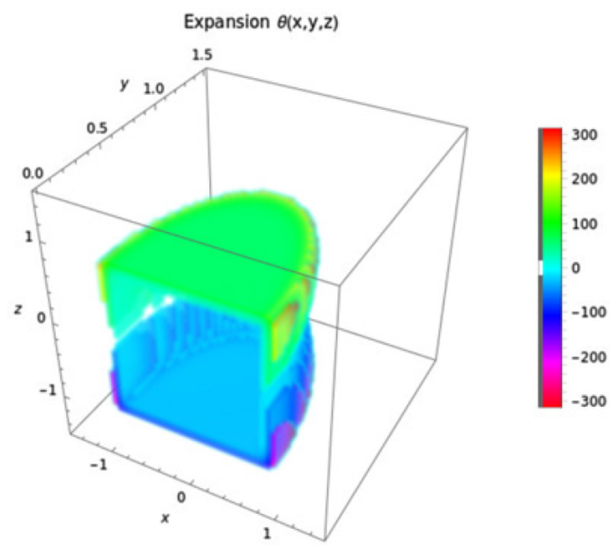

Figure 23 – Plot of expansion $\theta(x, y, z)$ [10]

The shift vector $\vec{\beta}$ is represented in figure (24). Using numerical calculations, Fell and Heisenberg calculated the shift vector at the spacecraft's position (around the position (0,0,0)). The magnitude of the shift vector in this case is $\left|\vec{\beta}\right| = 1,3$ (using geometric units). This shows that warp drive is superluminal. Furthermore, from Figure (24) we can see that the shift vector is a vector pointing towards the negative z region. Physically this implies that the spacecraft moves in the negative z direction. It also implies that the spacecraft moves from the region of spacetime that has expansion to the region of spacetime that has contraction. This feature is very similar to the mechanism used by the Alcubierre warp drive [10].

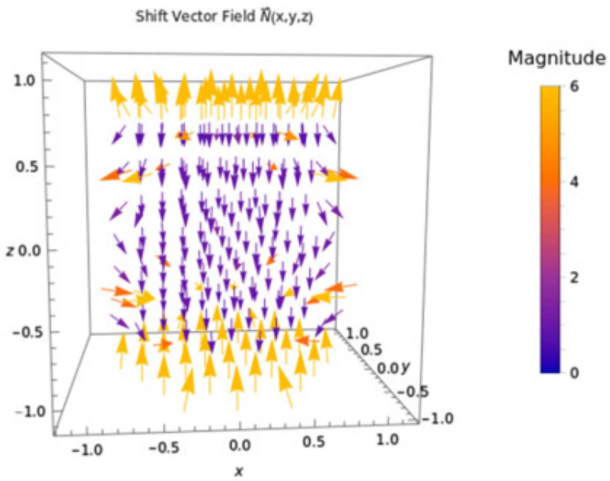

Figure 24 – Plot of shift-vector field $\vec{\nabla}\phi^{I}(x,y,z) = (\beta_x, \beta_y, \beta_z)$ [10]

As we can see, this first type of warp drive with $\vec{\beta} = \vec{\nabla}\phi^{I}(x,y,z)$ only uses positive Eulerian energy. However, as mentioned above, since the scalar field $\phi^{I}(x,y,z)$ is a piecewise function, the energy density will have discontinuities. Then we should think about using another type of scalar field. This scalar field should be more physically realistic.

5.3.2 Example II

With the aim of finding a warp drive with a physical nature, Fell and Heisenberg proposed the following C^2-differentiable scalar field:

$$\phi^{II}(x,y,z) = \frac{\sqrt{\sigma\pi}V}{m+n}\left\{-\beta(m+n)\mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right) + n(\beta\, m - r^{2\alpha})\mathrm{Erf}\left(\frac{\beta - \frac{r^{2\alpha}}{m}}{\sqrt{\sigma}}\right) + \right.$$
$$\left. + m(\beta\, n - r^{2\alpha})\mathrm{Erf}\left(\frac{\beta + \frac{r^{2\alpha}}{n}}{\sqrt{\sigma}}\right)\right\} + \frac{\sigma V}{m+n}\left\{m\, n\, e^{-\frac{\left(\beta - \frac{r^{2\alpha}}{m}\right)^2}{\sigma}} + m\, n\, e^{-\frac{\left(\beta + \frac{r^{2\alpha}}{n}\right)^2}{\sigma}} - (m+n)e^{-\frac{\beta^2}{\sigma}}\right\}, \tag{5.28}$$

where $m = m(x,y,z)$ and $n = n(x,y,z)$ are functions of the spatial coordinates. Also, α, β and V are free parameter. We also denote:

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{5.29}$$

Furthermore, σ is a Gaussian weight parameter. Also, function $\mathrm{Erf}(\chi)$ is the so-called Error Function and has the following definition:

$$\mathrm{Erf}(\chi) = \frac{2}{\sqrt{\pi}}\int_0^{\chi} e^{-t^2} dt. \tag{5.30}$$

The scalar field $\phi^{II}(x, y, z)$ seems to be too tedious, however, it has many interesting properties. First, let's see the values chosen by Fell and Heisenberg for the parameters α, β, V and σ [10].

$$(\alpha, \beta, V, \sigma) = \left(\frac{1}{4}, 6, 10, 1\right). \tag{5.31}$$

With the parameters fixed, we can now analyze $m(x, y, z)$ and $n(x, y, z)$. These functions are directly related to the speed of the warp drive (with respect to a distant observer).

First, let's analyze the case where we have a static warp drive (with speed equal to zero). To obtain a static warp drive then the functions $m(x.y.z)$ and $n(x, y, z)$ must necessarily be symmetrically spherical. As a consequence of this, the scalar field $\phi^{II}(x, y, z)$ will be smooth over $\mathbb{R}^3 - 0$ (considering the parameter values given in (5.31)). Also:

$$\vec{x} \to 0 \Rightarrow \phi^{II}(x, y, z) \to 0. \tag{5.32}$$

From (5.32) we can say the following: As we approach the point $(0, 0, 0)$ then the scalar field $\phi^{II}(x, y, z)$ will tend to zero. Replacing (5.32) with (5.28) we have:

$$\lim_{r \to 0} \phi^{II}(x, y, z) = 0. \tag{5.33}$$

Indeed:

$$\begin{aligned} 0 = \frac{\sqrt{\sigma\pi}V}{m+n}\left\{-\beta(m+n)\mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right) + n\,\beta\,m\,\mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right) + m\,\beta\,n\,\mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right)\right\} + \\ + \frac{\sigma V}{m+n}\left\{m\,n\,e^{-\frac{\beta^2}{\sigma}} + m\,n\,e^{-\frac{\beta^2}{\sigma}} - (m+n)e^{-\frac{\beta^2}{\sigma}}\right\}. \end{aligned} \tag{5.34}$$

Then:

$$0 = \frac{V}{m+n}\{2mn - (m+n)\}\left\{\sigma e^{-\frac{\beta^2}{\sigma}} + \beta\sqrt{\sigma\pi}\mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right)\right\}. \tag{5.35}$$

If we consider that parameters σ and β are always positive, from equation (5.35) we can deduce the following:

$$\beta > 0 \to \mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right) > 0. \tag{5.36}$$

Then, from (5.36) we have:

$$\sigma e^{-\frac{\beta^2}{\sigma}} + \beta\sqrt{\sigma\pi}\mathrm{Erf}\left(\frac{\beta}{\sqrt{\sigma}}\right) \neq 0. \tag{5.37}$$

Therefore, from (5.35) and (5.37) we have finally:

$$m + n = 2mn. \tag{5.38}$$

Furthermore, with this configuration of $m(x, y, z)$ and $n(x, y, z)$, the scalar field $\phi^{II}(x, y, z)$ will be spherically symmetric. As a consequence, the energy density $\rho(x, y, z)$ will be uniformly distributed in a spherical shell whose center will be the point $(0, 0, 0)$. In this case, parameter β would represent the radius of the spherical shell [10].

It is also important to mention that with the spherical symmetry conditions for $m(x, y, z)$ and $n(x, y, z)$, the spacetime in the region outside the warp bubble is no longer Minkowski spacetime (feature very different with respect to Alcubierre warp drive). The outer region will now be a Schwarzschild spacetime. This new feature could explain why negative energy densities do not exist for an Eulerian observer in this new warp drive spacetime [5].

Another interesting characteristic of the warp drive generated by $\phi^{II}(x, y, z)$ is that it is relatively easy to reduce the amount of energy it needs. To do this, the Gaussian weight must be calibrated with parameter σ. In this way, it would prevent our warp drive from becoming a black hole.

It is clear that if the energy density is distributed in a spherical shell, as mentioned above, the warp drive will have no speed. If we want our warp drive to work, then we must alter that energy density distribution. To do this we must manipulate the functions $m(x, y, z)$ and $n(x, y, z)$. Indeed, if we do in the $+z$ region:

$$m(x, y, z) > n(x, y, z). \tag{5.39}$$

And analogously, in the $-z$ region we do:

$$m(x, y, z) < n(x, y, z). \tag{5.40}$$

As a result of the conditions (5.39) and (5.40) we will have the scalar potential $\phi^{II}(x, 0, z)$ of the figure (25).

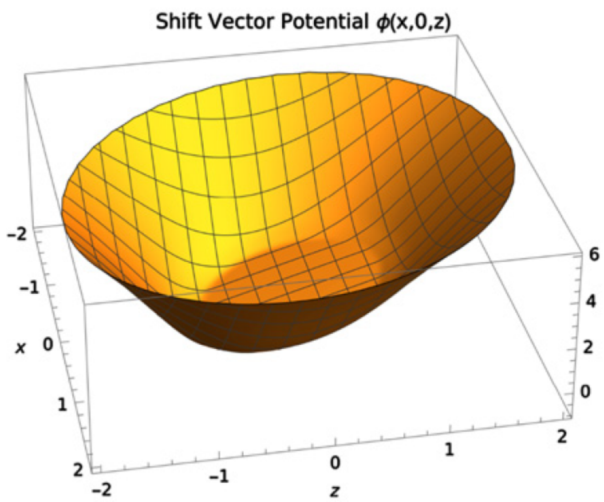

Figure 25 – Plot of scalar field $\phi^{II}(x, 0, z)$ [10]

Furthermore, we see from the energy density plot $\rho(x.0, z)$ and $\rho(x.y, z)$ (see figure (26) and figure (27), respectively), that the energy density behind the spacecraft $\rho^{-z}(x, y, z)$ is increased. Likewise, the energy density ahead of the spacecraft $\rho^{+z}(x, y, z)$ is decreased. Indeed:

$$\rho^{-z}(x, y, z) < \rho^{+z}(x, y, z). \tag{5.41}$$

Indeed, condition (5.41) implies that the spacecraft will move in the $+z$ direction (as will be seen later). Also, from figures (26) and (27) we can see that in the central area of the warp bubble, the energy density is zero. This is a perfect condition for the spaceship.

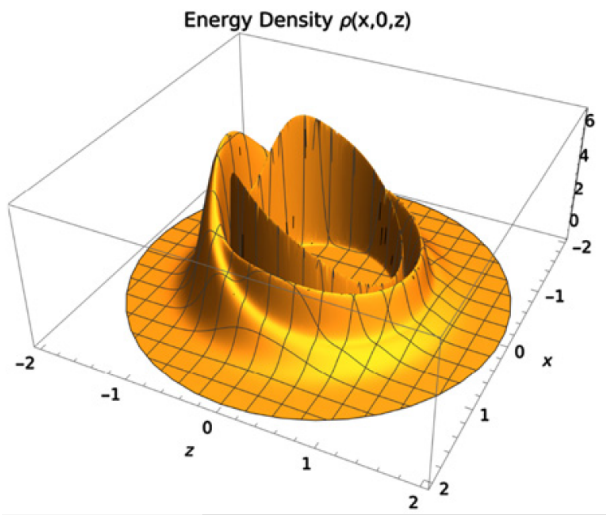

Figure 26 – Plot of energy density $\rho(x.0,z)$ [10]

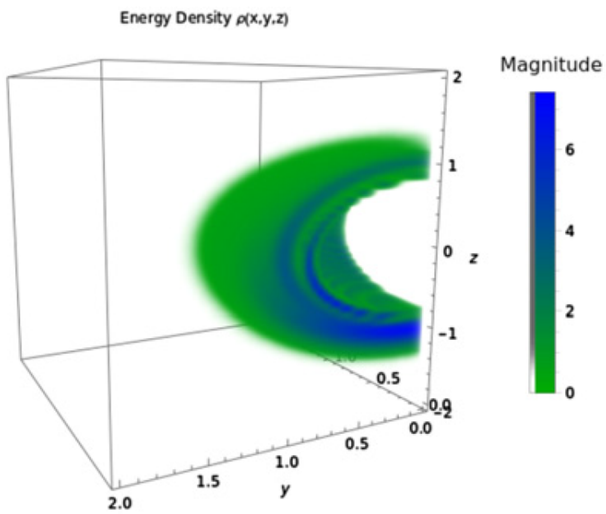

Figure 27 – Plot of energy density $\rho(x.y,z)$ [10]

Figure (28) shows us the expansion $\theta(x,y,z)$. We clearly see that $\theta(x,y,z)$ will always be positive throughout the space. Furthermore, we can see that the expansion in the $-z$ region $\theta^{-z}(x,y,z)$ is greater than the value of the expansion in the +z region $\theta^{+z}(x,y,z)$. Indeed:

$$\theta^{-z}(x,y,z) > \theta^{+z}(x,y,z) > 0. \tag{5.42}$$

Finally, from the figure (29) we can see the plot of the vector field shift vector $\vec{\nabla}\phi^{II}(x,0,z)$. The vector field will tend to be radial as we evaluate that vector field at points further away from the point $(0,0,0)$. However, the shift vector field will only have physical meaning in the center of the warp bubble. The vector field $\vec{\nabla}\phi^{II}(x,0,z)$ evaluated at the center of the warp bubble will give us the value of the speed $v(t)$ at which the warp bubble is moving (with respect to a distant observer). With the parameters given in (5.31), in this case, Fell and Heisenberg calculated that the speed of the warp bubble would be $v = 1,26$ in $+z$-direction (again, in geometric units). In this case, the warp drive given by the scalar field $\phi^{II}(x, y, z)$ is superluminal [10].

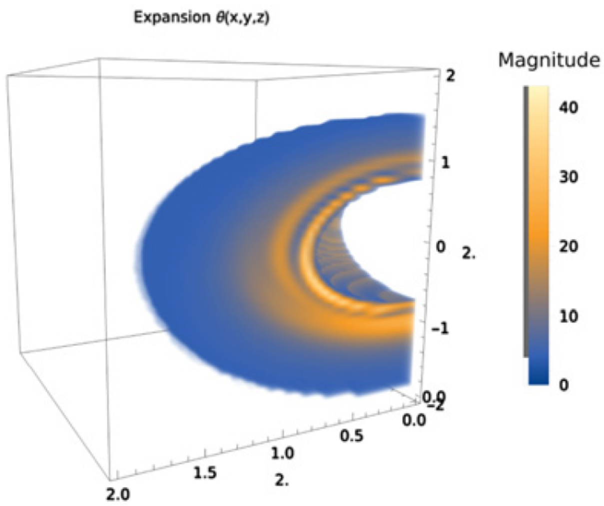

Figure 28 – Plot of expansion $\theta(x.y, z)$ [10]

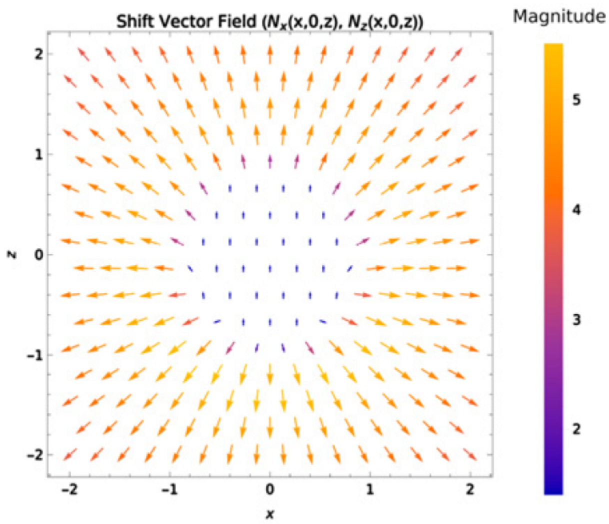

Figure 29 – Plot of shift-vector field $\vec{\nabla}\phi^{II}(x, 0, z) = (\beta_x, 0, \beta_z)$ [10]

As we have seen above, the warp drive generated by the scalar field $\phi^{II}(x, y, z)$ has many interesting characteristics. We can summarize the most notable features:

- Warp drive uses positive energy densities (with respect to an Eulerian observer). However, it is important to mention that Fell-Heisenberg warp drives do not solve the negative energy problem. Santiago, Schuster, and Visser [24] proved that Fell-Heisenberg warp drives violate the WEC (weak energy condition). That is, any Fell-Heisenberg warp drive is going to need negative energy density for certain types of observers.

- Varying the values of functions $m(x, y, z)$ and $n(x, y, z)$ will generate variations in the energy density distributions $\rho(x, y, z)$. As a consequence of this, a non-zero shift vector is generated in the center of the warp bubble. This means that our warp drive can vary in speed v as we vary the distribution of energy density $\rho(x, y, z)$. This is a unique feature very different from the Alcubierre warp drive [10].

The key to this type of warp drive is that it is possible to manipulate the energy density distribution $\rho(x, y, z)$. With this, we will control the speed of the warp drive v. For example, if we have a warp drive with zero speed, the energy will be distributed in a spherical shell (as studied above). However, if we want our warp drive to work, we need to alter the energy distribution. An energy distribution with low spherical symmetry will generate a high velocity in our warp bubble (relative, again, to a distant observer). Similarly, an energy distribution with high spherical symmetry will generate a low warp drive bubble velocity.

5.4 The horizon problem

As mentioned above, Fell and Heisenberg proposed a type of warp drive spacetime that does not require negative energy. In this case, they assumed that this new metric would be very different from Natario's generic metrics [10]. In this context, Fell and Heisenberg left it as an open problem whether these new warp drives have horizons [10].

However, it has already been shown that the Fell-Heisenberg warp drive violates the WEC [24]. Consequently, it would be a special type of generic Natario metric. Therefore it is very easy to see that Fell-Heinsenberg warp drives will present horizons.

In this section we will present the solution of the null geodesics for the warp drive with scalar field $\phi^I(x, y, z)$. Then we will present the Fell-Heisenberg warp drive horizons as a new constraint on any scalar field $\phi(x, y, z)$ that generates a warp drive metric with $\vec{\beta} = \nabla\phi$.

5.4.1 Null geodesics

We shall now analyze the case for $\phi^I(x, y, z)$. In particular, we will consider only the (x,z)-plane, due to the cylindrical symmetry. Thus, setting $y = 0$ and $dy/ds = 0$ for the null geodesics of the Fell-Heisenberg metric, we get

$$\ddot{t} - \Big[(\phi_{,x})^2\phi_{,xx} + 2\,\phi_{,x}\phi_{,z}\phi_{,xz} + (\phi_{,z})^2\phi_{,zz}\Big]\,\dot{t}^2 - 2\,(\phi_{,x}\phi_{,xx} + \phi_{,z}\phi_{,xz})\,\dot{t}\dot{x}$$
$$-2\,(\phi_{,x}\phi_{,xz} + \phi_{,z}\phi_{,zz})\,\dot{t}\dot{z} - \phi_{,xx}\dot{x}^2 - 2\,\phi_{,xz}\dot{x}\dot{z} - \phi_{,zz}\dot{z}^2 = 0, \tag{5.43}$$
$$\ddot{x} + \Big[(\phi_{,x})^3\phi_{,xx} + 2\,(\phi_{,x})^2\phi_{,z}\phi_{,xz} + \phi_{,x}(\phi_{,z})^2\phi_{,zz} - \phi_{,x}\phi_{,xx} - \phi_{,z}\phi_{xz}\Big]\,\dot{t}^2$$
$$+2\,\phi_{,x}(\phi_{,x}\phi_{,xx} + \phi_{,z}\phi_{,xz})\dot{t}\dot{x} + 2\,\phi_{,x}(\phi_{,x}\phi_{,xz} + \phi_{,z}\phi_{,zz})\dot{t}\dot{z} + \phi_{,x}\phi_{,xx}\dot{x}^2$$
$$+2\,\phi_{,x}\phi_{,xz}\dot{x}\dot{z} + \phi_{,x}\phi_{,zz}\dot{z}^2 = 0, \tag{5.44}$$
$$\ddot{z} + \Big[(\phi_{,z})^3\phi_{,zz} + 2\,(\phi_{,z})^2\phi_{,x}\phi_{,xz} + \phi_{,z}(\phi_{,x})^2\phi_{,xx} - \phi_{,z}\phi_{,zz} - \phi_{,x}\phi_{xz}\Big]\,\dot{t}^2$$
$$+2\,\phi_{,z}(\phi_{,x}\phi_{,xx} + \phi_{,z}\phi_{,xz})\dot{t}\dot{x} + 2\,\phi_{,z}(\phi_{,x}\phi_{,xz} + \phi_{,z}\phi_{,zz})\dot{t}\dot{z} + \phi_{,z}\phi_{,xx}\dot{x}^2$$
$$+2\,\phi_{,z}\phi_{,xz}\dot{x}\dot{z} + \phi_{,z}\phi_{,zz}\dot{z}^2 = 0, \tag{5.45}$$

where dot means derivative with respect to the affine parameter of the null geodesics and comma means partial derivative with respect to the subsequent set of spatial coordinates.

Since the corresponding Lagrangian is time-independent, there is an immediate first integral given by

$$\dot{t} = \frac{\dot{\phi}^I - C}{1 - |\nabla\phi^I|^2}, \tag{5.46}$$

where C is an integration constant. There is another first integral given by the fact that we are dealing with null-geodesics, that is

$$\Big(|\nabla\phi^I|^2 - 1\Big)\,\dot{t}^2 + \dot{x}^2 + \dot{z}^2 - 2\dot{\phi}\dot{t} = 0. \tag{5.47}$$

Substituting the equation (5.46) into the equation above, we find

$$\frac{C^2 - \dot{\phi}^2}{|\nabla\phi|^2 - 1} + \dot{x}^2 + \dot{z}^2 = 0, \tag{5.48}$$

which can be solved for $\dot{x}$ in terms $\dot{z}$, using that $\phi^I(x,z)$ given by the equation (5.25) is separable in a function of x plus a function of z, namely

$$\dot{x} = \left[\frac{F_{,x}G_{,z} \pm \sqrt{(F_{,x})^2(G_{,z})^2 + (1 - (F_{,x})^2)C^2}}{(F_{,x})^2 - 1}\right]\dot{z}, \tag{5.49}$$

where $F \equiv F(x)$ and $G \equiv G(z)$, with $\phi^I(x,z) = F(x) + G(z)$. Therefore, the remaining equation is

$$\ddot{z} + \frac{(1080\,x^{14}(2z+1) - 2 + 2\,(2z+1)^3 - 4\,z)\,(6\,x^5\dot{x} + (2z+1)\dot{z} + C)^2}{(-1 + 36\,x^{10} + (2z+1)^2)^2}$$
$$-360\,\frac{(2z+1)x^9\,(6\,x^5\dot{x} + (2z+1)\dot{z} + C)\,\dot{x}}{-1 + 36\,x^{10} + (2z+1)^2} + 30\,(2z+1)x^4\dot{x}^2 + 2\,(2z+1)\dot{z}^2$$
$$-2\,\frac{(2z+1)\,(4\,z+2)\,(6\,x^5\dot{x} + (2z+1)\dot{z} + C)\,\dot{z}}{-1 + 36\,x^{10} + (2z+1)^2} = 0. \tag{5.50}$$

Together with the equations (5.46) and (5.49), this equation closes the system of equations of motion for the light trajectory in the Fell-Heisenberg warp drive. In general, this system is not separable and, therefore, non-integrable.

5.4.2 The horizons

It has been mentioned that every Natario warp drive has a horizon. This horizon $\mathcal{H}$ is a one-way membrane $\mathcal{N}$. We have also easily noted that Fell-Heinsenberg's warp drive is a special case of Natario's warp drive. From the Fell-Heisenberg metric (5.4) for $\vec{\omega} = 0$ and the generic Natario metric (3.1), the shift vector we will be the following:

$$\vec{X} = -\partial_x \phi e_x - \partial_y \phi e_y - \partial_z \phi e_z. \tag{5.51}$$

Now, to find the horizon in the Fell-Heisenberg warp drive, we must find the metric tensor $g_{\hat{\alpha}\hat{\beta}}$ related to the observer inside the warp bubble. If we use the transformations given in (4.157). Then:

$$\partial_{\hat{x}} \phi = \partial_x \phi \qquad \partial_{\hat{y}} \phi = \partial_y \phi. \tag{5.52}$$

Also:

$$\partial_z \phi = \frac{\partial \hat{z}}{\partial z}\frac{\partial \phi}{\partial \hat{z}} = \frac{\partial}{\partial z}\left[z - z_0(t)\right]\frac{\partial \phi}{\partial \hat{z}} = \partial_{\hat{z}} \phi. \tag{5.53}$$

If we consider that our warp drive moves in the positive direction of the z axis, then from (4.94) the metric $g_{\hat{\alpha}\hat{\beta}}$ given for an observer who is inside the warp bubble will be (using (5.52) and (5.53)):

$$g_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} \left[-1 + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi + v)^2\right] & \partial_x\phi & \partial_y\phi & (v + \partial_z\phi) \\ \partial_x\phi & 1 & 0 & 0 \\ \partial_y\phi & 0 & 1 & 0 \\ (v + \partial_z\phi) & 0 & 0 & 1 \end{bmatrix}. \tag{5.54}$$

Indeed:

$$ds^2 = \left[-1 + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi + v)^2\right] dt^2 + 2\left[\partial_x\phi dx + \partial_y\phi dy + (\partial_z\phi + v)\, d\hat{z}\right] dt + \\ + dx^2 + dy^2 + d\hat{z}^2. \tag{5.55}$$

Transforming the line element (5.55) into cylindrical coordinates. Then:

$$x = \rho\cos(\theta), \qquad y = \rho\sin(\theta), \qquad \hat{z} = \hat{z}. \tag{5.56}$$

Also:

$$dx = \cos(\theta)d\rho - \rho\sin(\theta)d\theta, \qquad dy = \sin(\theta)d\rho + \rho\cos(\theta)d\theta \tag{5.57}$$

We know that $\rho = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. Then for the partial derivatives we have:

$$\partial_x\phi = \frac{\partial\phi}{\partial x} = \frac{\partial\rho}{\partial x}\frac{\partial\phi}{\partial\rho} + \frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial\theta}, \tag{5.58}$$

$$\partial_y \phi = \frac{\partial \phi}{\partial y} = \frac{\partial \rho}{\partial y}\frac{\partial \phi}{\partial \rho} + \frac{\partial \theta}{\partial y}\frac{\partial \phi}{\partial \theta}. \tag{5.59}$$

Then:

$$\partial_x \phi = \cos(\theta)\partial_\rho \phi - \frac{\sin(\theta)}{\rho}\partial_\theta \phi, \tag{5.60}$$

$$\partial_y \phi = \sin(\theta)\partial_\rho \phi + \frac{\cos(\theta)}{\rho}\partial_\theta \phi. \tag{5.61}$$

Then, replacing (5.57), (5.60) and (5.61) we have:

$$\left(\partial_x \phi\right)^2 + \left(\partial_y \phi\right)^2 = \left(\partial_\rho \phi\right)^2 + \frac{1}{\rho^2}\left(\partial_\theta \phi\right)^2, \tag{5.62}$$

$$dx^2 + dy^2 = d\rho^2 + \rho^2 d\theta^2, \tag{5.63}$$

$$\partial_x \phi dx + \partial_y \phi dy = \partial_\rho \phi d\rho + \partial_\theta \phi d\theta. \tag{5.64}$$

Replacing (5.62) - (5.64) in (5.55) then:

$$ds^2 = \left[-1 + \left(\partial_\rho \phi\right)^2 + \frac{1}{\rho^2}\left(\partial_\theta \phi\right)^2 + \left(\partial_z \phi + v\right)^2\right] dt^2 + 2\left[\partial_\rho \phi d\rho + \partial_\theta \phi d\theta + \left(\partial_z \phi + v\right) d\hat{z}\right] dt + \\ + d\rho^2 + \rho^2 d\theta^2 + d\hat{z}^2. \tag{5.65}$$

In matrix form:

$$g_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} \left\{-1 + \left(\partial_\rho \phi\right)^2 + \frac{1}{\rho^2}\left(\partial_\theta \phi\right)^2 + \left(\partial_z \phi + v\right)^2\right\} & \partial_\rho \phi & \partial_\theta \phi & \left(\partial_z \phi + v\right) \\ \partial_\rho \phi & 1 & 0 & 0 \\ \partial_\theta \phi & 0 & \rho^2 & 0 \\ \left(\partial_z \phi + v\right) & 0 & 0 & 1 \end{bmatrix}. \tag{5.66}$$

Indeed:

$$g^{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} -1 & \partial_\rho \phi & \frac{\partial_\theta \phi}{\rho^2} & \left(\partial_z \phi + v\right) \\ \partial_\rho \phi & \left[1 - \left(\partial_\rho \phi\right)^2\right] & -\frac{\partial_\theta \phi \partial_\rho \phi}{\rho^2} & -\partial_\rho \phi\left(\partial_z \phi + v\right) \\ \frac{\partial_\theta \phi}{\rho^2} & -\frac{\partial_\theta \phi \partial_\rho \phi}{\rho^2} & \frac{1}{\rho^2}\left[1 - \frac{\left(\partial_\theta \phi\right)^2}{\rho^2}\right] & -\frac{\partial_\theta \phi}{\rho^2}\left(\partial_z \phi + v\right) \\ \left(\partial_z \phi + v\right) & -\partial_\rho \phi\left(\partial_z \phi + v\right) & -\frac{\partial_\theta \phi}{\rho^2}\left(\partial_z \phi + v\right) & \left[1 - \left(\partial_z \phi + v\right)^2\right] \end{bmatrix}. \tag{5.67}$$

Now we are going to look at one-way membranes $\mathcal{N}$. Let us consider covector n_α as normal to the null hypersurface as follows (assuming cylindrical symmetry around the z axis):

$$n_\alpha = \left(0, \frac{\partial \mathcal{U}}{\partial \rho}, 0, \frac{\partial \mathcal{U}}{\partial z}\right), \tag{5.68}$$

where $\mathcal{U}$ represents the null hypersurface (which we call one-way membrane $\mathcal{N}$). Since we know that n_α is null then:

$$n^\alpha n_\alpha = 0. \tag{5.69}$$

Replacing (5.68) in (5.69) then:

$$n^1 n_1 + n^3 n_3 = 0. \tag{5.70}$$

Then:

$$\left[g^{1\alpha}n_\alpha\right]n_1+\left[g^{3\beta}n_\beta\right]n_3=0, \tag{5.71}$$

$$\left[g^{11}n_1+g^{13}n_3\right]n_1+\left[g^{31}n_1+g^{33}n_3\right]n_3=0. \tag{5.72}$$

Indeed:

$$g^{11}\left(n_1\right)^2+2g^{13}n_1n_3+g^{33}\left(n_3\right)^2=0. \tag{5.73}$$

Replacing (5.67) and (5.68) in (5.73) we have:

$$\left[1-(\partial_\rho\phi)^2\right]\left(\frac{\partial\mathcal{U}}{\partial\rho}\right)^2-2\partial_\rho\phi\left(\partial_z\phi+v\right)\frac{\partial\mathcal{U}}{\partial\rho}\frac{\partial\mathcal{U}}{\partial z}+\left[1-(\partial_z\phi+v)^2\right]\left(\frac{\partial\mathcal{U}}{\partial z}\right)^2=0. \tag{5.74}$$

Indeed:

$$\left(\frac{\partial\mathcal{U}}{\partial\rho}\right)^2+\left(\frac{\partial\mathcal{U}}{\partial z}\right)^2-\left[\partial_\rho\phi\frac{\partial\mathcal{U}}{\partial\rho}+\left(\partial_z\phi+v\right)\frac{\partial\mathcal{U}}{\partial z}\right]^2=0 \tag{5.75}$$

Any scalar potential ϕ must satisfy the differential equation (5.75). On the other hand, from (4.120) we have:

$$\mathcal{U}=\sin(\alpha)\left|\vec{X}_b\right|-1, \tag{5.76}$$

where in this case from (4.111):

$$\vec{X}_b=-(\partial_x\phi+v)e_x-\partial_y\phi e_y-\partial_z\phi e_z. \tag{5.77}$$

Then:

$$\left|\vec{X}_b\right|=\sqrt{\left(\partial_x\phi+v\right)^2+\left(\partial_y\phi\right)^2+\left(\partial_z\phi\right)^2}. \tag{5.78}$$

Therefore, equation (5.75) will represent a constraint for any scalar field ϕ of any warp drive with $\vec{\beta}=\nabla\phi$.

6 Conclusion

We have studied the characteristics of various types of warp drives (Alcubierre warp drive, Natario warp drive with $\theta = 0$). And we have seen several similarities and differences between these hypothetical spacetimes. In particular, in this thesis, we have investigated the horizon problem.

It is true that the Fell-Heisenberg warp drive does not solve the WEC and must also have all the anomalies of the Natario warp drive. However, when we have $\vec{\omega} = 0$ and therefore $\vec{\beta} = \nabla\phi$, the construction of different types of warp drive becomes more efficient. We only need to vary the scalar field ϕ. On the other hand, studying the characteristics of warp drives with $\vec{\omega} \neq 0$ is much more difficult. We will leave this problem for future studies.

In this thesis, we have shown that warp drives with a physical nature will be very difficult to obtain within the framework of general relativity. As many studies have already shown, these spacetimes need to be tested in alternative theories of gravitation. In that sense, we have $f(R)$ theories, and massive gravity, among others. In any of these theories, it would be interesting to investigate whether warp drives have a physical nature.

We believe that it is useful to investigate Fell-Heisenberg warp drives with $\vec{\beta} = \nabla\phi$ for two reasons: First, the construction of a spacetime only varying the scalar field ϕ (with its respective parameters) allows different types of warp drives to be investigated more efficiently using numerical methods. The second is the geometric interpretation given to Eulerian energy. Thus, with geometric intuition, we can more efficiently find different types of warp drives and try to reduce the amount of negative energy density ρ (as we saw in the warp drive with $\phi^{II}(x, y, z)$).

Also, we have avoided doing numerical calculations as much as possible for one reason only: Some topics about the physical foundation of warp drives presented in the literature are still very confusing. In particular, we have discussed the “observer problem” in great detail. The transformation (4.157) was very controversial for some people. Even in the literature presented, there is a kind of “confusion” between the change of observer and the change of coordinates. However, in effect, (4.157) does not imply a change of coordinates. The transformation (4.157) implies a change of observer. In this thesis, we have emphasized this problem. And in fact we have closed any misinterpretation.

We have analyzed the Natario warp drive with a single objective: to define the physical characteristics of a generic warp drive in general relativity. As we have shown in this thesis, we have placed constraints on any Fell-Heisenberg type warp drive with $\vec{\beta} = \nabla\phi$. In particular, we have used the horizon problem to place constraints for the scalar field ϕ. Also, we have seen that any generic Natario warp drive will always have two horizons: Horizon $\mathcal{H}$ and visibility horizon $\mathcal{H}_v$. We have shown explicitly that the horizon $\mathcal{H}$ is a one-way membrane $\mathcal{N}$ for the Alcubierre 2-D warp drive.

In summary, in this thesis we have made two main contributions: First, clarify the "observer problem" in warp drives. Second, establish constraints for scalar fields ϕ in Fell-Heisenberg warp drives with $\vec{\beta} = \nabla\phi$. And as we have mentioned before, we have emphasized the physical basis. We will leave the numerical calculations for a future study.

We believe that future studies of warp drives within the framework of general relativity will exist. The most logical thing would be to try to replace the definition of warp drive proposed by Natario (by varying the lapse function α or by varying the induced metric γ_{ij}). However, we must be careful and always ensure that our warp drive does not have closed timelike curves. That is, our warp drive must always be a globally hyperbolic spacetime. We want to travel in space, not travel in time!

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