Universidade Federal de Itajubá
Programa de Pós–Graduação em Matemática

Aditividade do posto de tensores pequenos de três fatores

Alejandro Camilo Vanegas Obregón

Orientador: Prof. Dr. Rick Antônio Rischter

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Abstract

This work explores the application of advanced concepts of algebra and algebraic geometry to understand some attempts to solve the so-called Strassen Conjecture, which consists of considering the union of two bilinear systems, each depending on different variables, and determining whether the multiplicative complexity of this union is equal to the sum of the multiplicative complexities of both systems. Our study relates these bilinear systems and their multiplicative complexities to tensor spaces and their ranks, respectively.

We will restrict our study to the case of three-factor tensor spaces, developing the theoretical knowledge needed to support the current conclusions and establish new research directions. It is common knowledge that the Conjecture is not true, however, we will study some special cases in which the Conjecture holds, using concepts and results relating to projective spaces, linear transformations and their properties.

Keywords: Tensors, projective spaces, linear transformations, multiplicative complexity, Strassen's Conjecture.

Resumo

O trabalho aprofunda na aplicação de conceitos avançados de álgebra e geometria algébrica para entender algumas tentativas de resolver a chamada Conjectura de Strassen, que consiste em considerar a união de dois sistemas bilineares, cada um deles dependendo de variáveis diferentes, e determinar se a complexidade multiplicativa dessa união é igual à soma das complexidades multiplicativas de ambos os sistemas. Nosso estudo relaciona esses sistemas bilineares e suas complexidades multiplicativas com os espaços de tensores e seus postos, respectivamente.

Restringiremos nosso estudo ao caso de espaços de tensores de três fatores, desenvolvendo os conhecimentos teóricos necessários para apoiar as conclusões atuais e estabelecer novos rumos de pesquisa. É de conhecimento geral que a Conjectura não é verdadeira, no entanto, estudaremos alguns casos especiais nos quais a Conjectura vale, usando conceitos e resultados relativos aos espaços projetivos, transformações lineares e suas propriedades.
Palavras chaves: Tensores, espaços projetivos, transformações lineares, complexidade multiplicativa, Conjectura de Strassen.

Contents

Page

List of Symbols

V^*	—	Dual vector space of V.
S_n	—	Symmetric group of n elements.
$\mathbf{a}$	—	Dimension of A.
I^c	—	Complement of I.
$M_{m,n,\ell}$	—	Multiplication of matrices tensor.
$\otimes$	—	Tensorial product.
$\langle A \rangle$	—	Span of A.
$\Lambda^k V$	—	Exterior product of k elements of V.
$S^k V$	—	Symmetric product of V.
$S^\bullet V$	—	Symmetric algebra of V.
$\mu(\mathbf{a}, \mathbf{b}, \mathbf{c})$	—	Tensor in $\mathbb{K}^{\mathbf{a}} \otimes \mathbb{K}^{\mathbf{b}} \otimes \mathbb{K}^{\mathbf{c}}$.
π	—	Projection function.
p	—	Tensor p.
$R(p)$	—	Rank of the tensor p.
$\underline{R}(p)$	—	Border rank of the tensor p.
$GL(V)$	—	Linear Group of V.
$I^\perp$	—	Span of I^c.

List of Figures

List of Tables

Introduction

Man's greatest asset is an inquisitive mind.
Isaac Asimov

One typical approach to solving mathematical problems is to seek the simplest solution. However, a more advanced framework is required to tackle seemingly straightforward issues. In this dissertation, we will get into the application of advanced algebra and algebraic geometry concepts in attempting to solve a computational problem. Building on this, we will explore the theoretical development needed to support the current conclusions and set a path for future research.

The *problem of multiplicative complexity of systems of bilinear forms*, or in more colloquial terms, *the fast multiplication of matrices* has been a source of work for mathematicians for more than 50 years, more specifically in Modern Computer Theory. In 1969, Strassen presented an algorithm showing that the multiplication of 2×2 matrices can be done in fewer steps than had been possible up to that point. Four years later, Strassen proposed the *Direct Sum Conjecture*, or *Strassen Conjecture*:

Considering the union of two systems of bilinear forms that depend on different variables, can we say that the multiplicative complexity of this union is equal to the sum of the multiplicative complexities of both systems?

Since then, efforts have been directed at reducing the number of multiplications needed to multiply two order n matrices (complexity) further.

Now, given a system of bilinear forms, we could take a look at the set of zeros of the system, which would lead us down the path of algebraic geometry (Chapter 2 provides enough context). For now, we will study the correspondence obtained by reinterpreting the problem in terms of the *algebra of tensors*. From this, a system of bilinear forms becomes an arrangement of elements in the form

$$p = \sum_{i \in I} a_i \otimes b_i \otimes c_i,$$

Figure 1: Volker Strassen (1936 -)

where I is a finite set of indexes, and $a = (a_i), b = (b_i), c = (c_i)$ are vectors over a field $\mathbb{K}$.

After describing the system, the next step is to give an equivalent of complexity in this new language. In our case, it's the *tensor's rank*, which we will denote by $R(p)$, where p is a tensor. We define the rank of a tensor after some required context in the next chapter. For now, we can think about the rank of a tensor as the rank of a matrix.

Definition 0.0.1. *We call the union of two bilinear systems depending on different sets of variables a direct sum. In the language of tensors, this means that if* $A = A' \oplus A''$, $B = B' \oplus B'', C = C' \oplus C''$ *are vector spaces and* $p' \in A' \otimes B' \otimes C', p'' \in A'' \otimes B'' \otimes C''$, *then* $p' \oplus p'' = p = p' + p''$.

With this definition, we can rewrite the direct sum conjecture as follows:

Conjecture 0.0.1. *Suppose that* $A = A' \oplus A'', B = B' \oplus B'', C = C' \oplus C''$, *where all* $A, B, C, \ldots, B'', C''$ *are finite-dimensional vector spaces over a field* $\mathbb{K}$. *Let* $p \in A' \otimes B' \otimes C'$, $p'' \in A'' \otimes B'' \otimes C''$ *and* $p = p' + p''$. *Is the equality*

$$R(p) = R(p') + R(p'')$$

satisfied?

$$\begin{array}{ccc} & & Arrangement\ of\ Elements \\ Multilinear\ system & \iff & p = \sum_{i \in I} a_i \otimes b_i \otimes c_i, \\ & & a_i, b_i, c_i \in \mathbb{K}^I \\ Multiplicative\ complexity & \iff & Rank \end{array}$$

A positive answer to this problem was previously known as Strassen's conjecture until recent counterexamples were proposed by Shitov (See [18]). The latter are not very explicit, and they are only known to exist asymptotically for very large tensor spaces.

We can then draw an analogy between direct sums of tensors and block diagonal matrices in a multidimensional sense. Knowing that the rank of a block-diagonal matrix is the sum of the ranks of its diagonal blocks, we can say that Strassen's conjecture asks an analog question for tensors.

After reading most of the available literature, we can say that using a tensor point of view made it possible to do calculations, but including a geometric point of view became fundamental in obtaining enough conditions for the conjecture to be satisfied. The main goal of this work is to show how some results and techniques can be mixed to prove the validity of Conjecture 0.0.1 in some cases.

We base our work mainly on the results and overviews from [6]. Going forward, all of our spaces will be finite-dimensional, and $\mathbb{K}$ will be an arbitrary field unless otherwise stated. We use V^* for the dual space associated with a vector space V, bold type for the dimensions of the spaces, and $e_i \otimes e_j \otimes e_k = e_{ijk}$.

Having dealt with the problem superficially, in Chapter 1 we will study the issues relating to tensors and their ranks with more detail. Chapter 2 will be dedicated to understanding the fundamentals of projective spaces and their connection with border rank.

Chapter 3 will be dedicated to exploring the relationship between the rank of a tensor and the rank of a subspace of a tensor product and establishing the first theorems concerning the additivity of the tensor rank. We also recall the concepts of minimal decomposition, projection, and conciseness to get inequalities related to this additivity. In the final part of this chapter, we study the Substitution method and go towards algebraic geometry to explore the concept of hook-shaped space and how in the event of a rank-one matrix we can guarantee additivity.

In Chapter 4 we will decompose our tensor space into smaller subspaces that allow us to get the main results of [6] on the additivity of the rank by using projections and their images. The main results of this chapter will be Theorem 4.3.1, Theorem 4.3.2, Theorem 4.3.3, and the following corollary:

Corollary 0.0.1. *(Theorem 4.16, [6]) If* $\mathbb{K} = \mathbb{C}$, $p' \in A' \otimes B' \otimes C', p'' \in A'' \otimes B'' \otimes C''$, *and* $R(p'') \leq 6$, *then independently of* p', *the additivity of the rank holds.*

The final chapter 5 will be about the additivity cases for the border rank of tensors belonging to product spaces of low dimensions. More specifically, we conclude the following:

Theorem 0.0.1. *Let $p \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ be concise tensors, with*

$$\mathbf{c}' + \mathbf{c}'' \leq \mathbf{b}' + \mathbf{b}'' \leq \mathbf{a}' + \mathbf{a}'' \leq 4.$$

Then the additivity of the border rank holds:

$$\underline{R}(p) = \underline{R}(p') + \underline{R}(p'').$$

Chapter 1

Additivity of tensor rank: preliminary concepts

In this first chapter, we start studying tensors, their algebraic properties, and their relationship to our interests. We will skip some of the basic tensor-related concepts of abstract algebra, in particular the construction of the tensor product and its universal property. The reader can find this explained in detail in Sections 10.4 and 11.5 of [7].

Also, we recall that V is a vector space that can be identified with $\mathbb{K}^n$ using one basis e_i, $i = 1, \ldots, n$ (n being the dimension of V) and there is an isomorphism of $GL(V)$ in the group of invertible matrices $n \times n$. Throughout this work, we will seamlessly use the fact that every linear transformation has an associated matrix, among other typical linear algebra concepts.

1.1 Tensor products

We present some general aspects of the tensor products and introduce some notation.

Notation 1.1.1. *Let us denote by $V^* \otimes W$ the vector space of the linear functions $V \to W$. Thus, $V \otimes W$ denotes the linear functions $V^* \to W$.*

Considering the previous notation, the space $V^* \otimes W$ can be seen in four different ways:

1. As the space of linear functions $V \to W$,
2. as the space of linear functions $W^* \to V$ (considering the isomorphism of the transpose),
3. as the dual space of $V \otimes W^*$,
4. as the space of bilinear functions $V \times W^* \to \mathbb{K}$.

If we fix a basis and represent $f \in V^* \otimes W$ as a matrix $X = (x_{ij})$, the first action is to multiply it by a column vector $v \longmapsto Xv$. The second is direct multiplication by a row vector $\beta \longmapsto \beta X$, the third is to take a matrix $n \times m$ $Y = (y^i_j)$, and do the following:

$$Y = (y^i_j) \longmapsto \sum_{i,j} x_{ij} y^i_j,$$

and the fourth is given by

$$(v, \beta) \longmapsto \sum_{i,j} x_{ij} v_i \beta_j.$$

Definition 1.1.1. *Let $V_1, \dots, V_k$ be vector spaces. The function*

$$f : V_1 \times V_2 \times \cdots \times V_k \to \mathbb{K} \tag{1.1}$$

is ***multi-linear*** *if it is linear in each factor V_ℓ, $\ell \in \{1, \cdots, k\}$. We call the* ***tensor product*** *of the spaces $V_1^*, V_2^*, \dots, V_k^*$ the space of multi-linear functions of the form (1.1), and denote it by $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$. The integer k is called the* ***order*** *of a tensor $p \in V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$. The sequence of numbers $(\mathbf{v}_1, \dots, \mathbf{v}_k)$, where $\mathbf{v}_j = dim(V_j)$, is called the sequence of* ***dimensions of*** *p. The definition does not change much if instead of $\mathbb{K}$ we consider a vector space W in (1.1). We will define $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^* \otimes W$ as the space of multilinear functions*

$$f : V_1 \times \cdots \times V_k \to W.$$

Notation 1.1.2. *We consider the space $V_1 \otimes \cdots \otimes V_n$. Let's denote $V_{\hat{j}} = V_1 \otimes \cdots \otimes V_{j-1} \otimes V_{j+1} \otimes \cdots \otimes V_n$. Given $p \in V_1 \otimes \cdots \otimes V_n$, we write $p(V_j^*) \subset V_{\hat{j}}$ for the image of the natural linear application $V_j^* \to V_{\hat{j}}$. Let's also denote $V^{\otimes k} := \underbrace{V \otimes \cdots \otimes V}_{k-times}$.*

For a better understanding on the tensor product, let us consider vector spaces A, B, C, each with bases a_i, b_j, c_k, $i = 1, \dots, m, j = 1, \dots, n, k = 1, \dots, \ell$, and let $p \in A \otimes B \otimes C$. So, in terms of bases, we have that

$$p = \sum_{i,j,k} p^{i,j,k} a_i \otimes b_j \otimes c_k.$$

It is then possible to form a rectangular solid arrangement of size $m \times n \times \ell$ whose entries will be the coefficients $p^{i,j,k}$. This solid can be decomposed into *slices*. Consider the collection of m matrices of size $n \times \ell$: $(p^{1,j,k}), \dots, (p^{m,j,k})$, which will be the horizontal slices. Similarly, we can consider the n matrices of size $m \times \ell$ $(p^{i,1,k}), \dots, (p^{i,n,k})$, which will be called side slices, or the collection of ℓ matrices of size $m \times n$, which we will call front slices. In the case where two indices are fixed, we call the resulting vector a **fiber**.

1.2 The rank of a tensor

In this section, we will define the rank of a tensor and give some examples of tensors we will use repeatedly. Given $\alpha_1 \in V_1^*, \alpha_2 \in V_2^*, \dots, \alpha_k \in V_k^*$, we define an element

$$\alpha_1 \otimes \cdots \otimes \alpha_k \in V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$$

by

$$\alpha_1 \otimes \cdots \otimes \alpha_k(v_1, \dots, v_k) = \alpha_1(v_1) \cdots \alpha_k(v_k) \tag{1.2}$$

Definition 1.2.1. *An element of* $V_1^* \otimes V_2^* \otimes \cdots \otimes V_k^*$ *is said to have* ***rank 1 (or be simple, decomposable)*** *if it can be written in the form (1.2). In general, we define the* ***rank*** *of a tensor* $p \in V_1 \otimes \cdots \otimes V_k$*, which we denote by* $\mathbf{R}(p)$*, as the minimum number* r *such that* $p = \sum_{u=1}^{r} Z_u$*, where each* Z_u *is a simple tensor.*

It is important to emphasize that the base field of the vector spaces to be treated is of paramount importance since the rank of a tensor could change if the entries of a decomposable tensor are taken from an extension of a field $\mathbb{K}$, as we will see in the following example:

Example 1.2.1. *[3] Let* A, B *be vector spaces over a non-algebraically closed field* $\mathbb{K}$ *, and bases* $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\}$, $n \geq 2$*, let* c *be a non-zero scalar of* $\mathbb{K}$*, and let* M *be the subspace of* $A \otimes B$ *generated by the elements*

$$u = \sum_{i=1}^{n} x_i \otimes y_i \qquad \text{and} \qquad v = \sum_{i=1}^{n} x_i \otimes y_{i+1} + c x_n \otimes y_1.$$

A general element $\alpha u + \beta v$ *of* M*, has the matrix representation*

$$\begin{pmatrix} \alpha & \beta & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \beta & \vdots \\ 0 & 0 & \dots & \alpha & \beta \\ c\beta & 0 & \dots & 0 & \alpha \end{pmatrix}$$

Lets suppose that $\alpha, \beta \neq 0$*. The determinant of this matrix is* $\alpha^n - (-\beta)^n c$*. Therefore, if we consider* c *as an element of* $\mathbb{K}$ *which is not an* n*-th power, all the nonzero elements of* M *will have rank* n*, and therefore* $R(M) = n$*, but it could decrease to* $n-1$ *if we consider an extension from* $\mathbb{K}$ *to* $\mathbb{K}(c^{1/n})$*. It will be smaller than* n *because the determinant is zero, but it will also be greater or equal to* $n-1$ *by looking at the smallest* $(n-1) \times (n-1)$ *minor in the top left corner.*

It will be handy to study the following three tensors, which are special because they are **invariant** under the action of $GL(V)$, which means they commute with the action of the group of changes of bases: tensor contraction, matrix multiplication, and the transposition of a tensor.

1. **Contraction of tensors:** Let's consider the bilinear function

$$\begin{aligned} \bowtie\colon (V_1\otimes\cdots\otimes V_k)\times(V_k^*\otimes B_1\otimes\cdots\otimes B_m) &\to V_1\otimes\cdots\otimes V_{k-1}\otimes B_1\otimes\cdots\otimes B_m \\ (v_1\otimes\cdots\otimes v_k,\alpha\otimes b_1\otimes\cdots\otimes b_m) &\longmapsto \alpha(v_k)v_1\otimes\cdots\otimes v_{k-1}\otimes b_1\otimes\cdots\otimes b_m \end{aligned}$$

The function $\bowtie\in((V_1\otimes\cdots\otimes V_k)\times(V_k^*\otimes U_1\otimes\cdots\otimes U_m))^*\otimes(V_1\otimes\cdots\otimes V_{k-1}\otimes U_1\otimes\cdots\otimes U_m)$ is called a contraction.

Observation 1.2.1. *If $p\in V_1\otimes\cdots\otimes V_k$ and $s\in B_1\otimes\cdots\otimes B_m$, and for some fixed i,j there is an association $V_i\simeq B_j^*$, we can contract $p\otimes s$, and we get an element of $V_{\widehat{i}}\otimes B_{\widehat{j}}$, which is called the* ***modal product*** *(i,j) of p and s.*

2. **The multiplication of matrices as a tensor:** Let A,B,C be vector spaces of dimensions m,n,ℓ and consider the matrix multiplication operator $M_{m,n,\ell}$. Remember that every matrix has an associated linear transformation, so the product of matrices can be seen as a composition of linear transformations. Thus, we have a composition of a linear transformation of A into B ($\in A^*\otimes B$), with one of B into C ($\in B^*\otimes C$), obtaining a transformation of A into C ($\in A^*\otimes C$). Let $V_1=A^*\otimes B$, $V_2=B^*\otimes C$, and $V_3=A^*\otimes C$, so $M_{m,n,\ell}\in V_1^*\otimes V_2^*\otimes V_3$. As we have seen, we can define $M_{m,n,\ell}$ on the simple elements and linearly extend the definition to obtain the definition of the tensor for all of the space. On a simple element, we have

$$\begin{aligned} M_{m,n,\ell}:(A^*\otimes B)\times(B^*\otimes C) &\longrightarrow A^*\otimes C, \\ (\alpha\otimes b)\times(\beta\otimes c) &\longmapsto \beta(b)\alpha\otimes c \end{aligned} \tag{1.3}$$

As a tensor,

$$M_{m,n,\ell}\in(A^*\otimes B)^*\otimes(B^*\otimes C)^*\otimes(A^*\otimes C)=A\otimes B^*\otimes B\otimes C^*\otimes A^*\otimes C$$

It is then possible to see $M_{m,n,\ell}$ as any of the three contractions:

$$\begin{gathered} A^*\otimes B\times B^*\otimes C\to A^*\otimes C, \\ A\otimes C^*\times B^*\otimes C\to A\otimes B^* \text{ or} \\ A\otimes C^*\times A^*\otimes B\to C^*\otimes B \end{gathered}$$

In the case where $A=B=C$, we have a symmetry by the action of the group S_3 of permutations of three elements.

3. **Transposition of tensors:** Now we consider the function

$$\sigma : V \otimes V \longrightarrow V \otimes V \tag{1.4}$$

$$a \otimes b \longmapsto b \otimes a$$

Note that $\sigma \in V^* \otimes V^* \otimes V \otimes V = End(V \otimes V)$, and it is an invariant tensor relative to the action of $GL(V)$.

Observation 1.2.2. *Now that we have mentioned $GL(V)$, we can also recall the following action: consider the group $GL(V)$, and its action on a simple tensor, that is, let $g \in GL(V)$ and $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ of rank 1, be defined by*

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_n).$$

The defined action can be extended linearly to obtain an action in $V^{\otimes n}$. Similarly, one can define the action of $GL(V_1) \times GL(V_2) \times \cdots \times GL(V_k)$ on $V_1, \ldots, V_k$.

1.3 The Strassen Algorithm

The algorithm that Strassen presented in 1969 consists of constructing, given two matrices

$$A = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix}$$

the matrix $C = AB$,

$$C = \begin{bmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{bmatrix}$$

using 7 multiplications and 4 sums:

MULTIPLICATIVE STEPS

$$m_1 = (a_1^1 + a_2^2) \cdot (b_1^1 + b_2^2)$$
$$m_2 = (a_1^2 + a_2^2) \cdot b_1^1$$
$$m_3 = a_1^1 \cdot (b_2^1 - b_2^2)$$
$$m_4 = a_2^2 \cdot (-b_1^1 + b_1^2)$$
$$m_5 = (a_1^1 + a_2^1) \cdot b_2^2$$
$$m_6 = (-a_1^1 + a_1^2) \cdot (b_1^1 + b_2^1)$$
$$m_7 = (a_2^1 - a_2^2) \cdot (b_1^2 + b_2^2)$$

SUMS

$$c_1^1 = m_1 + m_4 - m_5 + m_7$$
$$c_2^1 = m_2 + m_4$$

$$c_1^2 = m_3 + m_5$$
$$c_2^2 = m_1 + m_3 - m_2 + m_6$$

Now let's write this process in the language of tensors: Let V_1, V_2, V_3 be $V_i = M_{2\times 2}$ for all $i = 1, 2, 3$. We use for V_1, V_2, V_3 the canonical bases $(a_j^i), (b_j^i), (c_j^i)$ of matrices whose entries are 1 in the entry (i, j) and 0 elsewhere. So the standard algorithm can look like this

$$\begin{aligned} M_{2,2,2} = a_1^1 \otimes b_1^1 \otimes c_1^1 + a_2^1 \otimes b_1^2 \otimes c_1^1 + a_1^2 \otimes b_1^1 \otimes c_1^2 + a_2^2 \otimes b_1^2 \otimes c_1^2 \\ +a_1^1 \otimes b_2^1 \otimes c_2^1 + a_2^1 \otimes b_2^2 \otimes c_2^1 + a_1^2 \otimes b_2^1 \otimes c_2^2 + a_2^2 \otimes b_2^2 \otimes c_2^2 \end{aligned}$$

The Strassen Algorithm can be written as:

$$\begin{gathered} M_{2,2,2} = (a_1^1 + a_2^2) \otimes (b_1^1 + b_2^2) \otimes (c_1^1 + c_2^2) + (a_1^2 + a_2^2) \otimes b_1^1 \otimes (c_1^2 - c_2^2) \\ +a_1^1 \otimes (b_2^1 - b_2^2) \otimes (c_2^1 + c_2^2) + a_2^2 \otimes (-b_1^1 + b_1^2) \otimes (c_1^2 + c_1^1) + (a_1^1 + a_2^1) \otimes b_2^2 \otimes (-c_1^1 + c_2^1) \\ +(-a_1^1 + a_1^2) \otimes (b_1^1 + b_2^1) \otimes c_2^2 + (a_2^1 - a_2^2) \otimes (b_1^2 + b_2^2) \otimes c_1^1. \end{gathered}$$

The number of simple tensors in the tensor expression will be the number of multiplications required to develop the algorithm. So the rank gives an lower bound on the number of multiplications required to apply the corresponding bilinear function using the best possible algorithm.

1.4 Alternating and symmetric tensors

We will use the σ application as in (1.4) to define the spaces of symmetric and alternating tensors. These spaces will allow us to make an algebraic decomposition (in terms of direct sums) of a tensor product. Consider the function $\sigma : V^* \otimes V^* \to V^* \otimes V^*$. We also consider $V^{\otimes 2} = V \otimes V$ with basis $\{v_i \otimes v_j, 1 \leq i, j \leq n\}$. The subspaces

$$\begin{aligned} S^2V &:= \langle \{v_i \otimes v_j + v_j \otimes v_i : 1 \leq i, j \leq n\} \rangle = \langle \{v \otimes v : v \in V\} \rangle \\ &= \{X \in V \otimes V : X \circ \sigma = X\}, \\ \Lambda^2 V &:= \langle \{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i, j \leq n\} \rangle \\ &= \langle \{v \otimes w - w \otimes v : v, w \in V\} \rangle \\ &= \{X \in V \otimes V : X \circ \sigma = -X\} \end{aligned}$$

are, respectively, the **symmetric and alternating tensors** of $V^{\otimes 2}$. Note that if $p \in S^2V$ and $g \in GL(V)$, then $g \cdot p \in S^2V$ and the same occurs if $p \in \Lambda^2 V$. So S^2V and $\Lambda^2 V$ are invariant under linear-coordinate changes. From two vectors in V, we can define the vectors

$$v_1 v_2 := \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) \in S^2V \text{ and}$$
$$v_1 \wedge v_2 := \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \in \Lambda^2 V.$$

Let's consider the function

$$\begin{aligned} \pi_S : V^{\otimes n} &\longrightarrow V^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n &\longmapsto \frac{1}{n!} \sum_{\gamma \in S_n} v_{\gamma(1)} \otimes v_{\gamma(2)} \otimes \cdots \otimes v_{\gamma(n)}, \end{aligned}$$

where S_n denotes the group of permutations of n elements. We define

$$S^n V = \pi_S(V^{\otimes n})$$

the n-th symmetric power of V, and

$$S^\bullet V = \bigoplus_n S^n V$$

the symmetric algebra of V, where $\alpha\beta := \pi_S(\alpha \otimes \beta)$ for $\alpha \in S^s V$ and $\beta \in S^t V$.

By changing the space from V to V^*, we get what is known as the space of symmetric n-linear forms on V, but we can also see this as the space of homogeneous polynomials of degree n on V. Given a multilinear form $\overline{Q}$, the function $x \longmapsto \overline{Q}(x, \ldots, x)$ is a polynomial function of degree k. The process of passing from a homogeneous polynomial to a symmetric multilinear form is called **polarization**. For example, if Q is a homogeneous polynomial of degree 2 over V, then the bilinear form $\overline{Q}$ is defined by the equation

$$\overline{Q}(x, y) = \frac{1}{2}[Q(x + y) - Q(x) - Q(y)]$$

In general, the polarization identity is given by the equation

$$\overline{Q}(x_1, \ldots, x_k) = \frac{1}{n!} \sum_{I \subset J_n, I \neq \varnothing} (-1)^{n-|I|} Q\left(\sum_{i \in I} x_i\right) \tag{1.5}$$

where $J_n = \{1, \ldots, n\}$. Since when we talk about Q and $\overline{Q}$, we are only dealing with two interpretations for the same equation, we are not making any distinction between them.

We can see the space of antisymmetric (or alternating) k-tensors as the image $\Lambda^k V$ of the following map

$$\begin{aligned} \pi_\Lambda : V^{\otimes n} &\longrightarrow V^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n &\longmapsto v_1 \wedge \cdots \wedge v_n := \frac{1}{n!} \sum_{\sigma \in S_n} (sgn(\sigma)) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \end{aligned} \tag{1.6}$$

where $sgn(\sigma) = \pm 1$ is the sign of the permutation σ. Note that if $n = 2$, this definition agrees with the previous definition of $\Lambda^2 V$. In general,

$$\Lambda^n V = \{X \in V^{\otimes n} : X \circ \sigma = sgn(\sigma) X, \ \textit{for all } \sigma \in S_n\}$$

Observation 1.4.1. *The properties of S^nV and Λ^nV can be studied in Section 11.5 of [7]. There, it is proved that the space of alternating tensors is an algebra, and they called this algebra of alternating tensors the* ***exterior algebra****. For our purposes, it is enough to remark that*

1. $\Lambda^1V = S^1V = V$.
2. *The product $S^sV \times S^tV \longrightarrow S^{s+t}V$, where S^nV is the space of homogeneous polynomials over V^*, corresponds to the product of polynomials.*

It is straightforward to verify that the contraction

$$\begin{aligned} V^* \times V^{\otimes n} &\longrightarrow V^{\otimes n-1} \\ (\alpha, v_1 \otimes \cdots \otimes v_n) &\longmapsto \alpha(v_1)(v_2 \otimes \cdots \otimes v_n) \end{aligned}$$

preserves the subspaces of symmetric and alternating tensors.

Let $f : V \longrightarrow W$ be a linear transformation. It induces the function

$$\begin{aligned} f^{\otimes n} : V^{\otimes n} &\longrightarrow W^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n &\longmapsto f^{\otimes n}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n). \end{aligned}$$

$f^{\otimes n}$ can be restricted to obtain the induced functions

$$f^{\wedge n} : \Lambda^nV \longrightarrow \Lambda^nW \qquad \text{and} \qquad f^{\bullet n} : S^nV \longrightarrow S^nW.$$

Now let's look at symmetric tensors: Let S_n be the symmetry group of n elements. We can define an action of S_n on $V^{\otimes n}$. For $v_1, \dots, v_n \in V$,

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

We can linearly extend and redefine the symmetric tensors:

$$S^nV = \{p \in V^{\otimes n} : \sigma \cdot p = p \;\; for\ all\ \sigma \in S_n\}$$

In summary, S^nV is the subspace of $V^{\otimes n}$ whose elements are invariant by the action of S_n.

Now we will consider homogeneous polynomials in the space of matrices $n \times m$ and how they are affected when changing the bases of $\mathbb{K}^n$ and $\mathbb{K}^m$. Let $V = A \times B$, where A, B are vector spaces. The idea is to obtain an invariant description (under the change of bases), which will provide the explicit expression of these polynomials later on. The following will be to study the quadratic case.

Example 1.4.1. *In the case $A = B$, we know that every $M \in A^{\otimes 2}$ can be uniquely written as the sum of two matrices:*

$$M = M_1 + M_2,$$

where M_1 is symmetric and M_2 is antisymmetric. In fact, it is enough to take

$$M_1 = \tfrac{1}{2}(M + M^T) \text{ and } M_2 = \tfrac{1}{2}(M - M^T).$$

Example 1.4.2. *We have $V = A \otimes B$, the goal will be to find a decomposition for both $S^2(A \otimes B)$ and $\Lambda^2(A \otimes B)$. Now we move on to $S^2(A \otimes B)$. If we take $\alpha \in S^2A$ and $\beta \in S^2B$, $\alpha \otimes \beta$ is defined by*

$$\alpha \otimes \beta(x_1 \otimes y_1, x_2 \otimes y_2) = \alpha(x_1, x_2)\beta(y_1, y_2),$$

where $x_j \in A^*, y_j \in B^*$. The fact that α and β are symmetric guarantees that

$$\alpha \otimes \beta(x_1 \otimes y_1, x_2 \otimes y_2) = \alpha \otimes \beta(x_2 \otimes y_2, x_1 \otimes y_1),$$

so $\alpha \otimes \beta \in S^2(A \otimes B)$.

On the other hand, if $p \in S^2A \otimes S^2B$, then for all $g \in G = GL(A) \times GL(B)$, $g \cdot p \in S^2A \otimes S^2B$. We then have that $S^2A \otimes S^2B$ is an invariant subspace of $S^2(A \otimes B)$ over G. Now let's study the dimensions:

$$dim S^2V = \binom{ab+1}{2} = \frac{(ab+1)ab}{2}$$
$$dim(S^2A \otimes S^2B) = \binom{a+1}{2}\binom{b+1}{2} = \frac{(ab+a+b+1)ab}{4}$$

We still don't have all the possible elements of S^2V. We then consider $\alpha \in \Lambda^2A, \beta \in \Lambda^2B$ and define

$$\alpha \otimes \beta(x_1 \otimes y_1, x_2 \otimes y_2) = \alpha(x_1, x_2)\beta(y_2, y_1)$$
$$= (-\alpha(x_1, x_2))(-\beta(y_1, y_2)) = \alpha \otimes \beta(x_1 \otimes y_1, x_2 \otimes y_2).$$

So $\alpha \otimes \beta \in S^2(A \otimes B)$ and extending the function linearly, we get an inclusion

$$\Lambda^2A \otimes \Lambda^2B \subset S^2(A \otimes B).$$

To conclude the decomposition, we note that $dim(\Lambda^2A \otimes \Lambda^2B) = \dfrac{(ab-a-b+1)ab}{4}$, hence

$$dim(\Lambda^2A \otimes \Lambda^2B) + dim(S^2A \otimes S^2B) = S^2(A \otimes B)$$

and therefore

$$S^2(A \otimes B) = (\Lambda^2A \otimes \Lambda^2B) \oplus (S^2A \otimes S^2B)$$

is a decomposition of $S^2(A \otimes B)$. Using a similar idea, one can show that

$$\Lambda^2(A \otimes B) = (\Lambda^2A \otimes S^2B) \oplus (S^2A \otimes \Lambda^2B).$$

1.5 Generalities on tensor rank

Now that we have seen some relations of polynomials with matrices and tensors, let's begin studying a characterization of tensor rank in terms of matrices. We will consider, as in Section 1.1, tensors as functions $A^* \longrightarrow B \otimes C$, and see some elementary results, such as:

Theorem 1.5.1. *(Thm. 3.1.1.1, [12]) Let $p \in A \otimes B \otimes C$. Then the tensor rank $R(p)$ is the number of rank-one matrices needed to span a space containing $p(A^*) \subset B \otimes C$.*

Proof. If p is a rank r tensor, then there exists an expression of the form

$$p = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i,$$

where the vectors a_i need not to be linearly independent, and the same happens with the b_i, c_i. We have

$$p(A^*) \subset \langle b_1 \otimes c_1, b_2 \otimes c_2, \dots, b_r \otimes c_r \rangle$$

and then the number of rank-one matrices needed to span $p(A^*) \subset B \otimes C$ is at most $R(p)$. On the other hand, if $p(A^*)$ is spanned by rank-one elements, let's say $b_1 \otimes c_1, \dots, b_r \otimes c_r$, we consider the basis of A^*, $a^1, \dots, a^{\mathbf{a}}$, and the dual basis $a_1, \dots, a_{\mathbf{a}}$ (a basis for A). We obtain some constants x_{ij} such that

$$p(a^i) = \sum_{j=1}^{r} x_{ij} b_j \otimes c_j.$$

Rewriting p, we have

$$p = \sum_{j,i} a_i \otimes (x_{ij} b_j \otimes c_j) = \sum_{j=1}^{r} \left(\sum_i x_{ij} a_i \right) \otimes b_j \otimes c_j$$

and $R(p)$ is at most the number of rank-one matrices needed to span $p(A^*)$. □

Proposition 1.5.1. *(Corollary 3.1.2.1, [12]) Let $\mathbf{a} \geq \mathbf{b} \geq \mathbf{c}$. Then if $p \in A \otimes B \otimes C$, $R(p) \leq \mathbf{bc}$. In other words, if $p \in \mathbb{K}^{a_1} \otimes \mathbb{K}^{a_2} \otimes \mathbb{K}^{a_3}$, then $R(p) \leq \min\{a_1 a_2, a_1 a_3, a_2 a_3\}$.*

Theorem 1.5.2. *(Thm. 3.1.3.1, [12]) Let $n > 2$ and let $p \in A_1 \otimes \cdots \otimes A_n$ have rank r. Assume $p \in A'_1 \otimes \cdots \otimes A'_n$, where $A'_j \subset A_j$, and that there exists at least one j such that the inclusion $A'_j \subset A_j$ is proper. Then any expression for p of the form*

$$p = \sum_{i=1}^{\mu} u_i^1 \otimes \cdots \otimes u_i^n$$

with some $u_j^s \notin A'_s$ has $\mu > r$.

Proof. We will complete the spaces A_t considering a complement A_t'' such that $A_t = A_t' \oplus A_t''$. We put $u_j^t = u_j^{t\prime} + u_j^{t\prime\prime}$, where $u_j^{t\prime} \in A_t'$, and $u_j^{t\prime\prime} \in A_t''$. We can write $p = \sum_{i=1}^{\mu} u_i^{1\prime} \otimes \cdots \otimes u_i^{n\prime}$ so the terms with double prime involved must cancel (remember that p has no entries in any of the A_j''). As $R(p) = r$, $\mu \geq r$. Let's assume $\mu = r$ and that without lost of generality exists $j_0 \in \{1, \dots, n\}$, such that $u_{j_0}^{1\prime\prime} \neq 0$ (this can be assumed since $A_1 = A_1' \oplus A_1''$). Then the term

$$\sum_{j=1}^{r} u_j^{1\prime\prime} \otimes (u_j^{2\prime} \otimes \cdots \otimes u_j^{n\prime}) = 0,$$

but all the terms $(u_j^{2\prime} \otimes \cdots \otimes u_j^{n\prime})$ must be linearly independent in $A_2' \otimes \cdots \otimes A_n'$, because if not, this would contradict the minimality of r. This implies that the $u_j^{1\prime\prime}$ must all be zero, a contradiction because at least one of the inclusions $A_j' \subset A_j$ is proper. □

Definition 1.5.1. *A tensor p has* ***border rank (or edge rank)*** *r if it is a limit of tensors of rank r but is not a limit of tensors of rank s for any $s < r$. Let's denote $\underline{R}(p)$ the border rank of p.*

Observation 1.5.1. *The rank of a tensor and its border rank are not equal in general. In fact, $R(p) \geq \underline{R}(p)$ for every tensor p.*

Example 1.5.1. *(Sec. 2.4.5, [12]) We choose the canonical basis $\{e_i\}_{i=1,2}$ for A, B and C, where $A = B = C = \mathbb{C}^2$. Let $p = e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$ be a tensor and define*

$$\begin{aligned} y(t) &= \frac{1}{t}\left[(t-1)e_1 \otimes e_1 \otimes e_1 + (e_1 + te_2) \otimes (e_1 + te_2) \otimes (e_1 + te_2)\right] \\ &= \frac{1}{t}\left[t(e_{111} + e_{112} + e_{121} + e_{211}) + t^2(e_{122} + e_{212} + e_{221}) + t^3 e_{222}\right] \\ &= e_{111} + e_{112} + e_{121} + e_{211} + t(e_{122} + e_{212} + e_{221}) + t^2 e_{222} \\ &= p + t(e_{122} + e_{212} + e_{221}) + t^2 e_{222}. \end{aligned}$$

By taking the limit $t \to 0$, we have

$$\lim_{t \to 0} y(t) = p.$$

As $R(y(t)) = 2$ for all $t \neq 0$, we conclude by definition that $\underline{R}(p) \leq 2$. Note that $R(p) = 3$ by Theorem 1.5.1, as the rank-one matrices needed to span a space containing $p(A^*)$ are

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The last thing we say about p is that its border rank can not be 1 because a sequence p_j of rank-one tensors such that $\lim_{j \to \infty} p_j = p$ would imply that $R(p) = 1$, a contradiction.

Definition 1.5.2. *Let A_j, $j \in \{1, \ldots, n\}$ and V vector spaces over $\mathbb{C}$, and consider a Euclidean structure that induces Euclidean structures and measures on the spaces $A_1 \otimes \cdots \otimes A_n$ and $S^n V$. Then any r such that the set of tensors having rank r has a positive measure is known as a* ***typical rank****. We could also consider typical rank as the number r such that the set of tensors of rank r has a nonempty interior in the topology induced by the linear structure.*

Example 1.5.2. *For tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, we know that their typical rank is 2 (check Equation 5.1 in Section 5 of [9]). We would like to point out that the context and the notation in [9] will only be clear after reading the next chapter.*

This typical rank we just defined is not the same as the rank for a tensor. Note that in Example 1.5.1 we saw that p has rank 3 but p lives in a space with typical rank 2, again by [9]. The last two definitions in this chapter will be recalled mostly in the last part of this work, which is dedicated to studying cases of affirmative answers for the following question:

Conjecture 1.5.1. *Suppose that $A = A' \oplus A'', B = B' \oplus B'', C = C' \oplus C''$, where all $A, B, C, \ldots, B'', C''$ are finite-dimensional vector spaces over $\mathbb{C}$. Let $p \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ be $p = p' + p''$. Is the equality*

$$\underline{R}(p) = \underline{R}(p') + \underline{R}(p'')$$

satisfied?

If we want to be more accurate regarding the dimensions of sets that are more complicated than vector spaces, we would need to study some of the fundamental results on projective spaces. That would be the purpose of the next chapter.

Chapter 2

Projective space

This chapter's main goal is to expose the concepts needed to understand the space of tensors of rank one as a projective variety. For this, we will discuss some generalities on affine and projective varieties to show how the minors of a given matrix will provide equations that define the Segre variety. We need to start from the fundamentals: the concept of affine plane.

2.1 The affine plane

Definition 2.1.1. *An **affine plane** is a set Π endowed with a collection $\mathcal{L}$ of subsets of Π (called **lines**), whose elements (called **points**) satisfy the following axioms:*

1. *Given two points $P, Q \in \Pi$, there is a unique line containing P and Q.*
2. *Given a line ℓ and a point $P \notin \ell$, there is a unique line m such that $P \in m$ and $\ell \cap m = \varnothing$. .*
3. *There are three non-colinear points, which means they do not belong to the same line.*

Notation 2.1.1. *We will denote the affine plane Π, along with the collection of lines $\mathcal{L}$ for $(\Pi, \mathcal{L})$. Sometimes we will refer to $(\Pi, \mathcal{L})$ as Π, without specifying the collection of lines.*

Definition 2.1.2. *Let Π an affine plane. Two lines $\ell, m \in \Pi$ are called **parallels** if they are the same line or if they don't have any common points.*

Example 2.1.1. *The set $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$, along with the collection of lines $\ell = \{(x,y) \in \mathbb{R}^2 : ax + by + c = 0,\ a, b, c \in \mathbb{R}, (a,b,c) \neq (0,0,0)\}$ is an affine plane, called the real affine plane. It is easy to verify the axioms:*

1. *Given $P = (p_1, p_2)$, $Q = (q_1, q_2)$, where $P \neq Q$, the only line passing through them is the line with equation*

$$(q_2 - p_2)x + (p_1 - q_1)y + (p_2 - q_2)p_1 + (q_1 - p_1)p_2 = 0.$$

2. *Given* $P = (p_1, p_2)$ *and* ℓ *with equation* $ax + by + c = 0$*, where* a *or* b *are nonzero, such that* $P \notin \ell$*, the only parallel to* ℓ *passing by* P *is the line with equation* $ax + by - ap_1 - bp_2 = 0$*.*

3. $(0,0), (1,0)$ *and* $(0,1)$ *are non-colinears points.*

Example 2.1.2. *In general, if* $\mathbb{K}$ *is a field, the set*

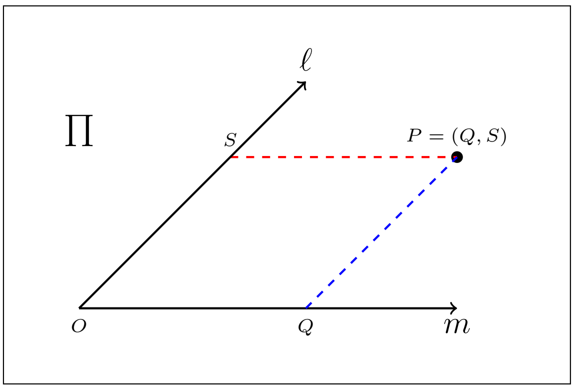
$$\mathbb{K}^2 = \{(x, y) : x, y \in \mathbb{K}\}$$

along with the lines of equation $ax + by + c = 0$*,* $a, b, c \in \mathbb{K}$*, and* $(a, b, c) \neq (0, 0, 0)$ *is an affine plane, called the affine* $\mathbb{K}$*-plane.*

Now, let Π be the Euclidean plane. If we assume the Euclidean line is in bijective correspondence with the real numbers, then choosing two lines ℓ and m which intersect in a single point O, we can establish a coordinate system, that is, a bijection between Π and $\mathbb{R}^2$.

The point O determines two semi-lines in ℓ and m, and choosing a point (distinct from O) in each one of them, we set the bijections

$$x : m \longrightarrow \mathbb{R} \text{ and } y : \ell \longrightarrow \mathbb{R}.$$



From this, we can associate each point $P \notin \ell \cup m$ with the vertices of the parallelogram determined by P and O, with sides parallel to ℓ and m (considering the parallels to ℓ and m passing through P). We call Q and S the final points of those sides. Then we have

$$\varphi_{\ell,m} : \Pi \longrightarrow \mathbb{R}^2$$
$$P \longmapsto \varphi_{\ell,m}(P) = (x(Q), y(S))$$

is a bijection between Π and $\mathbb{R}^2$, which depends on the choices of ℓ and m. Given two lines ℓ' and m' such that $\ell' \cap m' = \{O'\}$, we can establish the function of change of coordinates: Let's assume $O' \in \ell'$ and ℓ has direction (a, b), and m' has direction (c, d). Let $(x_0, y_0) = \varphi_{\ell',m'}(P)$ the coordinates of O' on the system given by ℓ' and m'. Then

$$\phi_{\ell,m}(P) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \varphi_{\ell',m'}(P) - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The previous discussion can be summarized in the following result:

Theorem 2.1.1. *(Prop. 1.15, [15]) Let $(\Pi, \mathcal{L})$ an affine plane, and let $\varphi : \Pi \longrightarrow \zeta$ a bijection. Then if $\varphi(\mathcal{L}) = \{\varphi(\ell) : \ell \in \mathcal{L}\}$ denotes the collection of images of lines of Π, then $(\zeta, \varphi(\mathcal{L}))$ is an affine plane.*

Definition 2.1.3. *Let $(\Pi, \mathcal{L})$ and $(\Pi', \mathcal{L}')$ be two affine planes. An **isomorphism** between Π and Π' is a bijection $\varphi : \Pi \longrightarrow \Pi'$ such that $\varphi(\mathcal{L}) = \mathcal{L}'$. When we have the case $\Pi = \Pi'$ in the previous definition, we call φ an **automorphism**.*

We denote the set of all automorphisms of Π as $Aut(\Pi)$ and also point out that an automorphism is an isomorphism that takes co-linear points in co-linear points, as we comment in the next observation.

Observation 2.1.1. *Consider $\varphi : \Pi \longrightarrow \Pi$ an automorphism. Then*

1. *For every pair of lines $\ell, m \subset \Pi$, we have*

$$\varphi(\ell \cap m) = \varphi(\ell) \cap \varphi(m).$$

2. *For every $P, Q \in \Pi$, $P \neq Q$, $\varphi(PQ) = \varphi(P)\varphi(Q)$.*

We could elaborate more on the properties of the automorphisms, but we would restrict to mention the following:

Proposition 2.1.1. *Let $Aut(\Pi)$ be the set of automorphisms of the affine plane Π. Then $Aut(\Pi)$ is a group, with identity element $Id : \Pi \longrightarrow \Pi$.*

Proof. The identity is an automorphism. If we take $\varphi, \varepsilon \in Aut(\Pi)$, we also have $\varphi \circ \varepsilon$ is also an automorphism. As $\varphi \in Aut(\Pi)$ is such that $\varphi(\mathcal{L}) = \mathcal{L}$, we have $\mathcal{L} = \varphi^{-1}(\varphi(\mathcal{L})) = \varphi^{-1}(\mathcal{L})$, so φ^{-1} is also an automorphism. This concludes the proof. □

We are going to talk about a special subgroup of $Aut(\Pi)$, the dilations of Π.

Definition 2.1.4. *A **dilation** of the affine plane Π is an automorphism $\gamma : \Pi \longrightarrow \Pi$ such that $PQ \| \gamma(P)\gamma(Q)$, being $P \neq Q$. We denote the set of all dilations of Π by $Dil(\Pi)$.*

Example 2.1.3. *Fix $(p_1, p_2) \in \mathbb{R}^2$. Every linear transformation of the form*

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + p_1, y + p_2) \end{aligned}$$

is a dilation of $\mathbb{R}^2$.

Example 2.1.4. *Fix $\lambda \in \mathbb{R}$. Every linear transformation of the form*

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (\lambda x, \lambda y) \end{aligned}$$

is a dilation of $\mathbb{R}^2$.

Proposition 2.1.2. *Let Π be an affine plane. Then $Dil(\Pi) \subset Aut(\Pi)$ is a subgroup.*

Proof. It is clear that $Id \in Dil(\Pi)$. If $\gamma, \theta \in Dil(\Pi)$, then $PQ \| \gamma(P)\gamma(Q)$ and $PQ \| \theta(P)\theta(Q)$. That way,

$$PQ \| \gamma(P)\gamma(Q) \| \theta(\gamma(P))\theta(\gamma(Q)) = (\theta \circ \gamma(P))(\theta \circ \gamma(Q).)$$

This tells us $\theta \circ \gamma \in Dil(\Pi)$. The fact that

$$PQ = \gamma(\gamma^{-1}(P))\gamma(\gamma^{-1}(Q)) \| \gamma^{-1}(P)\gamma^{-1}(Q)$$

gives us $\gamma^{-1} \in Dil(\Pi)$, and we have proved our proposition. □

2.2 Affine varieties

For this part, we state some definitions and results needed to study border rank. We also study some examples related to the definitions we present. Our main references will be Chapter 1 of [10] and we support some of the arguments in Chapter 7 of [2]. We will leave some of the results without proof for brevity reasons.

We define the **affine n-space over $\mathbb{K}$**, denoted $\mathbb{A}^n_k$ or simply $\mathbb{A}^n$, to be the set of all n-tuples of elements of $\mathbb{K}$. An element $P \in \mathbb{A}^n$ will be called a **point** and if $P = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{K}$, then a_i will be called the coordinates of P.

Now consider the ring $A = \mathbb{K}[x_1, \ldots, x_n]$ of polynomials in n variables over the field $\mathbb{K}$. Since $\mathbb{K}$ is infinite, the elements of A can be viewed as functions from the affine n-space to $\mathbb{K}$ by defining $f(P) = f(a_1, \ldots, a_n)$, where $f \in A$ and $P \in \mathbb{A}^n$ (see Theorem 29.18 of [8]). If f is a polynomial we can talk about the set of zeros of f,

$$Z(f) = \{P \in \mathbb{A}^n : f(P) = 0\}.$$

More generally, for any $T \subset A$, we define

$$Z(T) = \{P \in \mathbb{A}^n : f(P) = 0 \ \ for \ \ all \ \ f \in T\}.$$

Note that if I is the ideal of A generated by T, then $Z(T) = Z(I)$. Now, remember that A is a Noetherian ring (Theorem 7.5 of [2]), and by definition any ideal is finitely generated, so there exist $f_1, \ldots, f_r \in A$ such that $Z(T)$ is the set of zeros of $f_1, \ldots, f_r$.

Definition 2.2.1. *A subset Y of $\mathbb{A}^n$ is an* ***affine algebraic set*** *if there exists a subset $T \subset A$ such that $Y = Z(T)$.*

Proposition 2.2.1. *(Prop. 1.1, [10]) The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is also an algebraic set. The empty set and the whole space are algebraic sets.*

Definition 2.2.2. *We define the* ***Zariski topology*** *on $\mathbb{A}^n$ by taking the closed subsets to be the algebraic sets. This is a topology, as the previous proposition confirms that the axioms for a topology are fulfilled.*

Example 2.2.1. *Consider the space $\mathbb{A}^1$ with Zariski's topology, and $A = \mathbb{K}[x]$, $\mathbb{K}$ algebraically closed. We know that in particular, A is a principal ideal domain, then every algebraic set is the set of zeros of a single polynomial. Moreover, since $\mathbb{K}$ is algebraically closed, any $f \in A$ has the form*

$$f = c(x - a_1) \cdots (x - a_n)$$

with $c, a_1, \ldots, a_n \in \mathbb{K}$, so $Z(f) = \{a_1, \ldots, a_n\}$. Therefore any proper algebraic subset of $\mathbb{A}^1$ is a finite set.

Definition 2.2.3. *A nonempty subset Y of a topological space X is* ***irreducible*** *if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.*

Proposition 2.2.2. *(Cor. 1.4, [10]) An affine algebraic set is irreducible if and only if its ideal is a prime ideal.*

Example 2.2.2. *Let f be an irreducible polynomial in $\mathbb{K}[x, y]$. Then f generates a prime ideal in A because A is a unique factorization domain. By Proposition 2.2.2, $Y = Z(f)$ is irreducible. We call this zero set the* ***affine curve*** *defined by equation $f(x, y) = 0$. If f has degree d, we say that Y is a curve of degree d.*

Example 2.2.3. *Let $Y = Z(x^2 - yz, xz - x)$. We claim that Y is not irreducible. Indeed, Y is by definition the set of common zeros of $x^2 - yz$ and $xz - x = x(z - 1)$. Note that this is equivalent to saying that every point of Y is in the set of common zeros of $x^2 - yz$ and x, or in the set of common zeros of $x^2 - yz$ and $z - 1$. So we have that*

$$Y = Z(x^2 - yz, x) \cup Z(x^2 - yz, z - 1)$$

Now, to be a zero of x means $x = 0$, so a point in $Z(x^2 - yz, x)$ turns out to be in $Z(yz, x)$. Again, to be a point in $Z(yz, x)$ means that this point must be a common zero of z and x, or a common zero of y and x. So $Z(yz, x) = Z(y, x) \cup Z(z, x)$.

Summarizing, we can write Y as

$$Y = Z(y,x) \cup Z(z,x) \cup Z(x^2 - yz, z-1)$$

which is a union of irreducible closed sets in Zariski's topology. The irreducibility is granted for $Z(y,x)$ and $Z(z,x)$ by Proposition 2.2.2 as (y,x) and (z,x) are prime ideals of $\mathbb{K}[x,y,z]$ (remember that $\mathbb{K}[x,y,z]/(x,y) \simeq \mathbb{K}[z]$ and $\mathbb{K}[x,y,z]/(z,x) \simeq \mathbb{K}[y]$ are domains) and for $Z(x^2-yz, z-1)$ by Example 2.2.2, since $Z(x^2-yz, z-1) = Z(x^2-y)$ and x^2-y is irreducible.

Definition 2.2.4. *An **affine algebraic variety** (or simply **affine variety**) is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology).*

Definition 2.2.5. *For any subset $Y \subseteq \mathbb{A}^n$, let us define the **ideal** of Y in A by*

$$I(Y) = \{f \in A : f(P) = 0 \ for \ all \ \ P \in Y\}.$$

Underlying this definition is the existence of functions

$$Z : p \subset A \longmapsto Z(p) = Y$$

and

$$I : Y \subset \mathbb{A}^n \longmapsto I(Y).$$

These functions have the following properties:

Proposition 2.2.3. *(Prop. 1.2, [10])*

a) If $T_1 \subset T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.

c) For any two subsets Y_1, Y_2 of $\mathbb{A}^n$, we have that $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

d) For any subset $Y \subset \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the closure of Y.

Example 2.2.4. *Consider the variety $\{(0,0)\}$ consisting of the origin in $\mathbb{K}^2$. Then its ideal $I(\{(0,0)\})$ consists of all polynomials that vanish at origin. We claim that*

$$I(\{(0,0)\}) = \langle x, y \rangle.$$

It is clear that any polynomial of the form

$$A(x,y)x + B(x,y)y$$

vanishes at the origin. On the other side, let

$$f = \sum_{i,j} a_{ij} x^i y^j$$

such that $f(0,0) = 0$. Then every term of the sum is zero, in particular $a_{00} = 0 = f(0,0)$ and

$$f = a_{00} + \left(\sum_{i,j>0} (a_{i0}x^{i-1} + a_{ij}x^{i-1}y^j) \right) x + \left(\sum_{j>0} a_{0j}y^{j-1} \right) y \in \langle x, y \rangle.$$

Example 2.2.5. *$\mathbb{A}^n$ is irreducible since it corresponds to the zero ideal in A, which is prime.*

Definition 2.2.6. *A topological space X is called **Noetherian** if it satisfies the descending chain condition for closed subsets: for any sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \cdots$.*

Example 2.2.6. *$\mathbb{A}^n$ is a Noetherian topological space. Indeed, if*

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

is a descending chain of closed subsets, then

$$I(Y_1) \subseteq I(Y_2) \subseteq \cdots$$

is an ascending chain of ideals in $A = \mathbb{K}[x_1, \ldots, x_n]$. Since A is Noetherian as a ring, this chain is stationary. Moreover, for each i, $Y_i = Z(I(Y_i))$, so the chain of the Y_i's is also stationary.

The importance of being Noetherian is that we can guarantee an expression of every algebraic set in $\mathbb{A}^n$ as a union of varieties no one containing another.

Proposition 2.2.4. *(Prop. 1.5, [10]) In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = \bigcup_{i=1}^{r} Y_i$ of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$ then the Y_i's are uniquely determined, and we call the Y_i's the **irreducible components** of Y.*

Definition 2.2.7. *If X is a topological space, we define the **dimension** of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain*

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.

Proposition 2.2.5. *(Prop. 1.9, [10]) The dimension of $\mathbb{A}^n$ is n.*

Proposition 2.2.6. *(Prop. 1.10, [10]) If Y is an affine variety, then $\dim Y = \dim \overline{Y}$.*

Proposition 2.2.7. *A variety Y in $\mathbb{A}^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial $f \in A = \mathbb{K}[x_1, \ldots, x_n]$.*

2.3 Projective varieties

To get a wide overview (along with additional results) of our study, we would need to take advantage of the invariance under tensor rescalings of the properties we discussed in Chapter 1, so we consider a different perspective through an equivalence relation that allows us to quotient our space V and obtain a set of lines through the origin in V.

Definition 2.3.1. *Let V be a vector space of dimension $\mathbf{v}$. We define the **projective space** associated to V, which we denote $\mathbb{P}V = \mathbb{P}^{\mathbf{v}-1}$, as the set whose points $[v] \in \mathbb{P}V$ are equivalence classes of nonzero elements $v \in V$, where $[v] = [w]$ if and only if there exists a nonzero $\lambda \in \mathbb{K}$ such that $v = \lambda w$.*

Underlying the previous definition is the existence of a function

$$\begin{aligned} \pi : V \setminus \{0\} &\longrightarrow \mathbb{P}V \\ v &\longmapsto [v] \end{aligned}$$

We call π the projection map of V. This new space $\mathbb{P}V$ will inherit some aspects of the linear structure on V. For instance, if $U \subset V$ is a linear subspace, then $\mathbb{P}U \subset \mathbb{P}V$ is called a linear subspace.

Also, for any two different points in $\mathbb{P}V$, say $[x]$ and $[y]$, there exists a line, which we are denoting $\mathbb{P}^1_{xy}$ that contains $[x]$ and $[y]$. For every point $(x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ we denote its equivalence class in $\mathbb{P}^n$ by $(x_0 : x_1 : \cdots : x_n)$. We will call these coordinates the **homogeneous coordinates**.

If we go back to V, we can see this line as the result of applying the projection map to a bi-dimensional plane in V containing the origin and $x, y \in V$. When we look at $\mathbb{P}^2$ we see that any two distinct lines will intersect at a point.

Definition 2.3.2. *Let $Y \subset \mathbb{P}V$, and consider $\widehat{Y} := \pi^{-1}(Y)$, the inverse image of Y by π. We call $\widehat{Y}$ the **affine cone** over Z. If $X \subset V$ and we want to go back to $\mathbb{P}V$ through π, we call $\pi(X)$ the **projectivization** of X and we denote it by $\mathbb{P}X$.*

When working with projective spaces, we may need to clarify certain details: In the projective case, we need to make the polynomial ring $S = \mathbb{K}[x_0, \ldots, x_n]$ into a graded ring by taking S_d to be the set of all linear combinations of monomials of total degree d in $x_0, \ldots, x_n$. So we get an expression for S in the form

$$S = \bigoplus_{d \geq 0} S_d,$$

with

$$S_d \cdot S_e \subseteq S_{d+e},$$

and every element of S_d is an homogeneous element of degree d. The fact that S is a direct sum implies every element of S can be written uniquely as a finite sum of homogeneous elements.

In $\mathbb{P}^n$, we cannot use $f \in S$ to define a function because the lack of uniqueness of the homogeneous coordinates. However, if f is a homogeneous polynomial of degree d, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n),$$

and the property of being zero or not will depend on the equivalence class of $(a_0, \ldots, a_n)$. So, the best we have is a function

$$\mathbb{P}^n \longrightarrow \{0, 1\}$$

defined by $f(P) = 0$ if $f(a_0, \ldots, a_n) = 0$ and $f(P) = 1$ if $f(a_0, \ldots, a_n) \neq 0$. From this, we can define the zeros of a homogeneous polynomial f as the set of $P \in \mathbb{P}^n$ such that $f(P) = 0$. We can extend this idea to define the set of zeros for a collection of polynomials W and derive some results.

Definition 2.3.3. *Let $\pi : V \setminus \{0\} \longrightarrow \mathbb{P}V$ be the projection map and consider a collection W of homogeneous polynomials in V. Let $Z(W)$ the set of common zeros of W, that is,*

$$Z(W) = \{z \in V : P(z) = 0 \ \ for \ \ all \ \ P \in W\}.$$

*We say X is an **projective algebraic variety** if $X = \pi(Z(W)) \subset \mathbb{P}V$ for some W.*

Definition 2.3.4. *For a subset $Y \subset \mathbb{P}V$, define*

$$I(Y) := \{P \in S^\bullet V^* : P \mid_{\widehat{Y}} \equiv 0\}$$

*as the **ideal** associated to Y.*

Definition 2.3.5. *Let $X \subset \mathbb{P}V$ and $Y \subset \mathbb{P}V$ be two varieties. We say X and Y are **projectively equivalent** if there exist linear maps*

$$f : V \longrightarrow W \ and \ g : W \longrightarrow V$$

such that $f(\widehat{X}) = \widehat{Y}$ and $g(\widehat{Y}) = \widehat{X}$.

Definition 2.3.6. *We say that a collection of homogeneous polynomials $P_1, \ldots, P_r \in S^\bullet V^*$ **cuts out** X **in a set-theoretical sense** if the set of common zeros of the polynomials $P_1, \ldots, P_r$ is the set of points of X. We also define the notion of a generator set for the ideal of X as a set of polynomials $P_1, \ldots, P_r$ such that every $P \in I(X)$ can be written in the form*

$$P = q_1 P_1 + \cdots + q_r P_r$$

for some polynomials $q_j \in S^\bullet V^*$.

Observation 2.3.1. *This set of generators of the ideal of a variety has equations with the lowest possible degree. As a matter of example, take the line $x = 0$ in $\mathbb{C}^2$, that is the set of pairs in the form $(0, y) \in \mathbb{C}^2$. Note that this line is cut out in a set-theoretical sense by the homogeneous polynomial $x^2 = 0$, but the ideal is generated by $x = 0$ (the lowest possible degree). From this, we have in particular that the generators of the ideal cut out X set-theoretically.*

Definition 2.3.7. *A variety $X \subset \mathbb{P}V$ is said to be **reducible** if there exist varieties $Y, Z \neq X$ such that $X = Y \cup Z$. In terms of ideals, this means that there exist nontrivial ideals I_Y, I_Z such that $I_Z \cap I_Y = I_X$. Otherwise, we say X is **irreducible**.*

Example 2.3.1. *Let W be the zero set of equation $xyz = 0$. As W is the union of $\{x = 0\}, \{y = 0\}$, and $\{z = 0\}$, W is reducible.*

Theorem 2.3.1. *(Th. 1.15, [14]) Let $Y \subset \mathbb{P}^n$ be a projective algebraic set. Then Y is irreducible if and only if $I(Y)$ is prime.*

Proof. Suppose Y is irreducible. Let $f, g \in I(Y)$. Then $Z(f)$ and $Z(g)$ are projective algebraic sets, so Y can be expressed as

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g)).$$

As Y is irreducible, $Y = Y \cap Z(f)$ or $Y = Y \cap Z(g)$. But this implies that $f \in I(Y)$ or $g \in I(Y)$ and therefore $I(Y)$ is prime. The other direction is analog to the proof for affine varieties. See Corollary 1.4 of [10]. □

Example 2.3.2. *Consider the set Ω of points satisfying the equation $x^2 + y^2 + z^2 = 0$. The polynomial is irreducible, so $I(x^2 + y^2 + z^2)$ is a prime ideal and by Theorem 2.3.1, Ω is irreducible.*

Now let's focus on examples of varieties more related to our study. We will begin by studying the rank-one matrices, that is, two-factor tensors. Let (a_i) and (b_s) be bases of A and B, respectively, and $(a_i \otimes b_s)$ the induced basis of $A \otimes B$. We also consider the dual space of $A \otimes B$, $A^* \otimes B^*$ and its basis $(\alpha^i \otimes \beta^s)$. We can identify $a_i \otimes b_s$ with the matrix having a 1 in the (i, s)-entry and zero elsewhere.

Now consider the following quadratic polynomial on $A^* \otimes B^*$, (the space of $\mathbf{a} \times \mathbf{b}$ matrices) with coordinates $x^{i,s}$, that is,

$$X = \sum_{i,s} x^{i,s} \alpha^i \otimes \beta^s$$

corresponds to the matrix whose (i, s)-entry is $x^{i,s}$, so we define

$$P_{jk|tu}(X) := x^{j,t} x^{k,u} - x^{k,t} x^{j,u}.$$

Note that $P_{jk|tu}$ is the two by two minor $(jk|tu)$. In terms of tensors, we have

$$P_{jk|tu} = (a_j \otimes b_t)(a_k \otimes b_u) - (a_k \otimes b_t)(a_j \otimes b_u)$$
$$= \frac{1}{2}[a_j \otimes b_t \otimes a_k \otimes b_u + a_k \otimes b_u \otimes a_j \otimes b_t - a_k \otimes b_t \otimes a_j \otimes b_u - a_j \otimes b_u \otimes a_k \otimes b_t]$$

and as we know that $A \otimes B \otimes A \otimes B \simeq A \otimes A \otimes B \otimes B$, we can rearrange the expression, getting

$$P_{jk|tu} = \frac{1}{2}[a_j \otimes a_k \otimes b_t \otimes b_u + a_k \otimes a_j \otimes b_u \otimes b_t - a_k \otimes a_j \otimes b_t \otimes b_u - a_j \otimes a_k \otimes b_u \otimes b_t]$$
$$= \frac{1}{2}(a_j \otimes a_k - a_k \otimes a_j)(b_t \otimes b_u) + \frac{1}{2}(a_k \otimes a_j - a_j \otimes a_k)(b_u \otimes b_t)$$
$$= (a_j \wedge a_k) \otimes b_t \otimes b_u + (a_k \wedge a_j) \otimes b_u \otimes b_t$$
$$= (a_j \wedge a_k) \otimes b_t \otimes b_u - (a_j \wedge a_k) \otimes b_u \otimes b_t$$
$$= (a_j \wedge a_k) \otimes (b_t \otimes b_u - b_u \otimes b_t) = 2(a_j \wedge a_k) \otimes \frac{1}{2}(b_t \otimes b_u - b_u \otimes b_t)$$
$$= 2(a_j \wedge a_k) \otimes (b_t \wedge b_u).$$

From this, we conclude that a two-by-two minor, expressed as a tensor in $S^2(A \otimes B)$ is an element of $\Lambda^2 A \otimes \Lambda^2 B \subset S^2(A \otimes B)$. Note that the zero set of the two by two minors $P_{jk|tu}$ is the set of $\mathbf{a} \times \mathbf{b}$ matrices such that every $P_{jk|tu}$ is zero, that is, the set of rank-one matrices.

Definition 2.3.8. *Define the **two-factor Segre variety** $Seg(\mathbb{P}A \times \mathbb{P}B)$ to be the zero set of the ideal generated by the two by two minors as in the previous discussion.*

Observation 2.3.2. *The rank of a tensor p does not depend on the nonzero rescalings of p. From this, we can think about the rank as a function of the projective space*

$$\mathbb{P}(A_1 \otimes \cdots \otimes A_n) \longrightarrow \mathbb{N}$$
$$p \longmapsto R(p)$$

The set of rank-one tensors is isomorphic to $\mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_n$ in $\mathbb{P}(A_1 \otimes \cdots \otimes A_n)$. Its embedding in the tensor space is also called the Segre variety:

$$Seg = Seg_{A_1,A_2,\ldots,A_d} := \mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_d \subset \mathbb{P}(A_1 \otimes A_2 \otimes \cdots \otimes A_d).$$

Let A_j be vector spaces and let $V = A_1 \otimes A_2 \otimes \cdots \otimes A_n$. We define the n-factor Segre variety to be the image of the map

$$Seg : \mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_n \longrightarrow \mathbb{P}V$$
$$([v_1], \ldots, [v_n]) \longmapsto [v_1 \otimes v_2 \otimes \cdots \otimes v_n].$$

This map is called the Segre embedding. To show that $Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_n) \subset \mathbb{P}V$ is a variety, we will see it as the set of common zeros of $\Lambda^2 A_j^* \otimes \Lambda^2 A_{\hat{j}}^*$, for $1 \leq j \leq n$, where

$$A_{\hat{j}} = A_1 \otimes \cdots \otimes A_{j-1} \otimes A_{j+1} \otimes \cdots \otimes A_n.$$

As we did in the two-factor case, note that the zero set of $\Lambda^2 A_1^* \otimes \Lambda^2 A_{\hat{1}}^*$ is

$$Seg(\mathbb{P}A_1 \times \mathbb{P}(A_2 \otimes \cdots \otimes A_n)),$$

that is, the tensors of the form $a_1 \otimes M_1$, where $a_1 \in A_1$ and $M_1 \in A_2 \otimes \cdots \otimes A_n$. The second fact is that the zero set of $\Lambda^2 A_2^* \otimes \Lambda^2 A_{\hat{2}}^*$ is $Seg(\mathbb{P}A_2 \times \mathbb{P}A_{\hat{2}}^*)$. So the set of common zeros of $\Lambda^2 A_1^* \otimes \Lambda^2 A_{\hat{1}}^*$ and $\Lambda^2 A_2^* \otimes \Lambda^2 A_{\hat{2}}^*$ is

$$Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \mathbb{P}(A_3 \otimes \cdots \otimes A_n)).$$

Doing this recursively, we obtain that $Seg(\mathbb{P}A_1 \times \mathbb{P}A_2 \times \cdots \times \mathbb{P}A_n)$ is the set of rank-one matrices of $A_1 \otimes \cdots \otimes A_n$, and we denote it by $\widehat{\sigma}_1 \subset A_1 \otimes \cdots \otimes A_n$.

From our previous discussion regarding projective spaces, we can introduce the following notation and definitions:

Notation 2.3.1. *Let V a linear subspace of $A_1 \otimes \cdots \otimes A_n$, then*

$$V_{Seg} := \mathbb{P}V \cap Seg_{A_1, A_2, \ldots, A_n}.$$

V_{Seg} is (up to projectivization) the set of rank-one tensors in V.

Definition 2.3.9. *For $p \in A_1 \otimes \cdots \otimes A_n$, define $\underline{R}(p)$, the* ***border rank of*** *p, to be the minimal r such that $\langle p \rangle \in \sigma_r(Seg_{A_1,\ldots,A_n})$, and $\langle p \rangle$ is the underlying point of p in the projective space. We will say $\underline{R}(p) = 0$ if and only if $p = 0$.*

It can be proven that this definition of border rank coincides with our previous Definition 1.5.1. The last result we state is related to intersections of varieties and we will use it in the end of the next chapter:

Theorem 2.3.2. *(Ch.1, Sec.7 Th.7.2 [10])(Projective Dimension Theorem) Let Y, Z be varieties of dimensions r, s in $\mathbb{P}^n$. Then every irreducible component of $Y \cap Z$ has a dimension greater or equal than $r + s - n$. Furthermore, if $r + s - n \geq 0$, then $Y \cap Z$ is nonempty.*

Chapter 3

First additivity results

Now we begin the study of the central problem of our work: the additivity of the tensor rank, discussed in Chapter 1. More specifically, we will approach the following question:

Question 3.0.1. *Suppose $A = A' \oplus A'', B = B' \oplus B'', C = C' \oplus C''$ where all $A, \dots, C''$ are finite dimensional vector spaces over a field $\mathbb{K}$. Choose $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ and let $p = p' + p''$. Does the following equality hold*

$$R(p) = R(p') + R(p'')? \tag{3.1}$$

Regarding Question 3.0.1, the answer for the case where one of the vector spaces $A', A'', B', B'', C', C''$ is at most two dimensional, is that (3.1) holds. This conclusion is known as the Ja'Ja'-Takche theorem ([11]). We will see several recent approaches to this question. In [6], Buczynski, Postinghel, and Rupniewski worked with a variety of cases, more specifically, spaces involved in the tensor product with dimensions less or equal to 4. To address all of those cases, there was a mixture of perspectives and techniques that we will address in the remainder of this work. The main results for this chapter will be:

Theorem 3.0.1. *Let $p_1 \in A_1 \otimes B_1 \otimes C_1$ and $p_2 \in A_2 \otimes B_2 \otimes C_2$ be such that $R(p_1)$ can be determined by the substitution method. Then Strassen's additivity conjecture holds for $p_1 \oplus p_2$, that is, $R(p_1 \oplus p_2) = R(p_1) + R(p_2)$.*

Theorem 3.0.2. *Suppose $W'' \subset \langle x \rangle \otimes C'' + B'' \otimes \langle y \rangle$ and $W' \subset B' \otimes C'$ is an arbitrary subspace. Then the additivity of the rank holds for $W' \oplus W''$.*

Theorem 3.0.3. *Suppose $\mathbb{K}$ is an algebraically closed field, $W'' \subset \langle x \rangle \otimes C'' + B'' \otimes \mathbb{K}^2$, and $W' \subset B' \otimes C'$ is an arbitrary subspace. Then the additivity of the rank holds for $W' \oplus W''$.*

We first study a couple of results that show how the rank is independent of the choice of the vector spaces involved and explain the slice technique.

Lemma 3.0.4. *(Lemma 2.7, [6]) Let be $p \in A'_1 \otimes A'_2 \otimes \cdots \otimes A'_n$ for some linear subspaces $A'_i \subset A_i$. Then $R(p)$ (respectively under the condition $\mathbb{K} = \mathbb{C}$, $\underline{R}(p)$) measured as the rank (respectively, the border rank) in $A'_1 \otimes \cdots \otimes A'_n$ is equal to the rank (resp. the border rank) measured in $A_1 \otimes \cdots \otimes A_n$.*

Proof. The reader can check the proof for rank on Theorem 1.5.2. □

Lemma 3.0.5. *(Lemma 2.8, [6]) Suppose $p \in A' \otimes B \otimes C$ for a linear subspace $A' \subset A$, and that we have an expression for $p \in \langle s_1, \dots, s_r \rangle$, where $s_i = a_i \otimes b_i \otimes c_i$ are simple tensors. Then*

$$r \geq R(p) + dim\langle a_1, \dots, a_r \rangle - dimA'.$$

Proof. Note that we can replace A with a smaller subspace if needed. Set $d = dimA - dimA'$. We reorder the simple tensors s_i in such a way that the first d of the a_i's are L.I. and $\langle A' \sqcup \{a_1, \dots, a_d\} \rangle = A$. Let $A'' = \langle a_1, \dots, a_d \rangle$ so that $A = A' \oplus A''$ and consider the quotient map

$$\pi : A \longrightarrow A/A''.$$

We also consider the composition

$$A' \xrightarrow{i} A \xrightarrow{\pi} A/A'' \simeq A',$$

which is an isomorphism and denote it by ϕ. We can tensorize ϕ by $Id_B \otimes Id_C$ and along with a minor abuse of notation, set

$$\pi : A \otimes B \otimes C \longrightarrow (A/A'') \otimes B \otimes C \text{ and}$$
$$\phi : A' \otimes B \otimes C \longrightarrow A' \otimes B \otimes C.$$

We obtain

$$\begin{aligned} \phi(p) = \pi(p) &\in \pi(\langle a_1 \otimes b_1 \otimes c_1, \dots, a_r \otimes b_r \otimes c_r \rangle) \\ &= \langle \pi(a_1) \otimes b_1 \otimes c_1, \dots, \pi(a_r) \otimes b_r \otimes c_r \rangle \\ &= \langle \pi(a_{d+1}) \otimes b_{d+1} \otimes c_{d+1}, \dots, \pi(a_r) \otimes b_r \otimes c_r \rangle \end{aligned}$$

Now we take ϕ^{-1} to get a presentation of p as a linear combination of $(r - d)$ simple tensors, that is, $R(p) \leq r - d$, as we wanted. □

3.1 Slice technique and conciseness

Now we will define the conciseness of tensors to replace the calculation of the rank of three-way tensors with the rank calculation for linear spaces of matrices. For every tensor $p \in A_1 \otimes \cdots \otimes A_n$ there exists a linear map

$$p : A_1^* \longrightarrow A_2 \otimes \cdots \otimes A_n.$$

Definition 3.1.1. *The tensor $p \in A_1 \otimes \cdots \otimes A_n$ is called **concise** if each map*

$$p : A_i^* \longrightarrow A_1 \otimes \cdots \otimes A_{i-1} \otimes A_{i+1} \otimes \cdots \otimes A_n$$

is injective.

Example 3.1.1. *Let $A \otimes B \otimes C \simeq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with respective basis $(a_i), (b_j), (c_k)$, and $p \in A \otimes B \otimes C$ be the following tensor:*

$$\begin{aligned} p : A^* &\longrightarrow B \otimes C \\ a_1 &\longmapsto b_1 \otimes c_1 \\ a_2 &\longmapsto b_1 \otimes c_1 - b_2 \otimes c_2 \\ a_3 &\longmapsto b_2 \otimes c_2 \end{aligned}$$

Note that p as defined is not injective because

$$p(-a_1^* + a_2^* - a_3^*) = 0,$$

so $Ker(p) \neq 0$ and p is not concise, but rewriting p we have

$$\begin{aligned} p &= a_1 \otimes b_1 \otimes c_1 + a_2 \otimes (b_1 \otimes c_1 - b_2 \otimes c_2) + a_3 \otimes b_2 \otimes c_2 \\ &= (a_1 + a_2) \otimes (b_1 \otimes c_1) + (-a_2 + a_3) \otimes (b_2 \otimes c_2), \end{aligned}$$

and therefore $p = \widetilde{p} \in \widetilde{A} \otimes B \otimes C$, where $\widetilde{A} = \langle a_1 + a_2, -a_2 + a_3 \rangle \subsetneq A$ is concise. Note that we decreased A to get a concise tensor from a non-concise one. This process can be done in general.

Observation 3.1.1. *When p is not concise, we can change p for another concise one by decreasing the size of some of the spaces involved in the tensor product.*

Proposition 3.1.1. *(pp. 69, Section 3.1, [12]) If $p \in A_1 \otimes \cdots \otimes A_n$ is concise, then $\underline{R}(p) \geq \max\{\mathbf{a_i}\}$.*

We assume p being concise, and therefore consider $W = p(A_1^*) \subset A_2 \otimes \cdots \otimes A_n$. Choose a basis $\mathcal{B}$ of A_1^*, and consider $p(\mathcal{B})$. By the injectivity, we have a basis for W. We call the elements of this basis the **slices** of p. The key detail is that we can uniquely determine p from W (up to an action of $GL(A_1)$). This implies that W will capture the geometric information about p, particularly, its rank and border rank. We can rephrase our rank definition in terms of linear subspaces.

Definition 3.1.2. *We define the **rank** of a linear subspace W (denoted $R(W)$) of a product tensor space as the minimal number r, such that there exist simple tensors $s_1, \ldots, s_r$ with $W \subset \langle s_1, \ldots, s_r \rangle$.*

Lemma 3.1.1. *(Lemma 2.9, [6]) Lets assume $p \in A_1 \otimes \cdots \otimes A_n$ to be concise and $W = p(A_1^*)$ as above. Then for simple tensors $s_1, \ldots, s_r \in A_2 \otimes \cdots \otimes A_n$ there exist* *vectors $a_i \in A_i$ for $i = 1, \dots, r$ such that $p = a_1 \otimes s_1 + \cdots + a_r \otimes s_r$ if and only if $W \subset \langle s_1, \dots, s_r \rangle$. Moreover, $R(p) = R(W)$ and if $\mathbb{K} = \mathbb{C}$ $\underline{R}(p) = \underline{R}(W)$.*

Proof. See Theorem 2.5 of [5]. □

We can also replace A_1 with any of the A_i to define slices as images $p(A_i^*)$. Now we can use our previous lemmas to prove a statement for higher dimensional subspaces of the tensor space. We consider the case $n = 2$, as it is our main interest.

Example 3.1.2. *Let*

$$p = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 + e_3 \otimes e_2 \otimes e_2$$

and consider its associated subspace, where $a, b, c \in \mathbb{C}$

$$W = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Note that $dim(W) = 3$, and it can be proved that $R(W) = 4$, but $R(M) \leq 2$ for all $M \in W$, where $R(M)$ is the usual rank from linear algebra of the matrix M.

Lemma 3.1.2. *(Prop. 2.10, [6]) Suppose $W \subset B' \otimes C'$ for some linear subspaces $B' \subset B$, $C' \subset C$.*

i) The numbers $R(W)$ and $\underline{R}(W)$, measured as the rank and border rank of W in $B' \otimes C'$, are equal to the rank and border rank of W in $B \otimes C$ (for the border rank, we set $\mathbb{K} = \mathbb{C}$).

ii) Moreover, if we have an expression $W \subset \langle s_1, \dots, s_r \rangle$ where $s_i = b_i \otimes c_i$ are simple tensors, then

$$r \geq R(W) + dim\langle b_1, \dots, b_r \rangle - dimB'.$$

Proof. i) Let $k = R(W)$ measured in $B' \otimes C'$. Then there exist $u_1 = b_1 \otimes c_1, \dots, u_k = b_k \otimes c_k \in B' \otimes C'$ such that $W \subset \langle u_1, \dots, u_k \rangle$. Consider $A = \mathbb{K}^k$ and

$$p = e_1 \otimes b_1 \otimes c_1 + \cdots + e_k \otimes b_k \otimes c_k \in A \otimes B' \otimes C'.$$

Then $W = p(A^*)$.
By Lemma 3.0.4 the rank and border rank of p is the same measured in $A \otimes B' \otimes C'$ or $A \otimes B \otimes C$. Using Lemma 3.1.1 twice we get:

(a) $R(p) = R(W)$ in $A \otimes B' \otimes C'$ and $B' \otimes C'$.

(b) $R(p) = R(W)$ in $A \otimes B \otimes C$ and $B \otimes C$.

Moreover, $\underline{R}(W) = \underline{R}(p)$ in $A \otimes B' \otimes C'$, $B' \otimes C'$, $A \otimes B \otimes C$, and $B \otimes C$.

ii) As in i), we consider $p \in A \otimes B' \otimes C'$ such that $W = p(A^*) \subset \langle s_1, \dots, s_r \rangle$, where $s_i = b_i \otimes c_i$ are simple tensors. We are now under the hypothesis of Lemma 3.1.1, so we get $R(p) = R(W)$ and by Lemma 3.0.5 applied to B',

$$r \geq R(p) + dim\langle b_1, \dots, b_r \rangle - dimB' = R(W) + dim\langle b_1, \dots, b_r \rangle - dimB'$$

as we wanted.

□

To conclude this section, we will give another definition of conciseness for a tensor and a linear subspace. We can check that the definitions are equivalent, but we will use our definitions depending on what we need to deduce.

Definition 3.1.3. *Let $p \in A_1 \otimes \cdots \otimes A_n$ be a tensor or let $W \subset A_1 \otimes \cdots \otimes A_n$ be a linear subspace. We say that p or W is A_1-**concise** if for all linear subspaces $V \subset A_1$, if $p \in V \otimes A_2 \otimes \cdots \otimes A_n$ (respectively, $W \subset V \otimes \cdots \otimes A_n$) then $V = A_1$. Analogously, we define A_i-concise tensors and spaces for $i = 2, \dots, n$. We say p or W is **concise** if it is A_i-concise for all $i \in \{1, \dots, n\}$.*

Notation 3.1.1. *Let $A', A'', B', B'', C', C''$ be vector spaces over $\mathbb{K}$ with dimensions, respectively, $\mathbf{a}'$, $\mathbf{a}''$, $\mathbf{b}'$, $\mathbf{b}''$, $\mathbf{c}'$, $\mathbf{c}''$. Suppose $A = A' \oplus A''$, $B = B' \oplus B''$, $C = C' \oplus C''$, $\mathbf{a} = dimA = \mathbf{a}' + \mathbf{a}''$, $\mathbf{b} = dimB = \mathbf{b}' + \mathbf{b}''$, and $\mathbf{c} = dimC = \mathbf{c}' + \mathbf{c}''$*

We will approach the two-way tensors in $B \otimes C$ as matrices in $\mathbb{M}^{\mathbf{b} \times \mathbf{c}}$. For this, we choose bases for B and C, but we will refrain from naming the bases explicitly. In this section we will consider any element $w \in \mathbb{M}^{\mathbf{b} \times \mathbf{c}} \simeq B \otimes C$ as a $(\mathbf{b}' + \mathbf{b}'', \mathbf{c}' + \mathbf{c}'')$ partitioned matrix. So when we have a matrix $w \in \mathbb{M}^{\mathbf{b} \times \mathbf{c}}$, we will think of a matrix with four blocks of size $\mathbf{b}' \times \mathbf{c}', \mathbf{b}' \times \mathbf{c}'', \mathbf{b}'' \times \mathbf{c}'$, and $\mathbf{b}'' \times \mathbf{c}''$, respectively.
To complete our conventions, we need to introduce another notation:

Notation 3.1.2. *As in Notation 1.1, a tensor $p \in A \otimes B \otimes C$ can be seen as a linear map $p : A^* \longrightarrow B \otimes C$; we denote by $W := p(A^*)$, the image of A^* in the space of matrices $B \otimes C$. Similarly, if $p = p' + p'' \in (A' \oplus A'') \otimes (B' \oplus B'') \otimes (C' \oplus C'')$ is such that $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$, we set $W' := p'(A'^*) \subset B' \otimes C'$ and $W'' := p''(A''^*) \subset B'' \otimes C''$. In such a situation, we will say that $p = p' \oplus p''$ is a direct sum tensor.*

From above, we obtain the following direct sum decomposition:

$$W = W' \oplus W'' \subset (B' \otimes C') \oplus (B'' \otimes C'')$$

and there exists an induced matrix partition of size $(\mathbf{b}' + \mathbf{b}'', \mathbf{c}' + \mathbf{c}'')$ on every matrix $w \in W$ such that

$$w = \begin{pmatrix} w' & \underline{0} \\ \underline{0} & w'' \end{pmatrix}$$

where $w' \in W'$ and $w'' \in W''$ and the two $\underline{0}$'s denote zero matrices of size $\mathbf{b}' \times \mathbf{c}''$ and $\mathbf{b}'' \times \mathbf{c}'$, respectively.

Proposition 3.1.2. *(Prop. 3.3, [6]) Suppose that p and W are as in the notation above. Then $R(p) = R(p') + R(p'')$ if and only if $R(W) = R(W') + R(W'')$.*

Proof. Immediate from Lemma 3.1.1. □

3.2 Projections and decompositions

We explore decomposition aspects regarding direct sums. Our goal will be to exploit the concept of decomposition for the spaces involved to be able to analyze its rank and border rank. The concepts of projections and conciseness play a central role. We will mostly focus on useful inequalities, and explore additivity of the rank by giving conditions to the spaces in the decomposition.

Definition 3.2.1. *We define a **minimal decomposition** V for a subspace $W \subset B \otimes C$ such that*

1. *$dim V = R(W)$,*
2. *$\mathbb{P}W = \langle V_{Seg} \rangle$, and*
3. *$W \subset V$.*

Example 3.2.1. *Let $W = \langle Id_n \rangle$. Note that $dim(W) = 1$, $R(W) = n$. A minimal decomposition for W is the space*

$$V = \langle e_{ii}, i = 1, \ldots, n \rangle,$$

where e_{ii} is the $n \times n$ matrix with a 1 in the entry (i, i) and 0 elsewhere. We know that $dim(V) = n = R(W)$, $\mathbb{P}W = \langle V_{Seg} \rangle$ (as all of V's generators have rank 1), and $W \subset V$.

Now consider $\widetilde{W} = \langle Id_r \rangle + \langle Id_s \rangle \supset W$, where $r + s = n$. Then the same V is also a minimal decomposition for $\widetilde{W}$ and $\widetilde{W}$ has the same rank as W, but $dim(\widetilde{W}) = 2$ and $dim(W) = 1$.

Notation 3.2.1. *Under Notation 3.1.1, let $\pi_{C'}$ denote the projection*

$$\pi_{C'} : C \longrightarrow C''$$

whose kernel is C'. We can also tensorize $\pi_{C'}$ by $Id_B \otimes Id_A$ and keep the notation $\pi_{C'}$:

$$\pi_{C'} : B \otimes C \longrightarrow B \otimes C'' \quad \text{and} \quad \pi_{C'} : A \otimes B \otimes C \longrightarrow A \otimes B \otimes C''$$

with kernels, respectively, $B \otimes C'$ *and* $A \otimes B \otimes C'$. *From the projection we obtain a subspace of* $C, B \otimes C$, *or* $A \otimes B \otimes C$, *in such a way we can compose the projections, for instance*

$$\pi_{C'}\pi_{B''} : B \otimes C \longrightarrow B' \otimes C'' \text{ or}$$
$$\pi_{C'}\pi_{B''} : A \otimes B \otimes C \longrightarrow A \otimes B' \otimes C''.$$

We also choose $E' \subset B'$ *(respectively* $E'' \subset B''$*) as the minimal vector subspace such that* $\pi_{C'}(V)$ *(respectively* $\pi_{C''}(V)$*) is contained in* $(E' \oplus B'') \otimes C''$ *(respectively* $(B' \oplus E'') \otimes C'$*). If we swap the roles of* B *and* C, *we can define* $F' \subset C'$ *and* $F'' \subset C''$ *analogously. We denote the dimensions of the subspaces* E', E'', F', F'' *by* $\mathbf{e}', \mathbf{e}'', \mathbf{f}'$, *and* $\mathbf{f}''$.

We also recall the following result on linear algebra:

Theorem 3.2.1. *Let* $\pi : X \longrightarrow Y$ *a linear projection with* $Im(\pi) = Y \subset X$ *and* $Z = Ker(\pi) \subset X$. *Let's consider* $T \subset X$ *with* $T = \langle t_1, \ldots, t_r, \bar{t}_1, \ldots, \bar{t}_{\bar{r}} \rangle$, *where* $dimT = r + \bar{r}$, $t_1, \ldots, t_r \in Y$ *and* $\bar{t}_1, \ldots, \bar{t}_{\bar{r}} \in Z$. *Then*

$$\pi(T) = \langle t_1, \ldots, t_r \rangle \text{ and } dim(\pi(T)) = r.$$

Now we take a look at the differences $R(W') - dim(W')$ and $R(W'') - dim(W'')$. We will informally refer to these numbers as the *gaps*. If the gaps are large, that means the spaces E', E'', F', F'' could be large too, in particular, they can coincide with B', B'', C', and C'', respectively. These spaces give us an idea of how far a minimal decomposition V of a direct sum $W = W' \oplus W''$ is from being a direct sum of decompositions of W' and W''. Finally, we present the results for this section.

Lemma 3.2.2. *(Lemma 3.5, [6]) In Notation 3.2.1 with* $W = W' \oplus W'' \subset B \otimes C$, *the following inequalities hold:*

$$R(W') + \mathbf{e}'' \leq R(W) - dim(W''), \qquad R(W'') + \mathbf{e}' \leq R(W) - dim(W'),$$
$$R(W') + \mathbf{f}'' \leq R(W) - dim(W''), \qquad R(W'') + \mathbf{f}' \leq R(W) - dim(W').$$

We can assume W' is concise, as Lemma 3.1.2 guarantees there are no changes on $R(W')$ or $R(W)$ if we choose the minimal subspace B', and the minimal decomposition $V \subset B \otimes C$ of $W' \oplus W''$ cannot involve any tensor from outside of the minimal subspace.

Proof. Since every $b_i \otimes c_i$ is in $Im(\pi_{C''})$, and the rest of the generators are in $Ker(\pi_{C''})$, we have

$$\pi_{C''}(V) = \langle b_1 \otimes c_1, \ldots, b_r \otimes c_r \rangle,$$

where $r = dim\pi_{C''}(V)$ (here we are using Theorem 3.2.1). We also know $\pi_{C''}(W' \oplus W'') = \pi_{C''}(W') \oplus \pi_{C''}(W'') = W' \oplus \pi_{C''}(W'') \subset (B' \oplus B'') \otimes C' \subset \pi_{C''}(V)$, so $W' \subset \pi_{C''}(V)$. Now we claim that $B' \oplus E'' = \langle b_1, \ldots, b_r \rangle$. To prove this, note that

1. W' is concise,
2. $W' \subset \pi_{C''}(V) = \langle b_1 \otimes c_1, \ldots, b_r \otimes c_r \rangle \subset (B' \oplus E'') \otimes C'$ (by the definition of E''), and
3. $W' \subset V \cap (B' \otimes C')$.

From the definition of conciseness, we have $\langle b_1 \otimes c_1, \ldots, b_r \otimes c_r \rangle = (B' \oplus E'') \otimes C'$, so the inclusions $B' \subset \langle b_1, \ldots, b_r \rangle$, $E'' \subset \langle b_1, \ldots, b_r \rangle$, and $\langle b_1, \ldots, b_r \rangle \subset B' \oplus E''$ are granted. We can change the notation $s_i = b_i \otimes c_i$, for $i = 1, \ldots, r$, to get $W' \subset \langle s_1, \ldots, s_r \rangle$ and by Lemma 3.1.2.ii), we deduce

$$r = dim(\pi_{C''}(V)) \geq R(W') + \underbrace{dim\langle b_1, \ldots, b_r \rangle}_{\mathbf{b}'+\mathbf{e}''} - \mathbf{b}' = R(W') + \mathbf{e}''. \tag{3.2}$$

Since V contains W'' and $\pi_{C''}(W'') \subset \pi_{C''}(B'' \otimes C'') = \pi_{C''}(B'') \otimes \pi_{C''}(C'') = \pi_{C''}(B'') \otimes \{0\} = \{0\}$, we get $W'' \subset Ker(\pi_{C''})$, and the rank-nulity theorem gives us

$$r = dim \pi_{C''}(V) \leq dimV - dimW'' \overset{(1)}{=} R(W) - dimW''.$$

Combining these two last inequalities, we prove the first inequality of the lemma. The others follow using the same idea by swapping B and C or $'$ and $''$. □

Rephrasing the inequalities of Lemma 3.2.2, we obtain

Corollary 3.2.1. *(Cor. 3.6, [6]) If $R(W) < R(W') + R(W'')$, then*

$$\mathbf{e}'' < R(W'') - dim(W''), \qquad \mathbf{e}' < R(W') - dim(W'),$$
$$\mathbf{f}'' < R(W'') - dim(W''), \qquad \mathbf{f}' < R(W') - dim(W').$$

This immediately recovers the known case of additivity of the rank, when the gap is equal to zero, that is, if $R(W') = dim(W')$, then $R(W) = R(W') + R(W'')$ (as $\mathbf{e}' \geq 0$). In addition, it implies that if one of the gaps is equal to 1, for instance, $R(W') = dim(W') + 1$, then either the additivity of the rank holds or both E' and F' are trivial vector spaces. The latter assertion cannot occur without additivity of the rank. We end the section with a clarifying example.

Example 3.2.2. *Under Notation 3.1.1, we consider $A = \mathbb{C}^5$, $A' = \mathbb{C}^3$, $A'' = \mathbb{C}^2$, $B = \mathbb{C}^4$, $B' = \mathbb{C}^2 = B''$, $C = \mathbb{C}^4$, and $C' = \mathbb{C}^2 = C''$. We fix bases for A, B, C and consider*

$$A \otimes B \otimes C = (A' \oplus A'') \otimes (B' \oplus B'') \otimes (C' \oplus C'')$$

and

$$\begin{aligned} p &= (e_3 + e_1) \otimes (e_1 + 2e_3) \otimes (e_2 - e_4) \\ &= e_3 \otimes e_1 \otimes e_2 + 2e_3 \otimes e_3 \otimes e_2 - e_3 \otimes e_1 \otimes e_4 - 2e_3 \otimes e_3 \otimes e_4 \\ &\quad + e_1 \otimes e_1 \otimes e_2 - e_1 \otimes e_1 \otimes e_4 + 2e_1 \otimes e_3 \otimes e_2 - 2e_1 \otimes e_3 \otimes e_4 \end{aligned}$$

Now we take

$$\pi_{C'} : A \otimes B \otimes C \longrightarrow A \otimes B \otimes C''$$
$$\pi_{B''} : A \otimes B \otimes C \longrightarrow A \otimes B' \otimes C$$
$$\pi_{A''} : A \otimes B \otimes C \longrightarrow A' \otimes B \otimes C$$

and compute

$$\pi_{A''}\pi_{B''}\pi_{C'}(p) = \pi_{A''}\pi_{B''}((e_3 + e_1) \otimes (e_1 + 2e_3) \otimes (-e_4))$$
$$= \pi_{A''}((e_3 + e_1) \otimes e_1 \otimes (-e_4)) = -(e_3 + e_1) \otimes e_1 \otimes e_4 \in A' \otimes B' \otimes C''.$$

In addition,

$$W_1 = p(A^*) = \langle (e_1 + 2e_3) \otimes (e_2 - e_4) \rangle \subset B \otimes C, \text{ and}$$
$$W_2 = p(B^*) = \langle (e_3 + e_1) \otimes (e_2 - e_4) \rangle \subset A' \otimes C.$$

3.3 The substitution method

In this section, we use the previous results and adopt another perspective to get results on the additivity of the rank. We explore the substitution method, also known as the Alexeev–Forbes–Tsimerman method for bounding tensor rank, and show additivity of the rank for what we will define as *hook-shaped spaces* using an algorithm mainly consisting of the recursive use of three lemmas.

We need to clarify the following: Having fixed a basis for a vector space A, let's say $\{a_1, \dots, a_{\mathbf{a}}\}$, and $I \subset \{1, \dots, \mathbf{a}\}$, define $\langle a_i : i \in I \rangle^{\perp} := \langle a_i : i \in I^c \rangle$. For example, $a_1^{\perp}$ will denote the subspace $\langle a_2, \dots, a_{\mathbf{a}} \rangle$. The symbol $\perp$ is just a way to simplify the notation, we never assume our vector spaces as spaces with an inner product.

Proposition 3.3.1. *(Prop. 3.1, [13]) Fix a basis $a_1, \dots, a_{\mathbf{a}}$ of A. Write*

$$p = \sum_{i=1}^{\mathbf{a}} a_j \otimes m_j,$$

where $m_j \in B \otimes C$ for $j = 1, \dots, \mathbf{a}$. Let $R(p) = r > 0$ and $m_1 \neq 0$. Then there exist constants $\lambda_2, \dots, \lambda_{\mathbf{a}}$, such that the tensor

$$\widetilde{p} := \sum_{j=2}^{\mathbf{a}} a_j \otimes (m_j - \lambda_j m_1) \in a_1^{\perp} \otimes B \otimes C$$

has rank at most $r - 1$. Moreover, if $R(m_1) = 1$, then for any choice of $(\lambda_2, \dots, \lambda_{\mathbf{a}})$ we have $R(\widetilde{p}) \geq r - 1$.

Proof. By the first part of Lemma 3.1.1, there exist rank-one tensors $s_\mu \in B \otimes C$, $\mu \in \{1, \dots, r\}$ and scalars $x_{\mu j}$ such that

$$m_j = \sum_{\mu=1}^{r} x_{\mu j} s_\mu. \tag{3.3}$$

As $m_1 \neq 0$, we can assume without loss of generality that $x_{11} \neq 0$. More specifically, we have

$$\begin{aligned} m_1 &= x_{11}s_1 + \cdots + x_{r1}s_r \\ m_2 &= x_{12}s_1 + \cdots + x_{r2}s_r \\ &\vdots \\ m_{\mathbf{a}} &= x_{1\mathbf{a}}s_1 + \cdots + x_{r\mathbf{a}}s_r \end{aligned}$$

and we can see this as

$$m = XS,$$

where

$$X = \begin{bmatrix} x_{11} & \cdots & x_{r1} \\ \vdots & & \vdots \\ x_{1\mathbf{a}} & \cdots & x_{r\mathbf{a}} \end{bmatrix}_{\mathbf{a}\times r} \text{ and } S = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix}_{r\times 1}$$

Now let's consider

$$q = a_1 \otimes m_1 + \lambda_2 a_2 \otimes m_1 + \cdots + \lambda_{\mathbf{a}} a_{\mathbf{a}} \otimes m_1$$

and

$$\widetilde{p} = p - q = \sum_{j=2}^{\mathbf{a}} a_j \otimes (m_j - \lambda_j m_1) = \sum_{j=2}^{\mathbf{a}} a_j \otimes \widetilde{m}_j$$

Now let's take a look at $\widetilde{m_j}$:

$$\widetilde{m_j} = m_j - \lambda_j m_1 = (x_{1j} - \lambda_j x_{11})s_1 + (x_{2j} - \lambda_j x_{21})s_2 + \cdots$$

so we have $x_{1j} - \lambda_j x_{11} = 0$ if and only if $\lambda_j = \dfrac{x_{1j}}{x_{11}}$, and this is always possible because $x_{11} \neq 0$. Choose the λ_j's in the way we just mentioned and consider $W = \widetilde{p}(A^*)$. By the final part of Lemma 3.1.1, $R(\widetilde{p}) = R(\widetilde{W}) = R(\langle \widetilde{m}_2, \widetilde{m}_3, \dots, \widetilde{m}_{\mathbf{a}} \rangle) \leq R(\langle s_2, \dots, s_r \rangle) = r-1$, as $R(s_i) = 1$ for all $2 \leq i \leq r$. Note that in the last inequality of this paragraph, we obtain $R(\widetilde{p}) \leq r - 1$ and the $\leq$ means that is possible to get a coefficient of a particular s_j be zero when considering $\widetilde{m_j} = m_j - \lambda_j m_1$.

In the case that $R(m_1) = 1$, $m_1 = x_{11}s_1$, so we can ensure that only the coefficient corresponding to s_1 is vanished and from this follows that $R(\widetilde{m_j}) = R(m_j) - 1$ for all $j \geq 2$ and

$$R(\widetilde{p}) = r - 1 \text{ for all } j \geq 2.$$

□

Proposition 3.3.1 can be carried out by following these consecutive steps:

1. Identify A, choose a basis $\{a_j\}$ of A and consider bases $\{\beta^s\}$ and $\{\gamma^t\}$ of B^* and C^* respectively, and represent p as a matrix M with entries that are linear combinations of the a_i: $M_{s,t} = p(\beta^s \otimes \gamma^t)$.
2. Choose a subset of $\mathbf{b}'$ columns and $\mathbf{c}'$ rows of M.
3. Inductively, for elements of the chosen columns (resp. rows) remove the nonzero u-th column (resp. row) and add to all other columns (resp. rows) the u-th column (resp. row) times an arbitrary coefficient λ, regarding the a_j as formal variables. This step is to ensure that each time only nonzero columns or rows are removed.
4. Set all a_j that appeared in any of the selected rows or columns to zero, obtaining a matrix M'. Notice that M' does not depend on the choice of λ.
5. The rank of p is at least $\mathbf{b}'$ plus $\mathbf{c}'$ plus the rank of the tensor corresponding to M'.

The above steps can be iterated, interchanging the roles of A, B, and C.

Example 3.3.1. *Let*

$$p(A^*) = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & x_1 & 0 & 0 \\ x_3 & 0 & x_2 & 0 & 0 & 0 & x_1 & 0 \\ x_4 & x_3 & 0 & x_2 & 0 & 0 & 0 & x_1 \end{bmatrix}$$

Then $R(p) \geq 15$. Indeed, in the first iteration of the method, choose the first four rows and last four columns. One obtains a 4×4 matrix M' and the associated tensor p', so $R(p) \geq 8 + R(p')$. Iterating the method twice yields $R(p) \geq 8 + 4 + 2 + 1 = 15$.

Theorem 3.3.1. *(Thm. 4.1, [13]) Let $p_1 \in A_1 \otimes B_1 \otimes C_1$ and $p_2 \in A_2 \otimes B_2 \otimes C_2$ be such that $R(p_1)$ can be determined by the substitution method. Then Strassen's additivity conjecture holds for $p_1 \oplus p_2$, that is, $R(p_1 \oplus p_2) = R(p_1) + R(p_2)$.*

Proof. With each iteration of the substitution method, p_1 is modified to a tensor of lower rank living in a smaller space and o_2 is unchanged. After all applications, p_1 has been modified to zero and p_2 is still unchanged. □

Other examples can be found in [13]. This article suggests that if $\mathbf{a}' \leq 2$, then the rank of p' can be computed by the substitution method. To show that the substitution method can calculate the rank of $p \in \mathbb{K}^2 \otimes B' \otimes C'$, one needs to use the normal forms of such tensors and understand all of the cases, which turns out to be very demanding. We now rephrase Proposition 3.3.1 in terms of geometric and algebraic scopes to resume the discussion on hook-shaped spaces. We will see the same proposition in a coordinate-free manner, as well as linear spaces of tensors.

Proposition 3.3.2. *(Prop. 3.10, [6]) Let $p \in A \otimes B \otimes C$, $R(p) = r > 0$, and pick $\alpha \in A^*$ such that $p(\alpha) \in B \otimes C$ is nonzero. Consider two hyperplanes in A: the linear hyperplane $\alpha^\perp = (\alpha = 0)$ and the affine hyperplane $(\alpha = 1)$. For any $a \in (\alpha = 1)$ denote*

$$\widetilde{p}_a := p - a \otimes p(\alpha) \in \alpha^\perp \otimes B \otimes C.$$

Then

1. *there exists a choice of $a \in (\alpha = 1)$ such that $R(\widetilde{p}_a) \leq r - 1$;*
2. *if in addition, $R(p(\alpha)) = 1$, then for any choice of $a \in (\alpha = 1)$ we have $R(\widetilde{p}_a) \geq r - 1$.*

Proposition 3.3.3. *(Prop. 3.11, [6]) Suppose $W \subset B \otimes C$ is a linear subspace, $R(W) = r$. Assume $w \in W$ is a nonzero element. Then*

1. *there exists a choice of a complementary subspace $\widetilde{W} \subset W$ such that $\widetilde{W} \oplus \langle w \rangle = W$ and $R(\widetilde{W}) \leq r - 1$, and*
2. *if in addition $R(w) = 1$, then for any choice of the complementary subspace such that $\widetilde{W} \oplus \langle w \rangle = W$ we have $R(\widetilde{W}) \geq r - 1$.*

3.4 Hook-shaped spaces

We will now explore another technique for obtaining lower bounds on rank. The term "hook-shaped" spaces arises from the observation that, with the right choice of basis, the non-zero coordinates resemble a hook ⌜ in the top left corner of the matrix. The integers (e, f) indicate the width of the hook's arms. Below, we will present the formal definition along with a brief example.

Definition 3.4.1. *For nonnegative integers e, f, we say that a linear subspace $W \subset B \otimes C$ is (e, f)-**hook shaped**, if $W \subset \mathbb{K}^e \otimes C + B \otimes \mathbb{K}^f$ for some choices of linear subspaces $\mathbb{K}^e \subset B$ and $\mathbb{K}^f \subset C$.*

Example 3.4.1. *A $(1, 2)$-hook shaped subspace of $\mathbb{K}^4 \otimes \mathbb{K}^4$ has the only following possibly nonzero entries in some coordinates:*

$$\begin{bmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$$

Proposition 3.3.2 will be useful to prove the additivity of the rank for tensorial product of vector spaces if one of them is $(1,2)$-hook shaped. We start by proving a series of lemmas that we use along the proof of the additivity of the rank in the case where one of the subspaces is $(1,1)$-hook shaped. We will work with a sequence of tensors $p_0, p_1, \ldots$ in the space $A \otimes B \otimes C$, which are not necessarily direct sums. Nevertheless, for each i, we write $p'_i = \pi_{A''}\pi_{B''}\pi_{C''}(p_i)$ (that is, the "corner" of p_i relative to A', B', and C'). We define p''_i in the same way.

Lemma 3.4.1. *(Lemma 3.13, [6]) Suppose $W' \subset A' \otimes B' \otimes C'$ and $W'' \subset A'' \otimes B'' \otimes C''$ are two subspaces. Let $r'' = R(W'')$ and suppose that there exists a sequence of tensors $p_0, p_1, \ldots, p_{r''} \in A \otimes B \otimes C$ satisfying the following:*

1. *$p_0 = p$ is such that $p(A^*) = W = W' \oplus W''$ (where W, W', and W'' are associated to p, p', and p''),*
2. *$p'_{i+1} = p'_i$ for every $0 \le i < r''$,*
3. *$R(p''_{i+1}) \ge R(p''_i) - 1$ for every $0 \le i < r''$, and*
4. *$R(p_{i+1}) \le R(p_i) - 1$ for every $0 \le i < r''$,*

then the additivity of the rank holds for $W' \oplus W''$ and for each $i < r''$ we must have $p''_i \neq 0$.

Proof. From (4) we have that

$$R(p_0) - 1 \ge R(p_1) \text{ and } R(p_1) - 1 \ge R(p_2)$$

that means

$$R(p_0) \ge R(p_1) + 1 \text{ and } R(p_1) \ge R(p_2) + 1$$

Combining these two inequalities we obtain

$$R(p_0) \ge R(p_1) + 1 \ge R(p_2) + 2$$

We recursively use the same idea to get

$$R(p_0) \ge R(p_{r''}) + r''$$

so we have

$$R(W') + R(W'') = R(p'_{r''}) + r'' \le R(p_{r''}) + r'' \le R(p_0) = R(W).$$

The other inequality always holds. From (3) we conclude $p''_i \neq 0$. □

Lemma 3.4.2 tells us how to construct a single step in the sequence above.

Lemma 3.4.2. *(Lemma. 3.14, [6]) Suppose $\Sigma \subset A \otimes B \otimes C$ is a linear subspace, $p_i \in \Sigma$ is a tensor, and $\gamma \in (C'')^*$ is such that*

1. *$R(p_i''(\gamma)) = 1$,*
2. *γ preserves Σ, that is, $\Sigma(\gamma) \otimes C \subset \Sigma$, where $\Sigma(\gamma) = \{t(\gamma) : t \in \Sigma\} \subset A \otimes B$, and*
3. *$\Sigma(\gamma)$ does not have entries in $A' \otimes B'$, that is $\pi_{A''}\pi_{B''}(\Sigma(\gamma)) = 0$.*

Consider $\gamma^\perp \subset C$. Then there exists

$$p_{i+1} \in \Sigma \cap A \otimes B \otimes \gamma^\perp$$

that satisfies

i) $p'_{i+1} = p'_i$,

ii) $R(p''_{i+1}) \geq R(p''_i) - 1$ and

iii) $R(p_{i+1}) \leq R(p_i) - 1$

(for this fixed i).

Proof. For each $c \in \gamma = 1$, set

$$(\widetilde{p_i})_c = p_i - p_i(\gamma) \otimes c \in A \otimes B \otimes \gamma^\perp$$

as in Proposition 3.3.2 with the roles of A and $B \otimes C$ changed by C and $A \otimes B$. Choosing c properly, we will pick p_{i+1} among the $(\widetilde{p_i})_c$. By Proposition 3.3.2.1, there exists a choice of c such that $p_{i+1} = (\widetilde{p_i})_c$ has rank less than $R(p_i)$, that is, (iii) holds. On the other hand, since γ is in $(C'')^*$, we have

$$p''_{i+1} = (\widetilde{p_i}'')_{c''} = p''_i - p''_i(\gamma) \otimes c'',$$

where $c = c' + c''$ with $c' \in C'$ and $c'' \in C''$. By Proposition 3.3.2.2, as $R(p''_i(\gamma)) = 1$ by hypothesis, also (ii) is satisfied. Property (i) follows as $\Sigma(\gamma)$ (in particular $p_i(\gamma)$) has no entries in $A' \otimes B' \otimes C'$. Finally, $p_{i+1} \in \Sigma$ because γ preserves Σ and Σ is a linear subspace. □

Observation 3.4.1. *Note that in general is not true that every $\gamma \in (C'')^*$ we choose is preserving a previously chosen subspace Σ. Let's consider $A \otimes B \otimes C$, with $A' = A'' = B' = B'' = C' = C'' = \mathbb{C}^2$ and take $\Sigma = \langle e_{111}, e_{121}, e_{222}, e_{333}, e_{344} \rangle \subset A \otimes B \otimes C$. Pick $\gamma = e_4^* \in (C'')^*$. Now we look at $\Sigma(\gamma)$ obtained by applying the corresponding generators of Σ to γ, that is*

$$\Sigma(\gamma) = \{t(\gamma) : t \in \Sigma\} = \langle e_{34} \rangle \subset A \otimes B.$$

Now we tensorize by C,

$$\Sigma(\gamma) \otimes C = \langle e_{341}, e_{342}, e_{343}, e_{344} \rangle \not\subset \Sigma.$$

This shows that γ has to be carefully chosen for the process described in Lemma 3.4.2 to be valid.

Lemma 3.4.3. *(Lemma 3.15, [6]) Suppose $W'' \subset B'' \otimes C''$ is a $(1, f)$-hook shaped space for some integer f and $W' \subset B' \otimes C'$ is arbitrary. Fix $\mathbb{K}^1 \subset B''$ and $\mathbb{K}^f \subset C''$ as in Definition 3.4.1 for W''. Then there exists a sequence of tensors $p_0, p_1, \ldots, p_k \in A \otimes B \otimes C$ for some k that satisfies the properties of Lemma 3.4.1 and in addition $p''_k \in A'' \otimes B'' \otimes \mathbb{K}^f$ and for every i we have $p_i \in (A' \otimes B' \otimes C') \oplus (A'' \otimes (B'' \otimes \mathbb{K}^f + \mathbb{K}^1 \otimes C))$. In particular,*

1. *$p''_i((A'')^*)$ is a $(1, f)$-hook shaped space for every $i < k$,*
2. *$p''_k((A'')^*)$ is a $(0, f)$-hook shaped space, and*
3. *every p_i is on the form $p_i = (p'_i \oplus p''_i) + q_i$ where $q_i \in A'' \otimes \mathbb{K}^1 \otimes C' \subset A'' \otimes B'' \otimes C'$.*

Proof. We construct the sequence p_i by a recursive application of Lemma 3.4.2. By hypothesis, $p''_i \in A'' \otimes B \otimes \mathbb{K}^f + A'' \otimes \langle x \rangle \otimes C$ for some choice of $x \in B''$ and a fixed $\mathbb{K}^f \subset C''$. We let $\Sigma = A' \otimes B' \otimes C' \oplus A'' \otimes ((B'' \otimes \mathbb{K}^f) + \langle x \rangle \otimes C)$.

Now define p_0 corresponding to $W' \oplus W''$ by (1) of Lemma 3.4.1. Suppose we have already constructed $p_0, \ldots, p_i$ and that p''_i is not yet contained in $A'' \otimes B'' \otimes \mathbb{K}^f$. Then there exists a hyperplane $\gamma^\perp = (\gamma = 0) \subset C$ for some $\gamma \in (C'')^* \subset C^*$ such that $\mathbb{K}^f \subset \gamma^\perp$, but $p''_i \notin A'' \otimes B'' \otimes \gamma^\perp$. Equivalently, $p''_i(\gamma) \neq 0$ and $p''_i(\gamma) \subset A'' \otimes \langle x \rangle$. In particular, $R(p''_i(\gamma)) = 1$ and $\Sigma(\gamma) \subset A'' \otimes \langle x \rangle$. That means γ preserves Σ as in Lemma 3.4.2 and $\Sigma(\gamma)$ has no entries in $A' \otimes B' \otimes C'$. Thus we construct p_{i+1} using Lemma 3.4.2. Note that the dimension of the third factor of the tensor space containing p''_{i+1} is being gradually reduced, so we will eventually arrive at the case $p''_{i+1} \in A'' \otimes B'' \otimes \mathbb{K}^f$, as we wanted. □

Example 3.4.2. *We have already discussed how the p_i's are being chosen. Now we study a concrete example of the existence of this hyperplane $\gamma^\perp$. Set $A \otimes B \otimes C$ as in Observation 3.4.1, $f = 1$ and*

$$\Sigma = A' \otimes B' \otimes C' \oplus A'' \otimes (B'' \otimes \langle e_3 \rangle + \langle e_4 \rangle \otimes C)$$

Rewriting Σ we have

$$\Sigma = \langle e_{111}, e_{112}, e_{121}, e_{122}, e_{211}, e_{212}, e_{221}, e_{222} \rangle \oplus (\langle e_{333}, e_{343}, e_{433}, e_{443} \rangle + \langle e_{341}, e_{342}, e_{343}, e_{344}, e_{441}, e_{442}, e_{443}, e_{444} \rangle)$$

In this case, we take $\gamma = e_4^* \in (C'')^*$. Note that $\Sigma(\gamma) = \langle e_{34}, e_{44} \rangle = A'' \otimes \langle e_4 \rangle$, and then tensorizing by C, we get

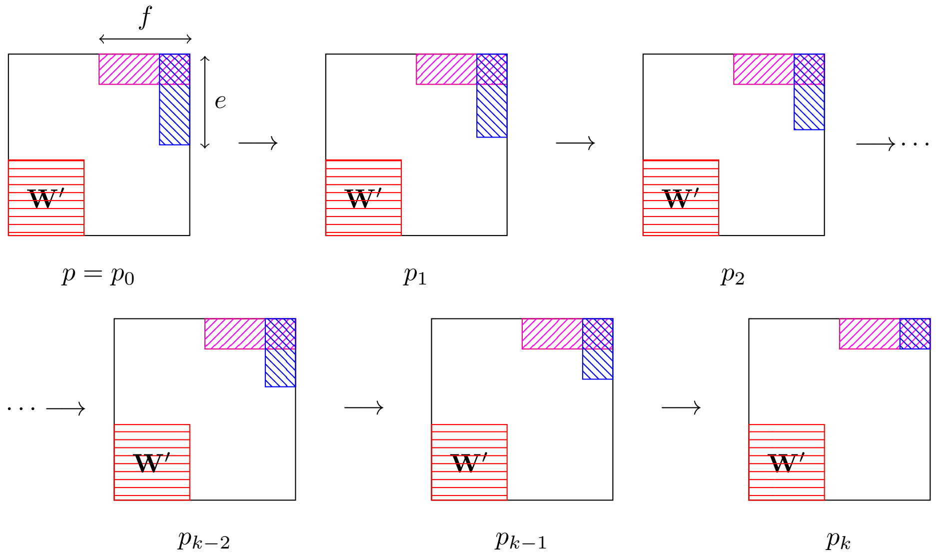


Figure 3.1: A representation of the subspaces W' and W'' (the (e,f)-hook shaped space), where $p = p_0$ as in Lemma 3.4.1.1, and the first k steps of the sequence, granted by Lemma 3.4.3. At each stage, the rank of one of the hook's sides is decreasing.

$$\Sigma(\gamma) \otimes C = \langle e_{341}, e_{342}, e_{343}, e_{344}, e_{441}, e_{442}, e_{443}, e_{444} \rangle \subset \Sigma,$$

as we expected.

Proposition 3.4.1. *(Prop. 3.12, [6]) Suppose $W'' \subset B'' \otimes C''$ is $(1,1)$-hook shaped and $W' \subset B' \otimes C'$ is an arbitrary subspace. Then the additivity of the rank holds for $W' \oplus W''$.*

Proof. The idea of the proof is to construct a sequence p_i as in Lemma 3.4.1, using Lemmas 3.4.2 and 3.4.3.

The initial elements $p_0, \dots, p_k$ of the sequence are given by Lemma 3.4.3. By this lemma and the fact that W'' is a $(1,1)$-hook shaped space, then by definition $W'' \subset \mathbb{K} \otimes C'' + B'' \otimes \mathbb{K}$, so we have in particular

$$p_i'' = \pi_{A'}\pi_{B'}\pi_{C'}(p_i) \in A'' \otimes B'' \otimes \langle y \rangle + A'' \otimes \langle x \rangle \otimes C''$$

for every $i < k$ and some choices of $x \in B''$ and $y \in C''$ and as p_k is a $(0,1)$-hook shaped space, then

$$p_k \in A' \otimes B' \otimes C' \oplus A'' \otimes B'' \otimes \langle y \rangle.$$

In the reminder of the proof, we use Lemma 3.4.2 with $\Sigma = A' \otimes B' \otimes C' \oplus A'' \otimes B \otimes (C' \oplus \langle y \rangle)$, and verify conditions (1)-(3) of the Lemma. Suppose we have constructed $p_{k+1}, \dots, p_j$ for some $j \geq k$ satisfying (2)-(4) of Lemma 3.4.1 such that

$$p_j \in \Sigma.$$

This implies $p''_j \in A'' \otimes B'' \otimes \langle y \rangle$. If $p''_j = 0$ then by Lemma 3.4.1 we are done, as $j = r''$. Suppose $p''_j \neq 0$, then there exists $\beta \in (B'')^* \subset B^*$ such that $p''_j(\beta) \neq 0$. Also, $p''_j(\beta) \in A'' \otimes \langle y \rangle$, so $R(p''_j(\beta)) = 1$.

We have checked the first condition to use Lemma 3.4.2. Let's check the second condition. Now consider $t \in \Sigma$ fixed but arbitrary. Note that $t = \sum_i t_i$, where

$$t_i = a'_i \otimes b'_i \otimes c'_i + a''_i \otimes b'^*_i \otimes c'^*_i + \widehat{a}''_i \otimes \widehat{b}''_i \otimes \widehat{c}'_i + \widetilde{a}''_i \otimes \widetilde{b}'_i \otimes y + \dot{a}''_i \otimes \dot{b}''_i \otimes y,$$

and $a'_i \in A', a''_i, \widehat{a}''_i, \widetilde{a}''_i, \dot{a}''_i \in A''$, $\widehat{b}''_i, \dot{b}''_i \in B'', b'^*_i, b'_i, \widetilde{b}'_i \in B'$, and $c'^*_i, c'_i, \widehat{c}'_i \in C'$. As β is a linear transformation, it is enough to check the preservation for t_i as above, and by the linearity we can conclude β preserves Σ. When calculating $t_i(\beta)$, we note that all summand that has an element from B' in its second factor goes to zero, and we have contractions in the ones whose second factor lies in B'', that is,

$$t_i(\beta) = \beta(\widehat{b}''_i)\widehat{a}''_i \otimes \widehat{c}'_i + \beta(\dot{b}''_i)\dot{a}''_i \otimes y \in A'' \otimes (C' + \langle y \rangle), \tag{3.4}$$

so we checked $\Sigma(\beta) \otimes B \subset \Sigma$, and β preserves Σ.

The third condition is satisfied since $\Sigma(\beta)$ does not have entries in $A' \otimes C'$ by equation (3.4). Then p_{j+1} is produced by Lemma 3.4.2 and we stop after having $p_{r''}$ constructed (condition (4) grants the end of the process), proving the additivity of the rank for $W' \oplus W''$. □

To complete our discussion on hook-shaped spaces we will see an analog of the previous proposition on $(1,1)$-hook-shaped spaces concerning $(1,2)$-hook-shaped spaces. We slightly modify our hypothesis regarding the base field, as we cannot go forward without an algebraically closed field. The following lemma guarantees the key condition to construct the desired sequence to prove additivity for hook-shaped spaces of bigger dimensions.

Lemma 3.4.4. *(Lemma 3.16, [6]) Suppose $\mathbb{K}$ is an algebraically closed field (of any characteristic) and $p \in A \otimes B \otimes \mathbb{K}^2$, $p \neq 0$. Then at least one of the following occurs:*

1. *there exists a rank-one matrix in $p(A^*) \subset B \otimes \mathbb{K}^2$ or*
2. *for any $x \in B$ there exists a rank-one matrix in $p(x^\perp) \subset A \otimes \mathbb{K}^2$, where $x^\perp \subset B^*$ is the hyperplane defined by x.*

Proof. Without loss of generality, we may suppose p is concise. We can just replace A and B with smaller spaces if needed. Then we have the following two cases:

1. $dim A \geq dim B$: Consider the linear variety $L = \mathbb{P}(p(A^*))$ and the Segre variety $X = Seg(\mathbb{P}(B) \times \mathbb{P}(\mathbb{K}^2))$, both contained in the projective space $\mathbb{P}(B \otimes \mathbb{K}^2)$. They

have dimensions $dim(L) = dim(A) - 1$ (as p is concise), $dim(X) = 2 + dim(B) - 2 = dim(B)$, and $dim(\mathbb{P}(B \otimes \mathbb{K}^2)) = 2dimB - 1$. Now,

$$dim(\mathbb{P}(p(A^*))) + dim(Seg(\mathbb{P}(B) \times \mathbb{P}(\mathbb{K}^2))) = dimA - 1 + dimB \geq 2dimB - 1.$$

By Theorem 2.3.2, we obtain

$$\mathbb{P}(p(A^*)) \cap Seg(\mathbb{P}(B) \times \mathbb{P}(\mathbb{K}^2) \neq \varnothing.$$

2. $dimA < dimB$: In this case, we take a look at $p(B^*)$. Again, as p is concise, Let $x \in B$. Then $L = \mathbb{P}(p(x^{\perp}))$ has dimension $dim(B) - 2$, $X = Seg(\mathbb{P}(A) \times \mathbb{P}(\mathbb{K}^2)$ has dimension equal to $dim(A)$ and $dim(\mathbb{P}(A \otimes \mathbb{K}^2)) = 2dimA - 1$. Then, using that $dimA + 1 \leq dimB$, we get

$$dim(X) + dim(L) = dimA + dimB - 2 \geq 2dimA - 1 = dim(\mathbb{P}(A \otimes \mathbb{K}^2)).$$

That means by Theorem 2.3.2, that $\mathbb{P}(p(B^*)) \cap Seg(\mathbb{P}(A) \times \mathbb{P}(\mathbb{K}^2) \neq \varnothing$.

□

Proposition 3.4.2. *(Prop. 3.17, [6]) Suppose $\mathbb{K}$ is an algebraically closed field, $W'' \subset B'' \otimes C''$ is a $(1,2)$-hook shaped space, and $W' \subset B' \otimes C'$ is an arbitrary subspace. Then the additivity of the rank holds for $W' \oplus W''$.*

Proof. We want a sequence $p_0, \dots, p_{r''} \in A \otimes B \otimes C$ with properties (1)-(4) of Lemma 3.4.1, and the initial elements $p_0, \dots, p_k$ are constructed in such a way that $p_k \in A' \otimes B' \otimes C' \oplus A'' \otimes (\langle x \rangle \otimes C'' \oplus B'' \otimes \mathbb{K}^2)$. Here $x \in B''$ is such that $W'' \subset \langle x \rangle \otimes C'' + B'' \otimes \mathbb{K}^2$. We can clean the part of the hook of size 1 as in Proposition 3.4.1.

Now we work with the remaining space of $\mathbf{b}'' \times 2$ matrices. We cannot use the same ideas as in Proposition 3.4.1 for p_i'' because the process could leave some wreck in the other parts of the tensor, so we need to control the wreck in such a way it does not affect p_i' (remember condition 2).

Note that what is left to do is not just the Strassen's additivity of the rank in the case $\mathbf{c}'' = 2$ since p_k may have already non-trivial entries in another block, the one corresponding to $A'' \otimes B'' \otimes C'$ (the small tensor q_k in the statement of Lemma 3.4.3). In other words, for the case of $(1,1)$-hook shaped spaces, we already know there is no wreck because the edge of the hook we are not dealing with has already rank one. In the $(1,2)$-case, we need to ensure a rank-one tensor to continue with the next term of the sequence (whose existence is granted by Lemma 3.4.2).

We set $\Sigma = A' \otimes B' \otimes C' \oplus A \otimes (B \otimes \mathbb{K}^2 \oplus \langle x \rangle \otimes C')$. To construct p_{j+1} we use Lemma 3.4.4. Thus either there exists $\alpha \in (A'')^*$ such that $R(p_j''(\alpha)) = 1$, or there exists

$\beta \in x^\perp \subset (B'')^*$ such that $R(p''_j(\beta)) = 1$. In both cases, we apply Lemma 3.4.2 with the roles of A and C swapped or the roles of B and C swapped (verify the conditions). We stop after constructing $p_{r''}$, so the desired sequence exists and this proves the statement. $\square$

In general, if we have the case of a $(1, f)$-hook shaped space, we can guarantee the existence of the rank-one matrix necessary to construct the sequence in Lemma 3.4.1 by setting $Z = \mathbb{P}(B \otimes \mathbb{K}^f)$, $X = Seg(\mathbb{P}(B) \times \mathbb{P}(\mathbb{K}^f))$ and $L = \mathbb{P}(p(A^*))$, where $dim(Z) = f\mathbf{b} - 1$, $dim(X) = (\mathbf{b} - 1) + (f - 1) = \mathbf{b} + f - 2$, and $dim(L) = \mathbf{a} - 1$. Using Theorem 2.3.2, we will get additivity of the tensor rank if we guarantee that

$$dim(X) + dim(L) = \mathbf{a} + \mathbf{b} + f \geq dim(Z) = f\mathbf{b} + 2.$$

Chapter 4

Additivity of the tensor rank for small tensors

In the previous chapter we concluded that if we wanted to prove the additivity of the rank, we could start by constructing a sequence of tensors satisfying some special conditions. In every step of the sequence, finding a rank-one matrix in one of the subspaces W'' or W' became fundamental to guarantee that the next element of the sequence exists. We will establish this formally, by proving that in the event of a rank-one matrix in one of the linear spaces, we have two possibilities: the additivity of the rank holds or there exists a "smaller" example where the additivity of the rank fails.

4.1 Combinatorial study of the decomposition

Recalling Notation 3.2.1, we now explore a combinatorial splitting of the decomposition by distinguishing seven types of rank-one matrix in a given vector space V:

Lemma 4.1.1. *(Lemma 4.1, [6]) Every element of $V \subset B \otimes C$ lies in one of the following subspaces of $B \otimes C$:*

i) $B' \otimes C'$, $B'' \otimes C''$ (Prime, Bis),

ii) $E' \otimes (C' \oplus F'')$, $E'' \otimes (F' \oplus C'')$ (HL, HR),
$(B' \oplus E'') \otimes F'$, $(E' \oplus B'') \otimes F''$ (VL, VR),

iii) $(E' \oplus E'') \otimes (F' \oplus F'')$ (Mix).

Proof. Let $b \otimes c \in V_{Seg}$ a rank-one matrix. We set $b = b' + b''$, $c = c' + c''$, where $b' \in B', b'' \in B'', c' \in C', c'' \in C''$. We take the projections

$$\pi_{B'}(b \otimes c) = b'' \otimes c \in B'' \otimes (F' \oplus C'') \tag{4.1}$$

$$\pi_{B''}(b \otimes c) = b' \otimes c \in B' \otimes (C' \oplus F'') \tag{4.2}$$

$$\pi_{C'}(b \otimes c) = b \otimes c'' \in (E' \oplus B'') \otimes C'' \tag{4.3}$$

$$\pi_{C''}(b \otimes c) = b \otimes c' \in (B' \oplus E'') \otimes C' \tag{4.4}$$

As $b \otimes c$ is a rank-one matrix, $(b', b'') \neq (0, 0) \neq (c', c'')$. Now we study the following cases:

1. If $b', b'', c', c'' \neq 0$, then $b \otimes c \in (E' \oplus E'') \otimes (F' \oplus F'')$ (Mix case). This is because if for instance $c' \neq 0$, we must have by (4.4) that $b'' \in E''$, and if $c'' \neq 0$, (4.3) gives us $b' \in E'$. If we use the same argument for $b' \neq 0$ and $b'' \neq 0$, we realize we are in the Mix case.

2. If $b', b'' \neq 0$ and $c' = 0$, then $b \otimes c = b \otimes c'' \in (E' \oplus B'') \otimes F''$ (VR case).

3. If $b', b'' \neq 0$ and $c'' = 0$, then $b \otimes c = b \otimes c' \in (B' \oplus E'') \otimes F''$ (VL case).

4. If $b' = 0$, then we have two subcases:

   (a) $c' = 0$ and therefore $b \otimes c = b'' \otimes c'' \in B'' \otimes C''$ (Bis case).

   (b) $c' \neq 0$ and with this $b \otimes c = b'' \otimes c \in E'' \otimes (F' \oplus C'')$ (HR case).

5. If $b'' = 0$, again we have two subcases:

   (a) $c'' = 0$ and therefore $b \otimes c = b' \otimes c' \in B' \otimes C'$ (Prime case).

   (b) $c'' \neq 0$ and with this $b \otimes c = b' \otimes c \in E' \otimes (C' \oplus F'')$ (HL case).

This concludes the proof. □

Observation 4.1.1. *Note that the spaces Prime and Bis are contained in the original direct summands, so they can be easily manipulated. It gets complex when we look at the HR, HL, VR, VL, and Mix subspaces because they stick out of the original summands in one particular direction (in the case of VR, VL, HR, HL) or all directions (Mix case). The other consideration is that the subspaces generally do not have an empty intersection.*

Notation 4.1.1. *We choose a basis $\mathcal{B}$ of V with the following conditions:*

1. *$\mathcal{B}$ consists of rank-one matrices only;*

2. *$\mathcal{B} = Prime \sqcup Bis \sqcup HL \sqcup HR \sqcup VL \sqcup VR \sqcup Mix$, where every element of this disjoint union is a finite set of rank-one matrices of the respective type.*

3. *$\mathcal{B}$ has as many elements of $Prime$ and Bis as possible, under the first two conditions.*

4. *$\mathcal{B}$ has as many elements of HL, HR, VL, and VR as possible, under the previous conditions.*

We set $\mathbf{prime}, \mathbf{bis}, \mathbf{vl}, \mathbf{vr}, \mathbf{hl}, \mathbf{hr}, \mathbf{mix}$ *as the dimension of the corresponding subspaces. The choice of* $\mathcal{B}$ *does not need to be unique, but we fix one going forward. We also note that the numbers* $\mathbf{prime}, \mathbf{bis}, \mathbf{mix}$ *we just defined are uniquely determined by* V*, and that is because of the way we choose our basis, but there may be some non-uniqueness between* $\mathbf{hl}, \mathbf{hr}, \mathbf{vl}$*, and* $\mathbf{vr}$*.*

Throughout this chapter, we will see how this manner of choosing our simple tensor generator set allows us to deduce some inequalities concerning the additivity of the rank. After the preliminary discussion, we ended up with seven nonnegative integers ($\mathbf{prime}, \mathbf{bis}, \dots, \mathbf{mix}$) for each decomposition.

Proposition 4.1.1. *(Prop. 4.3, [6]) The following inequalities hold:*

i) $\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \min\{\mathbf{mix}, \mathbf{e'f'}\} \geq R(W')$*;*

ii) $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \min\{\mathbf{e''f''}, \mathbf{mix}\} \geq R(W'')$*;*

iii) $\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \min\{\mathbf{mix} + \mathbf{hr}, (\mathbf{e'} + \mathbf{e''})\mathbf{f'}\} \geq R(W') + \mathbf{e''}$*;*

iv) $\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \min\{\mathbf{mix} + \mathbf{vr}, (\mathbf{f'} + \mathbf{f''})\mathbf{e'}\} \geq R(W') + \mathbf{f''}$*;*

v) $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \min\{\mathbf{mix} + \mathbf{hl}, (\mathbf{e'} + \mathbf{e''})\mathbf{f''}\} \geq R(W'') + \mathbf{e'}$*;*

vi) $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \min\{\mathbf{mix} + \mathbf{vl}, (\mathbf{f'} + \mathbf{f''})\mathbf{e''}\} \geq R(W'') + \mathbf{f'}$*.*

Proof. To prove the inequalities we work along with the projections to ensure the spaces involved contain the W', W'', E', E'', F', and F'' in the respective cases.

We consider $\pi_{B''}\pi_{C''}$. From the above notation we have $\pi_{B''}\pi_{C''}(\mathcal{B})$ span $\pi_{B''}\pi_{C''}(V)$, and it contains W' (remember that $W' \oplus W'' \subset V$ and therefore $W' \subset \pi_{B''}\pi_{C''}(V)$). From this we have that

$$dim(\pi_{B''}\pi_{C''}(V)) \geq R(W').$$

We also know that the elements of the basis whose components lie in B'' and C'' are in the kernel of the composition, so the surviving elements will be the ones from $Prime, Mix, VL$, and HL. This gives us

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{mix} \geq dim(\pi_{B''}\pi_{C''}(V)) \geq R(W').$$

Consider $\pi_{B''}\pi_{C''}(Mix) = \pi_{B''}\pi_{C''}((E' \oplus E'') \otimes (F' \oplus F''))$. It is contained in $E' \otimes F'$, so at most $\mathbf{e'f'}$ linearly independent matrices can be picked up to span $\pi_{B''}\pi_{C''}(V)$. We then obtain

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{e'f'} \geq dim(\pi_{B''}\pi_{C''}(V)) \geq R(W')$$

Combining these two inequalities we get i). To show ii) we use a similar argument, changing $\pi_{B''}\pi_{B''}$ by $\pi_{C'}\pi_{B'}$, W' by W'', $Prime$ by Bis, **hl** by **hr**, **vl** by **vr**, $\mathbf{e}'$ by $\mathbf{e}''$, and $\mathbf{f}'$ by $\mathbf{f}''$.

Now we prove iii). We can assume W' is concise. In the proof of Lemma 3.2.2 we got

$$dim(\pi_{C''}(V)) = R(W) - dim(W'') \geq R(W') + e'' \ (3.2)$$

As $\pi_{C''}$ kills all matrices from Bis and VR, we get

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{hr} + \mathbf{mix} \geq dim(\pi_{C''}(V)) \geq R(W') + e''$$

Note that $\pi_{C''}(HR \cup Mix) \subset (E' \oplus E'') \otimes F'$, so we can replace $\mathbf{hr} + \mathbf{mix}$ by $(\mathbf{e}' + \mathbf{e}'')\mathbf{f}'$, proving iii). The remaining inequalities are analogous to the proved ones swapping the roles of $B, C, '$ and $''$. □

Proposition 4.1.2. *(Prop. 4.4, [6]) If one among E', E'', F' or F'' is zero, then*

$$R(W) = R(W') + R(W'').$$

Proof. Let's assume without loss of generality that $E' = \{0\}$, so $\mathbf{e}' = 0$. Due to the way we chose the elements of $\mathcal{B}$, any candidate to become a member of HL, would be first elected to $Prime$, in a similar way VR is consumed by Bis and Mix is consumed by HR. So $\mathbf{hl} = \mathbf{vr} = \mathbf{mix} = 0$ and

$$R(W) = \mathbf{prime} + \mathbf{bis} + \mathbf{hr} + \mathbf{vl}.$$

and when we sum the inequalities 4.1.1.i and 4.1.1.ii of Proposition 4.1.1, we have

$$R(W') + R(W'') \leq (\mathbf{prime} + \mathbf{hr}) + (\mathbf{vl} + \mathbf{bis}) = R(W),$$

as we wanted. □

Corollary 4.1.1. *(Cor. 4.5, [6]) Assume that the additivity of the rank fails for W' and W'', that is,*

$$d = R(W') + R(W'') - R(W' \oplus W'') > 0.$$

Then the following inequalities hold:

a) $\mathbf{mix} \geq d \geq 1.$

b) $\mathbf{hl} + \mathbf{hr} + \mathbf{mix} \geq \mathbf{e}' + \mathbf{e}'' + d \geq 3.$

c) $\mathbf{vl} + \mathbf{vr} + \mathbf{mix} \geq \mathbf{f}' + \mathbf{f}'' + d \geq 3.$

Proof. We consider the inequalities i) and ii) from Proposition 4.1.1 and their sum

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{mix} \geq R(W')$$
$$\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \mathbf{mix} \geq R(W'')$$
$$\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \mathbf{prime} + \mathbf{hl} + \mathbf{vl} + 2\mathbf{mix} \geq R(W'') + R(W')$$

The way we chose our basis $\mathcal{B}$ gives us $\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{mix} = R(W)$, so we have

$$R(W) + \mathbf{mix} \geq R(W'') + R(W')$$

which implies

$$\mathbf{mix} \geq R(W') + R(W'') - R(W' \oplus W'') = d,$$

as we wanted. Using a similar idea, we can prove b) starting from inequalities 4.1.1.iii) and 4.1.1.v): We sum the two inequalities

$$\mathbf{prime} + \mathbf{hl} + \mathbf{vl} + \mathbf{mix} + \mathbf{hr} \geq R(W') + \mathbf{e}''$$
$$\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \mathbf{mix} + \mathbf{hl} \geq R(W'') + \mathbf{e}'$$

to get

$$\mathbf{bis} + \mathbf{hr} + \mathbf{vr} + \mathbf{prime} + \mathbf{hl} + \mathbf{vl} + 2\mathbf{mix} + \mathbf{hl} + \mathbf{hr} \geq R(W') + R(W'') + \mathbf{e}' + \mathbf{e}''$$

and from this

$$R(W) + \mathbf{hl} + \mathbf{hr} + \mathbf{mix} \geq R(W') + R(W'') + \mathbf{e}' + \mathbf{e}'',$$
$$\mathbf{hl} + \mathbf{hr} + \mathbf{mix} \geq R(W') + R(W'') - R(W) + \mathbf{e}' + \mathbf{e}'' = d + \mathbf{e}' + \mathbf{e}''$$

and as the additivity fails, $d > 0$ and $\mathbf{e}'$ and $\mathbf{e}''$ are non-zero by Proposition 4.1.2, so

$$\mathbf{hl} + \mathbf{hr} + \mathbf{mix} \geq d + \mathbf{e}' + \mathbf{e}'' \geq 3.$$

For c) we use 4.1.1.iv), 4.1.1.vi) and again Proposition 4.1.2. □

Example 4.1.1. *We can take a look at the following subspace $W \subset \mathbb{C}^4 \otimes \mathbb{C}^4$ seen as a matrix composed by four 2×2 blocks, which we can associate to a tensor $p \in \mathbb{C}^6 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$:*

$$W = \begin{bmatrix} a & b & b & 0 \\ c & f & f & d \\ c & f & f & d \\ 0 & e & e & f \end{bmatrix}$$

Here we can see that the 2×2 right bottom corner should be Bis, but is generated by elements of HL (the e's), Mix (the f's), and VR (the d's). So $\mathbf{bis} = 0$. *VL is the subspace corresponding to the c's and Prime is composed by the a in the left top corner. From this, we conclude that*

$$\mathbf{prime} = \mathbf{vl} = \mathbf{vr} = \mathbf{hl} = \mathbf{hr} = \mathbf{mix} = 1 \text{ and } \mathbf{bis} = 0.$$

4.2 Repletion and Digestion

Definition 4.2.1. *Consider a pair of linear spaces $W' \subset B' \otimes C'$ and $W'' \subset B'' \otimes C''$ with a fixed minimal decomposition $V = \langle V_{Seg} \rangle \subset B \otimes C$ and $Prime, \dots, Mix$ as defined above. We say (W', W'') is* ***replete*** *if $Prime \subset W'$ and $Bis \subset W''$.*
For any fixed W', W'' (and V) we also denote the repletion of (W', W'') as $({}^{\mathcal{R}}W', {}^{\mathcal{R}}W'')$, where

$$ {}^{\mathcal{R}}W' := W' + \langle Prime \rangle,\ {}^{\mathcal{R}}W'' := W'' + \langle Bis \rangle, $$
$$ \text{and } {}^{\mathcal{R}}W := {}^{\mathcal{R}}W' \oplus {}^{\mathcal{R}}W''. $$

Proposition 4.2.1. *(Prop 4.8, [6]) For any (W', W''), we have*

$$ R(W') \leq R({}^{\mathcal{R}}W') \leq R(W') + (dim({}^{\mathcal{R}}W') - dim W'), $$
$$ R(W'') \leq R({}^{\mathcal{R}}W'') \leq R(W'') + (dim({}^{\mathcal{R}}W'') - dim W''), $$
$$ R({}^{\mathcal{R}}W) = R(W). $$

In particular, if the additivity of the rank fails for (W', W''), then it also fails for $({}^{\mathcal{R}}W', {}^{\mathcal{R}}W'')$. Moreover,

i) V is a minimal decomposition of ${}^{\mathcal{R}}W$; in particular, the same distinguished basis

$$ Prime \sqcup Bis \sqcup \cdots \sqcup Mix $$

works for both W and ${}^{\mathcal{R}}W$;

ii) $({}^{\mathcal{R}}W', {}^{\mathcal{R}}W'')$ is a replete pair;

iii) the gaps $R({}^{\mathcal{R}}W') - dim({}^{\mathcal{R}}W')$, $R({}^{\mathcal{R}}W'') - dim({}^{\mathcal{R}}W'')$ and $R({}^{\mathcal{R}}W) - dim({}^{\mathcal{R}}W)$ are at most (respectively) $R(W') - dim(W')$, $R(W'') - dim(W'')$ and $R(W) - dim(W)$.

Proof. Since ${}^{\mathcal{R}}W'' \supset W''$, the inequality $R(W'') \leq R({}^{\mathcal{R}}W'')$ comes right away. Also, we have that ${}^{\mathcal{R}}W''$ is spanned by W'' and $dim({}^{\mathcal{R}}W'') - dim(W'')$ additional matrices that are chosen out of $Prime$ (rank-one matrices); That means

$$ R({}^{\mathcal{R}}W'') \leq R(W'') + dim({}^{\mathcal{R}}W'') - dim(W''). $$

The inequality regarding W' is analogous. In addition, since V is a decomposition of ${}^{\mathcal{R}}W$ we have ${}^{\mathcal{R}}W \subset V$, so $R({}^{\mathcal{R}}W) \leq dim(V) = R(W)$, and as ${}^{\mathcal{R}}W \supset W$, $R(W) \leq R({}^{\mathcal{R}}W)$, that way we have $R(W) = R({}^{\mathcal{R}}W)$.

From this we conclude V is also a minimal decomposition for ${}^{\mathcal{R}}W$, and the same basis works for both W and its repletion. From the inequalities we just showed, we obtain iii). □

Lemma 4.2.1. *(Lemma 4.9, [6]) If* $R(W') + e'' = R(W) - dimW''$*, then* $W'' =^{\mathcal{R}} W''$*. Analogous statements are true for the other equalities coming from replacing* $\leq$ *by* $=$ *in Lemma 3.2.2.*

Proof. By Lemma 3.2.2 applied to ${}^{\mathcal{R}}W =^{\mathcal{R}} W' \oplus^{\mathcal{R}} W''$ and by the previous proposition we have

$$R({}^{\mathcal{R}}W) - \mathbf{e}'' \geq R({}^{\mathcal{R}}W') + dim({}^{\mathcal{R}}W'') \geq R(W') + dim(W'') = R(W) - \mathbf{e}'' = R({}^{\mathcal{R}}W) - \mathbf{e}''$$

and the inequalities are in fact equalities. We also have that $dim({}^{\mathcal{R}}W'') = dim(W'')$, and as $W'' \subset^{\mathcal{R}} W''$ we get that W'' and its repletion are equal. □

Now we consider the complement of $\langle Prime\rangle$ in W' and $\langle Bis\rangle$ in W'', where (W', W'') is a replete pair.

Definition 4.2.2. *Suppose* S' *and* S'' *denote the following linear subspaces:*

$$S' = \langle Bis \sqcup HR \sqcup HL \sqcup VR \sqcup VL \sqcup Mix\rangle \cap W' \tag{4.5}$$

$$S'' = \langle Prime \sqcup HR \sqcup HL \sqcup VR \sqcup VL \sqcup Mix\rangle \cap W''. \tag{4.6}$$

We call (S', S'') *the* ***digested version*** *of* (W', W'')*.*

Lemma 4.2.2. *(Lemma 4.11, [6]) If* (W', W'') *is replete, then* $W' = \langle Prime\rangle \oplus S'$ *and* $W'' = \langle Bis\rangle \oplus S''$*.*

Proof. Note that $\langle Prime\rangle$ and S' are contained in W', so $\langle Prime\rangle + S' \subset W'$. As $\langle Prime\rangle$, $\langle Bis\rangle$, $\langle HR\rangle$, $\langle HL\rangle$, $\langle VL\rangle$, $\langle VR\rangle$, and $\langle Mix\rangle$ are all linearly independent, and span V, if we consider $S' + \langle Prime\rangle$ we are adding more elements than the needed to span W' (if we want exactly W', we would need to add $\langle Prime\rangle \cap W'$).

That means $W' \subset S' + \langle Prime\rangle$ and therefore $W' = S' + \langle Prime\rangle$. We also know that $S' \cap \langle Prime\rangle = \{0\}$ because they are linearly independent. Then, $W' = S' \oplus \langle Prime\rangle$. The result for W'' is analogous. □

Showing the additivity of the rank can be done in the following sense: If the additivity of the rank does not hold for (W', W''), then it also does not hold for (S', S''). Suppose (W', W'') is replete, define S' and S'' as above and set $S = S' \oplus S''$. Lemma 4.2.2 gives us

$$W = S \oplus \langle Prime, Bis\rangle, \tag{4.7}$$

so we must have

$$R(W) \leq R(S) + \mathbf{prime} + \mathbf{bis}$$

which is the same that

$$R(W) - \mathbf{prime} - \mathbf{bis} \leq R(S).$$

Again, by (4.7), we have $S \subset \langle HR, HL, VR, VL, Mix \rangle$, so

$$R(S) \leq \mathbf{hl} + \mathbf{hr} + \mathbf{vl} + \mathbf{vr} + \mathbf{mix} = R(W) - \mathbf{prime} - \mathbf{bis},$$

so we have $R(S) = R(W) - \mathbf{prime} - \mathbf{bis} = \mathbf{hl} + \mathbf{hr} + \mathbf{vl} + \mathbf{vr} + \mathbf{mix}$. Besides,

1. $\langle HR, HL, VR, VL, Mix \rangle \subset B \otimes C$,
2. $dim(\langle HR, HL, VR, VL, Mix \rangle) = R(S)$,
3. $\mathbb{P}(\langle HR, HL, VR, VL, Mix \rangle) = \langle HR, HL, VR, VL, Mix \rangle_{Seg}$, as the elements of $\mathcal{B}$ are rank-one matrices, and
4. $S' \oplus S'' = S \subset \langle HR, HL, VR, VL, Mix \rangle$,

that means $\langle HR, HL, VR, VL, Mix \rangle$ is a minimal decomposition of $S = S' \oplus S''$ (see Definition 3.2.1). For this minimal decomposition there is no tensor of type $Prime$ or Bis, so the pair (S', S'') is vacuously replete by definition. In addition, if S' contains a rank-one matrix, our choice of $\mathcal{B}$ would force this rank-one matrix to be in the span of $Prime$, a contradiction.

Let's analyze the additivity of the rank. Suppose that $R(S) = R(S') + R(S'')$. Then

$$R(W) = R(S) + \mathbf{prime} + \mathbf{bis} = (R(S') + \mathbf{prime}) + (R(S'') + \mathbf{bis}) \geq R(W') + R(W'')$$

so we conclude $R(W) = R(W') + R(W'')$. The previous discussion can be summarized in the following lemma:

Lemma 4.2.3. *(Lemma 4.12, [6]) Suppose (W', W'') is replete, define S' and S'' as above and let $S = S' \oplus S''$. Then*

i) $R(S) = R(W) - \mathbf{prime} - \mathbf{bis} = \mathbf{hl} + \mathbf{hr} + \mathbf{vl} + \mathbf{vr} + \mathbf{mix}$, and the space $\langle HR, HL, VR, VL, Mix \rangle$ determines a minimal decomposition of S. In particular (S', S'') is replete and both spaces S' and S'' contain no rank-one matrices.

ii) If the additivity of the rank $R(S) = R(S') + R(S'')$ holds for S, then it also holds for W, that is, $R(W) = R(W') + R(W'')$.

To end this section, we recall our results on additivity of the rank for $(1, f)$-hook shaped spaces and combine with the ones for repletion and digestion to get the next corollary:

Corollary 4.2.1. *(Cor. 4.13, [6]) Suppose that $W = W' \oplus W''$ is as in Notation 3.1.2, and $\mathbf{e}'', \mathbf{f}''$ as in Notation 3.2.1. If*

i) $\mathbb{K}$ is an arbitrary field, $\mathbf{e}'' \leq 1$ and $\mathbf{f}'' \leq 1$, or *ii)* $\mathbb{K}$ *is an algebraically closed field,* $\mathbf{e}'' \leq 1$ *and* $\mathbf{f}'' \leq 2$

then the additivity of the rank $R(W) = R(W') + R(W'')$ *holds.*

Proof. By Proposition 4.2.1 and Lemma 4.2.3, we can assume W is replete and equal to its digested version. Then $Bis = \varnothing$ and therefore we must have

$$W'' \subset E'' \otimes C'' + B'' \otimes F''.$$

In particular, W'' is either a $(1,1)$-hook shaped space or a $(1,2)$-hook shaped space. From Propositions 3.4.1 and 3.4.2 we deduce the corollary. □

Observation 4.2.1. *Note that we have another proof of the additivity of the rank for the cases we studied in Section 3.4, that is, the (1,1) and (1,2) hook-shaped spaces. It might seem that our result on (1,1)-hook shaped spaces turns out to be irrelevant, but our first result on (1,2)-hook shaped spaces involves ideas from algebraic geometry, this forces us to work under the hypothesis of* $\mathbb{K}$ *being algebraically closed (see the end of the Introduction).*

4.3 Three Main Theorems

We summarize our study of the tensor rank's additivity through the following results:

Theorem 4.3.1. *(Thm. 4.14, [6]) Let* $\mathbb{K}$ *an arbitrary field base, and let* $p' \in A' \otimes B' \otimes C'$ *any tensor,* $p'' \in A'' \otimes B'' \otimes C''$ *is a concise tensor, and* $R(p'') \leq \mathbf{a}'' + 2$*. Then*

$$R(p' \oplus p'') = R(p') + R(p'').$$

If we swap the roles of A *and* $'$ *with* B, C *and* $''$*, respectively, we get analogous results.*

Proof. The concision of p'' implies $W'' = p''(A''^{*})$ has dimension $\mathbf{a}''$. Corollary 4.2.1.i) gives us the additivity in the case $\mathbf{e}' \leq 1$ and $\mathbf{f}' \leq 1$, so we can assume $\mathbf{e}'' \geq 2$ or $\mathbf{f}'' \geq 2$. Let's say that $\mathbf{e}'' \geq 2$. Then by hypothesis $R(p'') - dim(W'') \leq \mathbf{a}'' + 2 - \mathbf{a}'' = 2 \leq \mathbf{e}''$. That is,

$$R(p'') - dim(W'') \leq \mathbf{e}''.$$

By Corollary 3.2.1 the additivity must hold. □

The next two theorems will be applications of Theorem 4.3.1 along with results from [19] and [11].

Theorem 4.3.2. *(Thm. 4.15, [6]) Suppose the base field is* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$*, and assume* $p' \in A' \otimes B' \otimes C'$ *is any tensor, while* $p'' \in A'' \otimes \mathbb{K}^3 \otimes \mathbb{K}^3$ *is concise for any* A'' *such that* $\mathbf{a}'' < 3$*. Then the additivity of the rank holds:* $R(p' \oplus p'') = R(p') + R(p'')$*.*

Proof. By Theorem 6 of [19], as $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we have that $R(p'') \leq \mathbf{a}'' + 2$ and from Theorem 4.3.1 the claim is proved. □

Theorem 4.3.3. *(Thm. 4.16, [6]) Suppose the base field $\mathbb{K}$ is such that the maximal rank of a tensor in $\mathbb{K}^3 \otimes \mathbb{K}^3 \otimes \mathbb{K}^3$ is at most 5. Assume $R(p'') \leq 6$. Then independently of p', the additivity of the rank holds.*

Proof. Without loss of generality, we may assume p'' is concise in $A'' \otimes B'' \otimes C''$. If any of $\mathbf{a}'', \mathbf{b}''$ or $\mathbf{c}''$ is at most 2, the claim follows by Ja'Ja'-Takche Theorem ([11]).

If any of $\mathbf{a}'', \mathbf{b}''$ or $\mathbf{c}''$ is at least 4, say $\mathbf{a}'' \geq 4$, then since $R(p'') \leq 6$ we have $R(p'') \leq \mathbf{a}'' + 2$, and therefore the result follows by Theorem 4.3.1.

The remaining case $\mathbf{a}'' = \mathbf{b}'' = \mathbf{c}'' = 3$ is granted by our assumption on the field, again using Theorem 4.3.1. □

Corollary 4.3.1. *(Theorem 4.16, [6]) If $\mathbb{K}$ is an algebraically closed field of characteristic $\neq 2$ or $\mathbb{R}$ (in particular, if $\mathbb{K} = \mathbb{C}$), $p' \in A' \otimes B' \otimes C'$, $p'' \in A'' \otimes B'' \otimes C''$, and $R(p'') \leq 6$, then independently of p', the additivity of the rank holds.*

The reader can check Theorem 5.1 of [4], where Hu and Bremner prove that the condition of the maximal rank of $\mathbb{K}^3 \otimes \mathbb{K}^3 \otimes \mathbb{K}^3$ being 5 holds for $\mathbb{K} = \mathbb{C}$.

Example 4.3.1. *Suppose $A' = A'' = \mathbb{C}^4$, $B' = B'' = \mathbb{C}^4$ and $C' = C'' = \mathbb{C}^3$. Suppose that both $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ are tensors of rank 7, p'' being concise ($p'' \in \widetilde{A''} \otimes B'' \otimes C''$ with $dim(\widetilde{A''}) < 3$), and that the additivity of the rank fails for $p = p' \oplus p''$. Then $R(p) = 13$.*

To prove this, note that we cannot have only one of the tensors being concise, because if that is the case, we would have additivity of the tensor rank by Theorem 4.3.2. So we will assume that both of the tensors are concise. Corollary 3.2.1 gives us $\mathbf{e}' < R(W'') - dim(W'') = 7 - 4 = 3$, which is the same as $\mathbf{e}' \leq 2$. The same argument works for $\mathbf{e}'', \mathbf{f}'$, and $\mathbf{f}''$. If one of them is strictly less than 2, then Corollary 4.2.1.ii is contradicted. From this we have $\mathbf{e}' = \mathbf{e}'' = \mathbf{f}' = \mathbf{f}'' = 2$.

We also have $R(W) < R(W') + R(W'') = 14$, that is, $R(W) \leq 13$, but from Lemma 3.2.2 we have

$$R(W') + \mathbf{e}'' \leq R(W) - dim(W''),$$

so $13 = 7 + 2 + 4 \leq R(W)$ and we have $R(W) = 13$.

Chapter 5

Additivity of the tensor border rank

In this chapter, we will study the additivity of the border rank restricted to the base field $\mathbb{C}$. The main question would be, if given

$$p' \in A' \otimes B' \otimes C' \text{ and } p'' \in A'' \otimes B'' \otimes C''$$

with $4 \geq \mathbf{a}' + \mathbf{a}'' \geq \mathbf{b}' + \mathbf{b}'' \geq \mathbf{c}' + \mathbf{c}''$, which conditions allow us to guarantee that

$$\underline{R}(p' \oplus p'') = \underline{R}(p') + \underline{R}(p'')?$$

In general, the answer is negative; in fact, there exist examples for which $R(p' \oplus p'') < R(p') + R(p'')$. Schönhage [17] proposed a family of counterexamples amongst which the smallest is

$$R(\mu(2,1,3)) = 6, R(\mu(1,2,1)) = 2, R(\mu(2,1,3) \oplus \mu(1,2,1)) = 7,$$

where $\mu(a,b,c)$ is a general tensor in $\mathbb{K}^a \otimes \mathbb{K}^b \otimes \mathbb{K}^c$ (see also Section 11.2.2 of[12]). Another interesting question is what is the smallest counterexample to the additivity of the border rank? The example of Schönhage lives in $\mathbb{C}^{2+2} \otimes \mathbb{C}^{3+2} \otimes \mathbb{C}^{6+1}$, that is, it requires using a seven dimensional vector space. Here we show that if all three spaces A, B, C have dimensions at most 4, then it is impossible to find a counterexample to the additivity of the border rank.

5.1 The variety of tensors of border rank at most r

We recall sections 5.1 and 5.2 of [12] to understand the variety of tensors of border rank at most r as a secant variety (more specifically, the r-th secant variety of the Segre variety of tensors of border rank one). We will explore the meaning of this phrase throughout this section.

Let $\sigma_{\leq r} \subset \mathbb{P}(A_1 \otimes \cdots \otimes A_n)$ denote the projectivization of the set of tensors of rank at most r. Define $\sigma_r := \overline{\sigma_{\leq r}}$, where the closure is the Zariski's closure. Then a tensor p has border rank r, that is, $\underline{R}(p) = r$ if and only if $[p] \in \sigma_r$ and $[p] \notin \sigma_{r-1}$.

Consider a curve (i.e., a one-dimensional variety) $C \subset \mathbb{P}V$ and $q \in \mathbb{P}V$ a fixed point. We define

$$J(q, C) = \overline{\bigcup_{x \in C,\ x \neq q} \mathbb{P}^1_{xq}}$$

as the closure of the set of points lying on all lines containing q and a point of C. We can call in this case $J(q, C)$ the **cone** over C with vertex q.

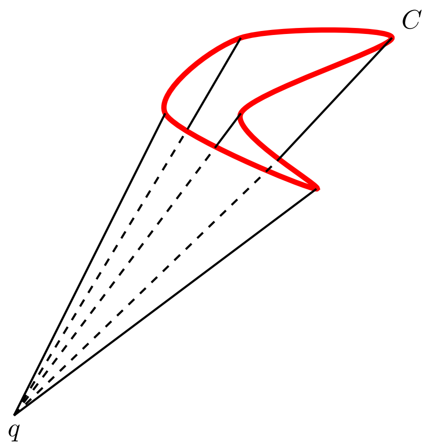


Figure 5.1: The cone over C with vertex q.

Observation 5.1.1. *The need of taking closure is just to guarantee that the case $q \in C$ also includes the points on the tangent line to C at q. We can think of the tangent line as the limit of secant lines $\mathbb{P}^1_{qx_j}$, where $(x_j)_{j \in \mathbb{N}}$ is a sequence of points such that $x_j \to q$ as $j \to \infty$.*

We define $J(q, Z)$ in a similar way for $Z \subset \mathbb{P}V$, where $dim(Z)$ is an arbitrary positive integer. This previous discussion gives us the idea of $dim(J(q, Z)) = dim(Z) + 1$. This intuition is correct, except for the case where Z is a linear space and $q \in Z$.

Observation 5.1.2. *When we talk about a cone in the affine space, we are talking about a set lying in the vector space V (see Definition 2.3.2). A projective variety which can be seen as a cone $J(q, X)$ is a set lying in the projective space $\mathbb{P}V$, so they are not the same thing.*

We can extrapolate our definition of $J(q, Z)$ to get a definition of $J(Y, Z)$, considering this last set as the closure of the union of the cones $J(q, Z)$, that is

Definition 5.1.1. *We call*

$$J(Y, Z) = \overline{\bigcup_{q \in Y} J(q, Z)} = \overline{\bigcup_{z \in Z} J(Y, z)} = \overline{\bigcup_{q \in Y,\ z \in Z,\ q \neq z} \mathbb{P}^1_{xy}}$$

the ***join*** *of two varieties* $Y, Z \subset \mathbb{P}V$. *In the case* $Y = Z$, *we define* $\sigma(Y) = \sigma_2(Y) := J(Y, Y)$, *the* ***secant variety*** *of* Y. *The variety* $\sigma_2(Y)$ *contains all points of all secant and tangent lines to* Y. *This allows us to generalize the definition.*

Definition 5.1.2. *The* ***join*** *of* k *varieties* $X_1, \dots, X_k \subset \mathbb{P}V$ *is defined to be*

$$J(X_1, \dots, X_k) = J(X_1, J(X_2, \dots, X_k)).$$

In the case $X_1 = X_2 = \dots = X_k = Y$, *we define the* k***-th secant variety*** *of* Y *as*

$$\sigma_k(Y) = J(Y, \dots, Y),$$

the join of k *copies of* Y.

Now we state two theorems that allow us to conclude that the Zariski and Euclidean closure for our case of study agree.

Theorem 5.1.1. *(Th.5.1.1.4.,[12]) Joins and secant varieties of irreducible varieties are irreducible.*

Theorem 5.1.2. *If* V *is a Zariski closed subset of the affine space* $\mathbb{C}^n$, *then* V *is closed in the Euclidean topology.*

Proof. It is enough to prove the result for a basic of Zarisky's topology, as every closed subset is the intersection of basic sets. Let $U_f = \{x \in \mathbb{C}^n : f(x) = 0\}$, where f is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$. We know that U_f by definition is a closed subset. But as f is continuous as a function $\mathbb{C}^n \longrightarrow \mathbb{C}$, $U_f = f^{-1}(0)$ is a closed subset of Euclidean topology. □

The projective version of this theorem is also true.

Theorem 5.1.3. *If* V *is a Zariski closed subset of the projective space* $\mathbb{P}^n$, *then* V *is closed in the Euclidean topology.*

Theorem 5.1.4. *A variety* X *is irreducible if and only if every Zariski open subset of* X *is dense.*

Proof. ($\Rightarrow$) Suppose X is irreducible and there exists a Zariski open subset U of X such that $\overline{U} \neq X$. Then $(\overline{U})^c \subset X$ is a Zariski open subset of X such that $\overline{(\overline{U})^c} \neq X$ (because no element of $\overline{U}$ lies in $\overline{(\overline{U})^c}$). But then $\overline{(\overline{U})^c} \cup \overline{U} \supset \overline{((\overline{U})^c \cup \overline{U}} = \overline{X} = X$. This implies X is reducible, an absurd.

($\Leftarrow$) Now suppose every open Zariski subset U of X is dense, that is, $\overline{U} = X$, and suppose $Y, Z \subset X$ is a pair of closed subvarieties such that $Y \cup Z = X$. We can assume without loss of generality that $Y \cap Z = \varnothing$. Then Y^c is an open Zariski subset of X and by hypothesis $\overline{Y^c} = X = Z$. Then $Y = \varnothing$ and $Z = X$, proving X is irreducible. □

Corollary 5.1.1. *The definitions of σ_r in terms of limits and Zariski closure agree.*

Proof. Call the closure of a set by taking limits the "Euclidean closure", and the closure by taking the common zeros of the polynomial vanishing on the set the "Zariski closure". As we already stablished in Theorem 5.1.3, any Zariski closed subset is an Euclidian closet subset. Now, if Z is an irreducible variety, we know by the previous theorem that $\overline{U} = Z$, in both closure senses. To complete the proof, it is enough to consider $Z = \sigma_r(X)$ and U the set of elements of rank at most r in X, that is,

$$U = \bigcup_{x_1, x_2, \ldots, x_r \in X} \mathbb{P}(\langle x_1, \ldots, x_r \rangle)$$

□

Now, we begin to study our results on the additivity of the border rank. Let's start by taking concise tensors $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ with $\underline{R}(p') \leq \mathbf{a}'$ and $\underline{R}(p'') \leq \mathbf{a}''$ (the conciseness implies $\underline{R}(p') = \mathbf{a}'$ and $\underline{R}(p'') = \mathbf{a}''$). Since p' and p'' are concise, the linear maps

$$p' : (A')^* \longrightarrow B' \otimes C' \text{ and } p'' : (A'')^* \longrightarrow B'' \otimes C''$$

are injective. From this

$$p : (A)^* \longrightarrow B \otimes C$$

is also injective and

$$\underline{R}(p) \geq dim(p(A)^*) = dim(p'(A')^*) + dim(p''(A'')^*) = \underline{R}(p') + \underline{R}(p'').$$

The other inequality always holds. From the previous discussion, we conclude the following lemma:

Lemma 5.1.5. *(Lemma 5.1, [6]) Consider concise tensors $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ with $\underline{R}(p') \leq \mathbf{a}'$ and $\underline{R}(p'') \leq \mathbf{a}''$ (Proposition 3.1.1 implies $\underline{R}(p') = \mathbf{a}'$ and $\underline{R}(p'') = \mathbf{a}''$). Let $p = p' \oplus p''$. Then the additivity of the border rank holds.*

Corollary 5.1.2. *(Corollary 5.2, [6]) Suppose that $(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ and $(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'')$ fall into one of the following cases:*

1. *$(a, b, 1), (a, 1, c), (a, b, 2)$, with $a \geq b \geq 2$,*
2. *$(a, 2, c)$, with $a \geq c \geq 2$, and*
3. *(a, b, c), with $a \geq bc$.*

Then for any concise tensors $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ the additivity of the border rank $\underline{R}(p) = \underline{R}(p') + \underline{R}(p'')$ is satisfied.

The reader might think the fact that we included analog cases such as $(a, b, 2)$ and $(a, 2, c)$ is weird; we cannot change the order of the factors because we would like to guarantee that $\mathbf{a}' + \mathbf{a}'' \geq \mathbf{b}' + \mathbf{b}'' \geq \mathbf{c}' + \mathbf{c}''$. Once this condition holds, there is no way we could choose the order of the factors in each block.

Proof. Let $q \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ be a concise tensor with $a \geq b$ and $c = 1$. This q is going to play the role of p' or p'' depending on the case.

As q is concise, the border rank is $\underline{R}(q) \geq a$ by Proposition 3.1.1 Using Observation 1.5.1 we know that $R(q) \geq \underline{R}(q)$. By Proposition 1.5.1, $R(q) \leq b$. Putting together the three inequalities above,

$$a \leq \underline{R}(q) \leq R(q) \leq b$$
$$a \leq b$$
$$a = b$$

Therefore, the only concise tensors in the case $(a, b, 1)$ with $a \geq b$ are in the form $(a, a, 1)$.

Now consider the case $(a, 1, c)$. Using last part of Proposition 1.5.1 and the same arguments as the previous case we get

$$\max\{a, c\} \leq \underline{R}(q) \leq R(q) \leq \min\{a, c\}$$
$$a = c$$

More generally, using the same ideas, if $q \in \mathbb{C}^{a_1} \otimes \mathbb{C}^{a_2} \otimes \mathbb{C}^{a_3}$ is concise, then

$$\max\{a_1, a_2, a_3\} \leq \underline{R}(q) \leq R(q) \leq \min\{a_1a_2, a_1a_3, a_2a_3\} \tag{5.1}$$

$$\max\{a_1, a_2, a_3\} \leq \min\{a_1a_2, a_1a_3, a_2a_3\}. \tag{5.2}$$

Case $(a, b, 2)$ with $a \geq b \geq 2$:

$$a \leq 2b$$

Case $(a, 2, c)$ with $a \geq c \geq 2$:

$$a \leq 2c$$

Case (a, b, c) with $bc \leq a$:

$$bc \leq a \leq \underline{R}(q) \leq R(q) \leq \min\{ab, ac, bc\} \leq bc, \tag{5.3}$$

Therefore all the inequalities in Equation 5.3 are in fact equalities and $a = bc$. The remaining cases are:

I. $(a, a, 1), (a, 1, a)$,

II. $(a, b, 2)$ with $2 \le b \le a \le 2b$,

III. $(a, 2, c)$ with $2 \le c \le a \le 2c$,

IV. (bc, b, c)

Now we prove that in each of the cases I-IV we have $\underline{R}(q) = a$. After we prove that, the result follows from Lemma 5.1.5.

We now prove $\underline{R}(q) = a$ in the case $(a, a, 1)$: From Equation 5.1, we get $\underline{R}(q) = a$ since $a = b$ forces the inequalities to be equalities. The case $(a, 1, a)$ is similar and $\underline{R}(q) = a$.

Now let $q \in \mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$. It corresponds to a two-dimensional subspace $W = q(\mathbb{C}^{2*}) \subset \mathbb{C}^a \otimes \mathbb{C}^a$ of squared matrices.

By subsection 3.8.2 of [12] or by the discussion before Lemma 5.6 of [5] in an open (and dense) Zariski's set of $\mathbb{C}^a \otimes \mathbb{C}^a$ we have

$$W = \{sId + tF, s, t \in C\},$$

where F is a diagonal matrix. Such subspace W has rank a as in Example 3.2.1, then $R(q) = a$ for $q \in \mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$ general.

Now, as a general tensor of $\mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$ has rank a, then the border rank of any tensor in $\mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$ is less or equal than a, where every Zariski's open is dense (by Theorems 5.1.1 and 5.1.4) and therefore every tensor living there can be aproximated by generic tensors. From this, we prove that $\underline{R}(q) \le a$, and $\underline{R}(q) = a$, for $q \in \mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$.

After proving $\underline{R}(p) = \mathbf{a}$ for the auxiliar case $(a, a, 2)$, we work with the case $(a, b, 2)$ with $2 \le b \le a \le 2b$. If $q \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbf{C}^2$ is concise, then considering $q \in \mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^2$, q is still concise and by the auxiliar case $(a, a, 2)$ its border rank is a. Case $(a, 2, c)$ is analogous.

For the case (bc, b, c) the border rank is bc by the same argument in the case $(a, a, 1)$. Remember that the inequalities in Equation (5.3) are equalities. This concludes the proof. □

The main objective of this chapter is to proof additivity of the border rank for all the cases $\mathbf{a}, \mathbf{b}, \mathbf{c} \le 4$, that is,

Theorem 5.1.6. *Let $p' \in A' \otimes B' \otimes C'$ and $p'' \in A'' \otimes B'' \otimes C''$ be concise tensors, with*

$$\mathbf{c}' + \mathbf{c}'' \le \mathbf{b}' + \mathbf{b}'' \le \mathbf{a}' + \mathbf{a}'' \le 4.$$

Then the additivity of the border rank holds:

$$\underline{R}(p) = \underline{R}(p') + \underline{R}(p'')$$

$(\mathbf{a}',\mathbf{b}',\mathbf{c}')$	$(\mathbf{a}'',\mathbf{b}'',\mathbf{c}'')$
$(1,1,1)$	$(1,1,1)$
$(2,2,2)$	$(1,1,1)$
$(2,2,1)$	$(1,1,2)$
$(2,1,2)$	$(1,2,1)$
$(2,1,1)$	$(1,2,2)$
$(2,2,1)$	$(1,1,1)$
$(2,1,1)$	$(1,2,1)$
$(2,1,1)$	$(1,1,1)$
$(3,3,3)$	$(1,1,1)$
$(3,3,2)$	$(1,1,2)$

Table 5.1: Some cases with $\mathbf{a},\mathbf{b},\mathbf{c} \leq 4$.

$(\mathbf{a}',\mathbf{b}',\mathbf{c}')$	$(\mathbf{a}'',\mathbf{b}'',\mathbf{c}'')$
$(3,3,3)$	$(1,1,1)$
$(2,3,3)$	$(2,1,1)$
$(2,3,2)$	$(2,1,2)$
$(2,2,3)$	$(2,2,1)$
$(2,3,2)$	$(2,1,1)$

Table 5.2: Cases after using the Corollary 5.1.2.

Proof. We can assume that $(\mathbf{a}',\mathbf{b}',\mathbf{c}') \geq (\mathbf{a}'',\mathbf{b}'',\mathbf{c}'')$ (with respect to the lexicographic order). Satisfying this condition, we have 47 pairs of triples. Some of them are listed on Table 5.1. With the help of Lemma 5.1.5 and Corollary 5.1.2, it remains the cases listed in Table 5.2.

Note that in Table 5.2 there are two isomorphic cases, the third and the fourth, it is enough to permute the second and the third factor. Besides, the second and the last cases in Table 5.2 have no concise tensors (recall Equation (5.2)).

After changing the order of the factors, to finish the proof of the current Theorem, it remains to prove the cases $(3+1, 2+2, 2+2)$ and $(3+1, 3+1, 3+1)$. This will follow from Propositions 5.3.1 and 5.4.1.

□

Definition 5.1.3. *Let $p, q \in A \otimes B \otimes C$ be two tensors. We say that p is* ***more degenerated*** *than q if $p \in \overline{GL(A) \times GL(B) \times GL(C) \cdot q}$.*

We stress here that $\overline{GL(A) \times GL(B) \times GL(C) \cdot q}$ denotes the closure of the orbit of the extended usual action of the group $GL(V)$ on a vector of V, more specifically

$$\begin{aligned}(GL(A) \times GL(B) \times GL(C)) \times (A \otimes B \otimes C) &\longrightarrow A \otimes B \otimes C \\ ((M_1, M_2, M_3), q) &\longmapsto M_1\pi_{B\oplus C}(q) \otimes M_2\pi_{A\oplus C}(q) \otimes M_3\pi_{A\oplus B}(q),\end{aligned}$$

where $\pi_{B\oplus C}, \pi_{A\oplus C}$, and $\pi_{A\oplus B}$ are the projections with kernel $B \oplus C$, $A \oplus C$ and $A \oplus B$, respectively.

Lemma 5.1.7. *If p, q are tensors and p is more degenerated than q, then $\underline{R}(p) \leq \underline{R}(q)$.*

Proof. Let $r = \underline{R}(q)$. Then $q = \lim_{n\to\infty} p_n$, where the p_n's are tensors of rank less or equal than r. As p is more degenerated than q, then $p \in \overline{GL(A) \times GL(B) \times GL(C) \cdot q}$. Note that

$$p \in \overline{GL(A) \times GL(B) \times GL(C) \cdot q} = \overline{GL(A) \times GL(B) \times GL(C) \cdot \lim_{n\to\infty} p_n}$$
$$= \overline{\lim_{n\to\infty} GL(A) \times GL(B) \times GL(C) \cdot p_n} = \overline{\lim_{n\to\infty} q_n}$$

where each q_n has rank at most r, therefore $\underline{R}(p) \leq r$. □

Example 5.1.1. *Let*

$$p = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes (b_2 \otimes c_1 + b_1 \otimes c_2) + a_3 \otimes b_2 \otimes c_2 \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.$$

We claim that p *is more degenerated than the tensor*

$$q = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_3 \otimes b_2 \otimes c_1.$$

To prove this, we check the definition. Consider the respective action

$$\begin{aligned}(GL_3 \times GL_2 \times GL_2) \times (\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) &\longrightarrow \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \\ (M_1, M_2, M_3, t_1 \otimes t_2 \otimes t_3) &\longmapsto M_1 t_1 \otimes M_2 t_2 \otimes M_3 t_3.\end{aligned}$$

Now, p *and* q *can be seen as* $A^* \longrightarrow B \otimes C$ *linear transformations defined by*

$$\begin{aligned} p : A^* &\longrightarrow B \otimes C \\ a_1^* &\longmapsto b_1 \otimes c_1, \\ a_2^* &\longmapsto b_2 \otimes c_1 + b_1 \otimes c_2, \\ a_3^* &\longmapsto b_2 \otimes c_2, \\ q : A^* &\longrightarrow B \otimes C \\ a_1^* &\longmapsto b_1 \otimes c_1, \\ a_2^* &\longmapsto b_1 \otimes c_2, \\ a_3^* &\longmapsto b_2 \otimes c_1. \end{aligned}$$

Note that $p(a_1^*) = q(a_1^*)$*, so in this case the action is the trivial one. The required action to get* $p(a_2^*)$ *from* $q(a_2^*)$ *is to take the associated matrix and add the transposed. As the identity and the linear transformation that consists on taking the transposed of a matrix are linear and invertible, we can conclude that the action is invertible and therefore,* $p(a_2^*)$ *belongs to the orbit of* $q(a_2^*)$*. The last action will consist on elementary rows and columns operations, again invertible. This proves that* $p \in \overline{GL_3 \times GL_2 \times GL_2 \cdot q}$*.*

Lemma 5.1.8. *(Lemma 5.6, [6]) Suppose* $p' \in A' \otimes B' \otimes C'$ *is an arbitrary tensor and* $p'', q'' \in A'' \otimes B'' \otimes C''$ *are such that* $\underline{R}(p'') = \underline{R}(q'')$ *and* p'' *is more degenerated than* q''*. If the additivity of the border rank holds for* $p' \oplus p''$*, then it also holds for* $p' \oplus q''$*.*

Proof. Since p'' is more degenerated than q'', $p' \oplus p''$ is more degenerated than $p' \oplus q''$. Then, by Lemma 5.1.7 we have

$$\underline{R}(p' \oplus q'') \geq \underline{R}(p' \oplus p'') = \underline{R}(p') + \underline{R}(p'') = \underline{R}(p') + \underline{R}(q'').$$

The other inequality always holds. □

5.2 Strassen's equations of secant varieties

Saying that a tensor is or is not of a given border rank is often the same as saying that the tensor satisfies the corresponding secant variety equations. In the small cases we are considering, studying one type of these equations will be enough. We will set $\mathbf{b} = \mathbf{c}$, so we are considering $B \otimes C$ as a space of square matrices. Let $f_{\mathbf{b}}$ be the map

$$\begin{aligned} f_{\mathbf{b}} : (B \otimes C)^{\times 3} &\longrightarrow B \otimes C \\ (x, y, z) &\longmapsto x adj(y) z - z adj(y) x \end{aligned}$$

where $adj(y)$ denotes the adjoint matrix of y. Now consider a tensor $p = \sum_{i=1}^{\mathbf{a}} a_i \otimes w_i$, where $w_i \in W := p(A^*) \subset B \otimes C$ for $i \in \{1, \ldots, \mathbf{a}\}$ are $\mathbf{b} \times \mathbf{c}$ matrices and $\{a_1, \ldots, a_{\mathbf{a}}\}$ is a basis of A.

Proposition 5.2.1. *(Prop. 5.7, [6]) Let $p \in A \otimes B \otimes C$ be a tensor.*

i) Suppose $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$. Then $\underline{R}(p) \leq 3$ if and only if $f_{\mathbf{3}}(x, y, z) = 0$ for every $x, y, z \in W$.

ii) Suppose $\mathbf{a} = \mathbf{b} = \mathbf{c}$ and $\underline{R}(p) \leq \mathbf{a}$. Then $f_{\mathbf{a}}(x, y, z) = 0$ for every $x, y, z \in W$.

So one way to determine if a tensor p has border rank at most $\mathbf{a}$ (in the case $\mathbf{a} = \mathbf{b} = \mathbf{c}$), is to check that every triple (x, y, z) lies in the kernel of $f_{\mathbf{a}}$. Another way comes from the following discussion:

Consider the tensor $p : B^* \longrightarrow A \otimes C$, and the contraction operator

$$\widehat{p_A} : A \otimes B^* \longrightarrow \Lambda^2 A \otimes C$$

obtained as a composition of

$$Id_A \otimes p : A \otimes B^* \longrightarrow A^{\otimes 2} \otimes C$$

and

$$\pi_\wedge \otimes Id_C : A^{\otimes 2} \otimes C \longrightarrow \Lambda^2 A \otimes C,$$

with $\pi_\wedge$ defined as in equation (1.6).

Proposition 5.2.2. *(Prop. 5.8, [6]) Assume* $3 \leq \mathbf{a} \leq \mathbf{b}, \mathbf{c}$. *If* $\underline{R}(p) \leq r$, *then rank*$(\widehat{p_A}) \leq r(\mathbf{a}-1)$.

Now, let's think a bit more about this $\widehat{p_A}$ function. Suppose $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$, and choose basis for A, B, C. We write

$$p = b_1 \otimes w_1 + b_2 \otimes w_2 + b_3 \otimes w_3,$$
$$w_i = \sum_{j=1}^{3} a_j \otimes c_{ji},$$
$$c_{ji} = \sum_{\ell=1}^{3} \lambda_{ji\ell} c_\ell$$

Considering p as a linear transformation

$$p : B^* \longrightarrow A \otimes C,$$
$$b_i^* \longmapsto \sum_{j=1}^{3} a_j \otimes c_{ji}$$

we get

$$\widehat{p_A} : A \otimes B^* \longrightarrow \Lambda^2 A \otimes C$$
$$a_k \otimes b_i^* \longmapsto \sum_{j=1}^{3} (a_k \wedge a_j) \otimes c_{ji}.$$

Calculating the associated matrix of this linear transformation, we get

$$M(\widehat{p_A}) = \left(\begin{array}{c|c|c} \displaystyle\sum_{j=1}^{3} (a_1 \wedge a_j) \otimes c_{j1} & \cdots & \displaystyle\sum_{j=1}^{3} (a_3 \wedge a_j) \otimes c_{j3} \end{array} \right),$$

where every $\sum_{j=1}^{3} (a_i \wedge a_j) \otimes c_{ij}$ is a 9×1 matrix. In terms of the basis (we list the basis in this particular order to make easy for the reader to identify the images of the $a_k \otimes b_i^*$)

$a_1 \otimes b_1^*, a_1 \otimes b_2^*, a_1 \otimes b_3^*, a_2 \otimes b_1^*, a_2 \otimes b_2^*, a_2 \otimes b_3^*, a_3 \otimes b_1^*, a_3 \otimes b_2^*, a_3 \otimes b_3^*$ for $A \otimes B^*$

and

$(a_2 \wedge a_3) \otimes c_1, (a_2 \wedge a_3) \otimes c_2, (a_2 \wedge a_3) \otimes c_3, (a_1 \wedge a_3) \otimes c_1, (a_1 \wedge a_3) \otimes c_2, (a_1 \wedge a_3) \otimes c_3, (a_1 \wedge a_2) \otimes c_1, (a_1 \wedge a_2) \otimes c_2, (a_1 \wedge a_2) \otimes c_3$ for $\Lambda^2 A \otimes C$

we have

$$\widehat{p_A}(a_1 \otimes b_1^*) = \sum_{j=1}^{3} (a_1 \wedge a_j) \otimes c_{j1},$$
$$= (a_1 \wedge a_2) \otimes c_{21} + (a_1 \wedge a_3) \otimes c_{31}$$

(remember that $a_1 \wedge a_1 = 0$). We know that c_{21} and c_{31} can be expressed in terms of the basis $\{c_1, c_2, c_3\}$ of C, therefore

$$c_{21} = \sum_{\ell=1}^{3} \lambda_{21\ell} c_\ell \text{ and } c_{31} = \sum_{\ell=1}^{3} \lambda_{31\ell} c_\ell.$$

So

$$(a_1 \wedge a_2) \otimes c_{21} = (a_1 \wedge a_2) \otimes \left(\sum_{\ell=1}^{3} \lambda_{21\ell} c_\ell \right) = \sum_{\ell=1} \lambda_{21\ell}((a_1 \wedge a_2) \otimes c_\ell)$$

and

$$(a_1 \wedge a_3) \otimes c_{31} = (a_1 \wedge a_3) \otimes \left(\sum_{\ell=1}^{3} \lambda_{31\ell} c_\ell \right) = \sum_{\ell=1}^{3} \lambda_{31\ell}((a_1 \wedge a_3) \otimes c_\ell),$$

and from this

$$\widehat{p_A}(a_1 \otimes b_1^*) = \sum_{\ell=1} \lambda_{21\ell}((a_1 \wedge a_2) \otimes c_\ell) + \sum_{\ell=1}^{3} \lambda_{31\ell}((a_1 \wedge a_3) \otimes c_\ell).$$

Since the order we listed the basis is: $(a_2 \wedge a_3) \otimes c_1, (a_2 \wedge a_3) \otimes c_2, (a_2 \wedge a_3) \otimes c_3, (a_1 \wedge a_3) \otimes c_1, (a_1 \wedge a_3) \otimes c_2, (a_1 \wedge a_3) \otimes c_3, (a_1 \wedge a_2) \otimes c_1, (a_1 \wedge a_2) \otimes c_2, (a_1 \wedge a_2) \otimes c_3$, the coordinates of $\widehat{p_A}(a_1 \otimes b_1^*)$ in the basis are

$$(0, 0, 0, \lambda_{311}, \lambda_{312}, \lambda_{313}, \lambda_{211}, \lambda_{212}, \lambda_{213}).$$

Using the same process to calculate $\widehat{p_A}(a_1 \otimes b_2^*)$ and $\widehat{p_A}(a_1 \otimes b_3^*)$ we get the coordinates for them in the basis ordered as above:

$$(0, 0, 0, \lambda_{321}, \lambda_{322}, \lambda_{323}, \lambda_{221}, \lambda_{222}, \lambda_{223})$$

and

$$(0, 0, 0, \lambda_{331}, \lambda_{332}, \lambda_{333}, \lambda_{231}, \lambda_{232}, \lambda_{233})$$

respectively. Following this process, the matrix ends up being

$$M(\widehat{p_A}) = \begin{pmatrix} 0 & 0 & 0 & -\lambda_{311} & -\lambda_{321} & -\lambda_{331} & -\lambda_{211} & -\lambda_{221} & -\lambda_{231} \\ 0 & 0 & 0 & -\lambda_{312} & -\lambda_{322} & -\lambda_{332} & -\lambda_{212} & -\lambda_{222} & -\lambda_{232} \\ 0 & 0 & 0 & -\lambda_{313} & -\lambda_{323} & -\lambda_{333} & -\lambda_{213} & -\lambda_{223} & -\lambda_{233} \\ \lambda_{311} & \lambda_{321} & \lambda_{331} & 0 & 0 & 0 & -\lambda_{111} & -\lambda_{121} & -\lambda_{131} \\ \lambda_{312} & \lambda_{322} & \lambda_{332} & 0 & 0 & 0 & -\lambda_{112} & -\lambda_{122} & -\lambda_{132} \\ \lambda_{313} & \lambda_{323} & \lambda_{333} & 0 & 0 & 0 & -\lambda_{113} & -\lambda_{123} & -\lambda_{133} \\ \lambda_{211} & \lambda_{221} & \lambda_{231} & \lambda_{111} & \lambda_{121} & \lambda_{131} & 0 & 0 & 0 \\ \lambda_{212} & \lambda_{222} & \lambda_{232} & \lambda_{112} & \lambda_{122} & \lambda_{132} & 0 & 0 & 0 \\ \lambda_{213} & \lambda_{223} & \lambda_{233} & \lambda_{113} & \lambda_{123} & \lambda_{133} & 0 & 0 & 0 \end{pmatrix}$$

$$M(\widehat{p_A}) = \begin{pmatrix} \underline{0} & w_3 & -w_2 \\ -w_3 & \underline{0} & w_1 \\ w_2 & -w_1 & \underline{0} \end{pmatrix}$$

Proposition 5.2.3. *(Prop. 5.9, [6]) If* $\mathbf{a} = \mathbf{b} = \mathbf{c} = 3$*, the degree nine equation*

$$det(\widehat{p_A}) = 0$$

defines the variety $\sigma_4(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}(A \otimes B \otimes C)$*.*

For $\mathbf{a} = 4$ and $p = \sum_{i=1}^{4} a_i \otimes w_i \in A \otimes B \otimes C$ with $w_i \in B \otimes C$, then the matrix representation of $\widehat{p_A}$ in block matrices is the $4\mathbf{b} \times 6\mathbf{c}$ partitioned matrix (we just need to follow the same reasoning as above):

$$M_4(w_1, \ldots, w_4) = \begin{pmatrix} \underline{0} & w_3 & -w_2 & w_4 & \underline{0} & \underline{0} \\ -w_3 & \underline{0} & w_1 & \underline{0} & -w_4 & \underline{0} \\ w_2 & w_1 & \underline{0} & \underline{0} & \underline{0} & w_4 \\ \underline{0} & \underline{0} & \underline{0} & -w_1 & w_2 & -w_3 \end{pmatrix}$$

5.3 Case $(3 + 1, 2 + 2, 2 + 2)$

Proposition 5.3.1. *(Prop. 5.10, [6]) For any* $p' \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ *and* $p'' \in \mathbb{C}^1 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ *the additivity of the border rank holds.*

Proof. We can assume p'' is concise since p'' is just a matrix. From this, $R(p'') = \underline{R}(p'') = 2$. If p' is not concise, then we grant additivity of the border rank by Corollary 5.1.2. The case $(3, 2, 2)$ is a particular case of case $(a, b, 2)$ with $2 \leq b \leq a \leq 2b$ discussed in the proof of Corollary 5.1.2. In that situation we concluded that $\underline{R}(p') = R(p') = a = 3$. We write

$$p' = a_1 \otimes w_1 + a_2 \otimes w_2 + a_3 \otimes w_3, \text{ and}$$
$$p'' = a_4 \otimes w_4,$$

where w_1, w_2, and w_3 are 2×2 matrices and as p'' is concise, w_4 is an invertible 2×2 matrix.

We now use Lemma 5.1.8 and Table 10.3.1 of [12] and choose a tensor that is more degenerated than p'. This more degenerated tensor has the form $a_1 \otimes w_1' + a_2 \otimes w_2' + a_3 \otimes w_3'$, where

$$w_1' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, w_2' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, w_3' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Being that said, p would have the following form:

$$p = \sum_{i=1}^{4} a_i \otimes w_i,$$

where w_i are the following $2 + 2 \times 2 + 2$ partitioned matrices

$$w_i = \begin{pmatrix} w_i' & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix}, i = 1,2,3 \text{ and } w_4 = \begin{pmatrix} \underline{0} & \underline{0} \\ \underline{0} & w_4'' \end{pmatrix}$$

Now we are going to use Proposition 5.2.2 to show additivity of the border rank in this case. Recall the contraction operator $\widehat{p_A}$, defined above. As $\underline{R}(p) \leq \underline{R}(p') + \underline{R}(p'')$ always holds, we want to prove the other inequality, that would be

$$\underline{R}(p) \geq \underline{R}(p') + \underline{R}(p'') = 3 + \mathbf{b}'' = 3 + 2 = 5.$$

To do this, by Proposition 5.2.2, it is enough to show that $rank(\widehat{p_A}) > r(\mathbf{a}-1) = 3r$, from this we would have $\underline{R}(p) > r = 4$ which implies $\underline{R}(p) \geq 5$ and we conclude additivity. So in this case, $r = 4$. In other words, the additivity of the border rank in this case is reduced to prove $rank(\widehat{p_A}) > 12$. We consider the associated matrix for $\widehat{p_A}$:

$$M_4(w_1, w_2, w_3, w_4) = \begin{pmatrix} \underline{0} & w_3 & -w_2 & w_4 & \underline{0} & \underline{0} \\ -w_3 & \underline{0} & w_1 & \underline{0} & -w_4 & \underline{0} \\ w_2 & w_1 & \underline{0} & \underline{0} & \underline{0} & w_4 \\ \underline{0} & \underline{0} & \underline{0} & -w_1 & w_2 & -w_3 \end{pmatrix}.$$

With all we know so far about w_i, the matrix $M_4(w_1, w_2, w_3, w_4)$ can be transformed by swapping rows and columns into the following $(6+3\mathbf{b}''+2+\mathbf{b}'', 6+3\mathbf{b}''+2+2+2+3\mathbf{b}'') = (6+6+2+2, 6+6+2+2+2+6)$ partitioned matrix

$$\begin{pmatrix} M_3(w_1', w_2', w_3') & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & N & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & -w_1' & w_2' & -w_3' & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{pmatrix},$$

where

$$N = \begin{pmatrix} w_4'' & \underline{0} & \underline{0} \\ \underline{0} & w_4'' & \underline{0} \\ \underline{0} & \underline{0} & w_4'' \end{pmatrix}$$

is a 6×6 invertible matrix (note that w_4'' is invertible) and

$$M_3(w_1', w_2', w_3') = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank 5, N has rank $3\mathbf{b}'' = 6$ and the block corresponding to

$$\left(-w_1' \,\middle|\, w_2' \,\middle|\, w_3' \right)$$

has rank 2. So $M_4(w_1, w_2, w_3, w_4)$ has rank equal to $5 + 2 + 3\mathbf{b}'' = 5 + 2 + 6$, then $rank(\widehat{p_A}) = 13 > 12$ and we have proved the additivity of the border rank in this case. $\square$

5.4 Case $(3 + 1, 3 + \mathbf{b}'', 3 + \mathbf{c}'')$

Proposition 5.4.1. *(Prop. 5.11, [6]) For any $p' \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ concise, and any $p'' \in \mathbb{C}^1 \otimes B'' \otimes C''$, we have that $\underline{R}(p) = \underline{R}(p') \oplus \underline{R}(p'')$.*

Proof. By the proof of Corollary 5.1.2, $\mathbf{b}'' = \mathbf{c}''$ and $\underline{R}(p'') = R(p'')$.

Now, we have $p' \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, that means its projectivization $\widehat{p'}$ lies in $X = Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$. Now, by the last part of the proof of Lemma 3.16 of [1], we know $dim(\sigma_4(X)) = 25$. In addition, by Proposition 1.2.2 of [16], as $\sigma_4(X)$ is a hypersurface of $\mathbb{P}^{26}$, $dim(\sigma_5(X)) = 26$, and this implies the maximum border rank for a tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is 5. That is, $\underline{R}(p') \leq 5$.

To prove the border rank additivity, we will need to prove it in the two subcases: $\underline{R}(p') = 4$ and $\underline{R}(p') = 5$. Before doing that, we choose a basis $\{a_1, a_2, a_3\}$ of $\mathbb{C}^3$, $\{a_4\}$ a basis for $\mathbb{C}$. We have then

$$p' = a_1 \otimes w'_1 + a_2 \otimes w'_2 + a_3 \otimes w'_3$$

where $w'_1, w'_2, w'_3 \in W' := p((\mathbb{C}^3)^*) \subset \mathbb{C}^3 \otimes \mathbb{C}^3$ are 3×3 matrices. We write p in a similar way

$$p = a_1 \otimes w_1 + a_2 \otimes w_2 + a_3 \otimes w_3 + a_4 \otimes w_4$$

where $w_1, w_2, w_3, w_4 \in W := p(A^*) \subset B \otimes C$ are $3 + \mathbf{b}'' \times 3 + \mathbf{b}''$ partitioned matrices:

$$w_i = \begin{bmatrix} w'_i & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}, i = 1, 2, 3, \text{ and } w_i = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & w''_4 \end{bmatrix},$$

where w''_4 is a $\mathbf{b}'' \times \mathbf{b}''$ invertible matrix. We are now ready to analyze our two cases:

1. $\underline{R}(p') = 4$. By contradiction, suppose that $\underline{R}(p') = 4$ but the additivity is not satisfied. Then $\underline{R}(p) \leq \underline{R}(p') + \underline{R}(p'') - 1 = 3 + \mathbf{b}''$. Using Proposition 5.2.1.ii), we obtain

$$f_{\mathbf{b}''+3}(x', y' + y'', z') = x' adj(y' + y'')z' - z' adj(y' + y'')x' = \underline{0},$$

where $x', y', z' \in W' = p((A')^*)$ and $0 \neq y'' \in W'' = p''((A'')^*)$. Note that

$$adj(y' + y'') = adj\left(\begin{bmatrix} y' & \underline{0} \\ \underline{0} & y'' \end{bmatrix}\right) = \begin{bmatrix} det(y'')adj(y') & \underline{0} \\ \underline{0} & det(y')adj(y'') \end{bmatrix}$$

and therefore we have

$$\begin{bmatrix} x' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} adj(y'+y'') \begin{bmatrix} z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$$
$$= \begin{bmatrix} x' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} det(y'')adj(y') & \underline{0} \\ \underline{0} & det(y')adj(y'') \end{bmatrix} \begin{bmatrix} z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$$
$$= \begin{bmatrix} det(y'')x'adj(y')z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}.$$

Since p'' is concise, $det(y'') \neq 0$. As $f_{\mathbf{b''+3}}(x', y'+y'', z')$ is zero, we must have that

$$\begin{bmatrix} x' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} adj(y'+y'') \begin{bmatrix} z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} = \begin{bmatrix} z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} adj(y'+y'') \begin{bmatrix} x' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$$
$$\begin{bmatrix} det(y'')x'adj(y')z' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} = \begin{bmatrix} det(y'')z'adj(y')x' & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix},$$

and this implies $x'adj(y')z' - z'adj(y')x' = 0 = f_{\mathbf{3}}(x', y', z')$, then by Proposition 5.2.1.i) we conclude $\underline{R}(p') \leq 3$, a contradiction.

2. $\underline{R}(p') = 5$. For this case, consider the projection

$$\begin{aligned} \pi : A \otimes B \otimes C &\longrightarrow A' \otimes B \otimes C \\ a_i &\longmapsto a_i, \text{ for } i = 1, 2, 3, \\ a_4 &\longmapsto a_1 + a_2 + a_3. \end{aligned}$$

Consider $\overline{p} = \pi(p) \in A' \otimes B \otimes C$. We obtain the following expression for $\overline{p}$:

$$\overline{p} = a_1 \otimes \overline{w_1} + a_2 \otimes \overline{w_2} + a_3 \otimes \overline{w_3},$$

where

$$\overline{w_i} = \begin{bmatrix} w_i' & \underline{0} \\ \underline{0} & w_4'' \end{bmatrix} \text{ for } i = 1, 2, 3.$$

Now, consider $\overline{p}$ as a linear transformation

$$\overline{p} : B^* \longrightarrow A' \otimes C$$

and construct the map

$$\widehat{\overline{p}_{A'}} : A' \otimes B^* \longrightarrow \Lambda^2 A' \otimes C.$$

Consider its associated matrix

$$M_3(\overline{w_1},\overline{w_2},\overline{w_3}) = \begin{bmatrix} \underline{0} & \overline{w_3} & -\overline{w_2} \\ -\overline{w_3} & \underline{0} & \overline{w_1} \\ \overline{w_2} & -\overline{w_1} & \underline{0} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{0} & \underline{0} & w_3' & \underline{0} & -w_2' & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & w_4'' & \underline{0} & -w_4'' \\ -w_3' & \underline{0} & \underline{0} & \underline{0} & w_1' & \underline{0} \\ \underline{0} & -w_4'' & \underline{0} & \underline{0} & \underline{0} & w_4'' \\ w_2' & \underline{0} & -w_1' & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & w_4'' & \underline{0} & -w_4'' & \underline{0} & \underline{0} \end{bmatrix}.$$

Now, by swapping rows and columns appropriately, we obtain

$$\begin{bmatrix} \underline{0} & w_3' & -w_2' & \underline{0} & \underline{0} & \underline{0} \\ -w_3' & \underline{0} & w_1' & \underline{0} & \underline{0} & \underline{0} \\ w_2' & -w_1' & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & w_4'' & -w_4'' \\ \underline{0} & \underline{0} & \underline{0} & -w_4'' & \underline{0} & w_4'' \\ \underline{0} & \underline{0} & \underline{0} & w_4'' & -w_4'' & \underline{0} \end{bmatrix} = \begin{bmatrix} \widehat{p'}_{A'} & \underline{0} \\ \underline{0} & M_3(w_4'', w_4'', w_4'') \end{bmatrix}.$$

Since p' is concise, $\widehat{p'}_{A'}$ has rank 9, as it is and invertible 9×9 matrix. On the other hand, $M_3(w_4'', w_4'', w_4'')$ has rank $2\mathbf{b}''$ since w_4'' has rank $\mathbf{b}''$. So

$$rank(\widehat{\overline{p}_{A'}}) = 9 + 2\mathbf{b}'' > 8 + 2\mathbf{b}'' = (4 + \mathbf{b}'')(3 - 1) = (r - 1)(\mathbf{a} - 1),$$

and by Proposition 5.2.2, $\underline{R}(\overline{p}) \geq 5 + \mathbf{b}'' = \underline{R}(p') + \underline{R}(p'')$. Note that in the calculation of $rank(\widehat{\overline{p}_{A'}})$, we have no tensor or associated subspace, we are talking about the rank of the associated matrix to the linear transformation $\widehat{\overline{p}_{A'}}$. So we are not using additivity of the tensor rank. To conclude the proof, remember that $\overline{p} = \pi(p)$, so $\underline{R}(p) \geq \underline{R}(\overline{p})$. The other inequality always holds.

$\square$

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