UNIVERSIDADE FEDERAL DE ITAJUBÁ - UNIFEI
INSTITUTO DE FÍSICA E QUÍMICA - IFQ

QUANTUM FLUCTUATIONS AND PARTICLE DYNAMICS TEST IN HARMONIC CONFINEMENT

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Then you will know the truth, and the truth will set you free.
John 8:32

Abstract

We investigate how a reservoir modifies the quantum Brownian motion of a particle by adopting a canonical quantization of the total system. Starting from a Lagrangian model describing a harmonically bound particle linearly coupled to a continuum of oscillators, we derive exact analytical solutions for the quantum correlations characterizing the system's dynamics. This approach enables a complete, with no approximations, treatment of the quantum Brownian motion, including the late-time regime and the positivity of the particle's kinetic energy. The generality of our formalism allows it to be extended to a broad class of coupling functions, offering a robust framework for exploring dissipative quantum dynamics and energy conservation mechanisms in open quantum systems.

Key-words: Quantum Theory; Quantum Brownian Motion; Quantum Field Theory; Mathematical Physics.

Contents

1 INTRODUCTION

Since its first application to explain the black body radiation [1], quantum field theory has been responsible for some of the most interesting and counterintuitive predictions made by the scientific community, and it remains as a paradigm within the standard model of elementary fields. Among the impressive hallmarks of quantum theory applied to fields is the implementation of squeezed light in the LIGO detector [2], that culminated in the first measurement of gravitational wave signals in 2016 [3]. Furthermore, quantum fields are expected to have a prominent role in black hole physics through the (semiclassical) mechanism of spontaneous Hawking radiation [4], that was also recently probed in an analogue black hole [5].

Despite the numerous successful applications of quantum field theory, some problems of fundamental importance are still not fully addressed, specially in connection to gravity. For instance, energy conservation dictates that black holes loose mass due to Hawking radiation, and yet the question of how this energy extraction ends remains unanswered. In general, the problem of determining how quantum fluctuations affect their environment is convoluted and solutions valid only in certain regimes can be found. For instance, recent findings include quantum corrections due to a black hole formation [6] and a solution to the backreaction problem in a Bose-Einstein condensate [7].

Conversely, the random movement of a particle immersed in a fluid in thermal equilibrium was first noticed by Robert Brown in 1827, when he was observing, utilizing a microscope, pollen grains in the water, and became known as Brownian motion [8]. Accordingly with the kinetic theory [9], fluids are formed by molecules that move freely, with a random pattern, and the collisions between these molecules are responsible for the thermal equilibrium. Thus, if the fluid possesses an external particle, that we may call as Brownian particle, those fluid molecules will collide with it, in the same manner that they collide with themselves, thus implying a random movement that Robert Brown has observed. Hence, the Brownian motion serves as probe to identify statistical fluctuations occurring in a system that has reached equilibrium.

The so called stochastic field is the responsible for random movement of the Brownian particles. In fact, tiny particles interacting with a finite temperature field display a Brownian motion due to thermal fluctuations. However, in the same sense that a classical stochastic field causes the random movement of a test particle, it is possible to expect a Brownian motion from quantum fluctuation as well [10]. In fact, even with zero temperature, where the thermal fluctuations are not present, quantum fields showcase vacuum fluctuations [11]. Therefore, if the fluctuations are present even in the vacuum state, a tiny particle interacting with the given field exhibits a quantum Brownian motion [12].

In this line of reasoning, another phenomenon which is important to our analysis and that can be studied only approximately is the quantum Brownian motion of [13], in which charged test particles were shown to acquire velocity fluctuations due to their interaction with a quantum field. Specifically, in the regime of non-relativistic dynamics, a particle of mass m and charge q is released at $t = 0$ and at a distance d from a plane perfect mirror. If the particle position does not vary appreciably, then its velocity undergoes fluctuations analytically given by

$$\langle v_\perp^2 \rangle = \frac{q^2}{\pi^2 m^2} \frac{t}{32 d^3} \ln\left(\frac{2d+t}{2d-t}\right)^2,$$
$$\langle v_\parallel^2 \rangle = \frac{q^2}{8\pi^2 m^2} \left[\frac{t}{8d^3} \ln\left(\frac{2d+t}{2d-t}\right)^2 - \frac{t^2}{d^2(t^2-4d^2)}\right],$$

where $v_\perp$ and $v_\parallel$ are the velocity components orthogonal and parallel to the mirror, respectively, and units are such that $\varepsilon_0 = \hbar = c = 1$.

Apart from the divergences in the above formulas, whose sources are well-understood (see, for instance, [14] and references therein), two features deserve mentioning. First, the assumption on the particle movement being negligible forbids the analysis of the late-time (ballistic) regime. Indeed, electromagnetic vacuum fluctuations lead to a diffusive stochastic dynamics [15], showing that larger particle displacements become relevant as time passes. Second, and more important, $\langle v_\parallel^2 \rangle < 0$ for $t > 2d$. This kind of phenomenon, where a classically positive definite quantity becomes negative upon quantization, is not uncommon in quantum field theory, e.g., the electromagnetic energy density in the Casimir effect, and it is usually called a *subvacuum* effect. However, for a particle's velocity, one always expects positive dispersions, as these are linked to uncertainties in measurements. For the particular case of the quantum Brownian motion of [13], it is conjectured that, once the test particle quantum features are taken into account, the overall velocity dispersion becomes a positive number and the effect of the mirror is to diminish the magnitude of this number.

In this work we discuss how the quantum Brownian motion of a particle is modified as it starts to interact with a reservoir. Specifically, we consider a Lagrangian model for a particle under a harmonic potential and a reservoir modeled by a continuum of oscillators, using for the latter the dielectric model of [16] as motivation, while the interaction term is turned on at $t_0 = 0$. By performing the canonical quantization of the system, two vacuum states are defined, since we have two regimes, and related to each other by a Boguliubov transformation (see section 2.3), and analytical expressions for the quantum correlations are found. Therefore, the late-time regime and the positivity of the particle kinetic energy can be fully addressed within our model, as no approximation is needed in determining the quantum dynamics.

It should be stressed that the damped harmonic oscillator is one of the most studied systems of quantum optics and it serves as a paradigm for the description of various systems in nature, and, in particular, for the study of the quantum Brownian motion [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Included in the vast literature on the subject, we cite, for instance, [30, 31], where a master equation for the density operator was developed and analytical solutions were found, and [32], where the method of Langevin equation is developed. For a recent account on the subject, see [33] and references therein. Our approach follows a distinct path, to the extent that the total system is quantized canonically. Although this procedure is in general involved, it is necessary when one needs to have full control of the quantum correlations. For instance, as no approximations are assumed, the quantum correlations are valid in general and problems linked to approximations of the density operator are not present [34]. Furthermore, we are interested in determining how the conservation of energy occurs in the system, which can become convoluted or even impossible to address if reduced density operator and Langevin methods are adopted. Throughout the analysis units are such that $\hbar = 1$.

The reading of this work is intended to begin with Chapter 3, where the main problem is introduced. Chapter 2, titled Basic Concepts, provides a review and connection of the theoretical foundations necessary to understand the work. It is therefore recommended that the reader refer back to Chapter 2 as needed, since references to specific sections will appear throughout the development of Chapter 3.

2 BASICS CONCEPTS

2.1 Coherent States

Let us start this section by recalling a good example, the quantum harmonic oscillator. A harmonic oscillator is an object that is subject to a quadratic potential energy, which produces a restoring force against any displacement from equilibrium that is proportional to the displacement [35]. The corresponding Hamiltonian is

$$\mathrm{H} = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2}\,, \tag{2.1}$$

where P is the momentum, Q is the spatial position and m is the mass. By (classically) deriving the motion equations, one can easily verify that ω is the natural frequency of the system.

2.1.1 Algebraic solution of the quantum harmonic oscillator

Since we are interested in the quantum harmonical oscillator, we promote the variables P and Q to operators $\hat{P}$ and $\hat{Q}$ and impose a commutation relation between them, namely

$$[\hat{Q}, \hat{P}] = i\hbar, \tag{2.2}$$

alongside with the self-adjointness of $\hat{P}$ and $\hat{Q}$,

$$\hat{P} = \hat{P}^\dagger,\ \hat{Q} = \hat{Q}^\dagger. \tag{2.3}$$

It is useful to introduce dimensionless operators as

$$\begin{aligned} \hat{q} &= \left(\frac{m\omega}{\hbar}\right)^{1/2} \hat{Q}, \\ \hat{p} &= \left(\frac{1}{m\omega\hbar}\right)^{1/2} \hat{P}, \end{aligned} \tag{2.4}$$

which satisfy $[\hat{q}, \hat{p}] = i$ such that the Hamiltonian can be written as

$$\hat{\mathrm{H}} = \frac{1}{2}\hbar\omega(\hat{p}^2 + \hat{q}^2). \tag{2.5}$$

Now, in order to solve algebraically, we introduce two more operators[1]:

$$\begin{aligned} a &= \frac{q + ip}{\sqrt{2}}, \\ a^\dagger &= \frac{q - ip}{\sqrt{2}}. \end{aligned} \tag{2.6}$$

[1] From now on, when no ambiguity is presented, the hat above an operator will be omitted.

These operators are not Hermitian, as seen by (2.3), and from the commutation relation between q and p, one easily sees that

$$[a, a^\dagger] = 1, \tag{2.7}$$

and the Hamiltonian can be rewritten as

$$\mathrm{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{2.8}$$

and the problem of finding the eigenvalues of H is reduced to that of finding the spectrum of $N \coloneqq a^\dagger a$, where N is the number operator. Using the Lie identity, we obtain

$$[N, a] = [a^\dagger a, a] = a^\dagger \cancelto{0}{[a, a]} + [a^\dagger, a]a = -a, \tag{2.9}$$

and similarly $[N, a^\dagger] = a^\dagger$. Now we are ready to calculate the spectrum of N. Let $N|\nu\rangle = \nu|\nu\rangle$, with $\langle\nu|\nu\rangle \neq 0$. Then, it follows that

$$Na|\nu\rangle = a(N-1)|\nu\rangle = (\nu-1)a|\nu\rangle, \tag{2.10}$$

and hence $a|\nu\rangle$ is an eigenvector of N with eigenvalue $(\nu - 1)$. Also, $\nu \geq 0$ because

$$\langle\nu|a^\dagger a|\nu\rangle = \langle\nu|N|\nu\rangle = \nu\langle\nu|\nu\rangle, \tag{2.11}$$

and this norm cannot be negative. Thus, if $a|0\rangle \neq 0$, we would be able to construct negative eigenvalues, which is not possible, and then $a|0\rangle = 0$ is a logical necessity. Using the same reasoning, we conclude that

$$Na^\dagger|\nu\rangle = (\nu+1)a^\dagger|\nu\rangle, \tag{2.12}$$

and that the squared norm of $a^\dagger|\nu\rangle$ is

$$\langle\nu|aa^\dagger|\nu\rangle = \langle\nu|(N+1)|\nu\rangle = (\nu+1)\langle\nu|\nu\rangle, \tag{2.13}$$

which never vanishes. By repeatedly applying $a^\dagger$, one can construct an unlimited sequence of eigenvectors, each having an eigenvalue that is one unit greater than that of its predecessor, and this sequence begins with $\nu = 0$. Therefore, the spectrum of N consists of all non-negative integers $\nu = n$. The orthonormal eigenvector of N will be denoted as $|n\rangle$. We have already shown that $a^\dagger|n\rangle$ is proportional to $|n+1\rangle$. If C_n is the constant of proportionality, we may define it by the norm of this vector, which was calculated above:

$$|C_n|^2 = \langle n|aa^\dagger|n\rangle = n+1 \implies |C_n| = \sqrt{(n+1)}. \tag{2.14}$$

The phase of $|n+1\rangle$ is arbitrary, and we choose so that C_n is real and positive. From this discussion, it follows from induction that

$$|n\rangle = \frac{(a^\dagger)^n|0\rangle}{\sqrt{n!}}. \tag{2.15}$$

From the considerations above and the orthogonality of eigenvectors, we obtain the matrix elements of $a^\dagger$:

$$\langle n'|a^\dagger|n\rangle = \sqrt{(n+1)}\delta_{n',n+1}. \tag{2.16}$$

Because a is the adjoint of $a^\dagger$, its matrix elements must be the transpose of (2.16):

$$\langle n'|a|n\rangle = \sqrt{n}\delta_{n',n-1}. \tag{2.17}$$

From here, we conclude that $a|n\rangle = \sqrt{n}|n-1\rangle$, if $n \neq 0$, and $a|0\rangle = 0$, as we obtained before. Finally, the eigenvector of the harmonic oscillator Hamiltonian are

$$\mathrm{H}|n\rangle = E_n|n\rangle, \tag{2.18}$$

with $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.

2.1.2 Definitions of coherent states

The coherent states are defined as eigenvectors of the operator a:

$$a|\mu\rangle = \mu|\mu\rangle, \tag{2.19}$$

where μ is a complex parameter. If we expand on the energy basis $|n\rangle$, $|\mu\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, the coherent state can be reconstructed from the equation (2.19):

$$a|\mu\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} \mu c_n|n\rangle. \tag{2.20}$$

Comparing coefficients, we obtain a recursion relation $c_n = \frac{\mu}{\sqrt{n}}c_{n-1}$, which by induction leads to

$$c_n = \frac{\mu^n}{\sqrt{n!}}c_0. \tag{2.21}$$

This coefficient c_0 can be determined from normalization by

$$1 = \sum_{n=0}^{\infty}|c_n|^2 = \sum_{n=0}^{\infty}\frac{|\mu|^{2n}}{n!}|c_0|^2 = \mathrm{e}^{|\mu|^2}|c_0|^2. \tag{2.22}$$

Finally,

$$|\mu\rangle = \mathrm{e}^{-|\mu|^2/2}\sum_{n=0}^{\infty}\frac{\mu^n}{\sqrt{n!}}|n\rangle = \mathrm{e}^{-|\mu|^2/2}\sum_{n=0}^{\infty}\frac{\mu^n}{\sqrt{n!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = \mathrm{e}^{\mu a^\dagger - |\mu|^2/2}|0\rangle. \tag{2.23}$$

As an example, we show that the distribution of the quanta number N in a coherent state follows a Poisson distribution with mean $\bar{n} = |\mu|^2$. In fact, since $N|n\rangle = n|n\rangle$, the distribution (on the energy basis) defined as $\frac{\langle\mu|N|\mu\rangle}{\langle n|N|n\rangle}$ is given by

$$\frac{\langle\mu|N|\mu\rangle}{n} = |c_n|^2 = \mathrm{e}^{-|\mu|^2}\frac{|\mu|^{2n}}{n!}. \tag{2.24}$$

2.1.3 Coordinate and momentum uncertainty

Let us calculate the coordinate uncertainty for a given coherent state μ. First, if $\lambda = \sqrt{\frac{\hbar}{m\omega}}$, then $Q = \lambda q$ and $P = \frac{\hbar}{\lambda} p$, so that

$$\langle Q^2 \rangle = \langle \mu | Q^2 | \mu \rangle = \frac{\lambda^2}{2} \langle \mu | (a + a^\dagger)^2 | \mu \rangle = \frac{\lambda^2}{2} \left[(\mu + \bar{\mu})^2 + 1 \right], \tag{2.25}$$

where we used the definition of a coherent state and $[a, a^\dagger] = 1$. Similarly,

$$(\langle \mu | Q | \mu \rangle)^2 = \frac{\lambda^2}{2} \left(\langle \mu | a + a^\dagger | \mu \rangle \right)^2 = \frac{\lambda^2}{2} (\mu + \bar{\mu})^2, \tag{2.26}$$

such that

$$\langle \delta Q^2 \rangle = \langle Q^2 \rangle - \langle Q \rangle^2 = \frac{\lambda^2}{2} = \frac{\hbar}{2m\omega}. \tag{2.27}$$

Proceeding in the same manner for $\langle \delta P^2 \rangle$, we find that

$$\langle \delta P^2 \rangle = \langle P^2 \rangle - \langle P \rangle^2 = \frac{\hbar^2}{2\lambda^2} = \frac{\hbar m \omega}{2}, \tag{2.28}$$

and thus both (2.27) and (2.28) does not depend on μ and their product is

$$\sqrt{\langle \delta Q^2 \rangle \langle \delta P^2 \rangle} = \frac{\hbar}{2}, \tag{2.29}$$

which is the lower bound of the Heisenberg uncertainty principle. We will see that coherent states can be naturally generalized to a more generic class of states that minimizes the uncertainty product.

2.2 Squeezed States

Of great importance for our analysis is the notion of a squeezed state, and it is often omitted on undergraduate text books. Here, we will define a squeezed state from the physical point of view and also from the mathematical one. The interpretation of such a state is given since the beginning and, to conclude this section, we show how a squeezed state arises naturally from a Hamiltonian that has a time dependence.

2.2.1 Definition of a squeezed state

Squeezed states can be introduced using the idea of coherent states. Like the latter, they are closely related to the properties of a and $a^\dagger$. As we saw above, coherent states have the property of minimizing the dispersions q and p. In order to understand a bit more about this property and define the squeezed states, let us recall the proof of the uncertainty relation $\delta q \delta p \geq \frac{\hbar}{2}$.

Given two hermitian operators A and B, and any state $|\mu\rangle$, consider the quantity

$$J(a) = \langle \mu | (\alpha A - iB)(\alpha A + iB) | \mu \rangle = \langle A^2 \rangle \alpha^2 + \alpha \langle i[A, B] \rangle + \langle B^2 \rangle, \tag{2.30}$$

which is the squared norm of $(\alpha A + iB)|\mu\rangle$, and by definition it is a nonnegative quadratic polynomial in α. Thus it does not have real roots besides zero possibly, and therefore its discriminant is less or equal to 0. Hence

$$\langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2. \tag{2.31}$$

Now, consider $A = q$ and $B = p = -i\hbar\partial_q$, with $[q,p] = i\hbar$. Then $\langle q^2\rangle\langle p^2\rangle \geq \frac{1}{4}\hbar^2$, and for a state that $\langle q\rangle = \langle p\rangle = 0$, we conclude

$$\langle\delta q^2\rangle\langle\delta p^2\rangle \geq \frac{1}{4}\hbar^2, \tag{2.32}$$

since $\langle\delta X^2\rangle = \langle(X - \langle X\rangle)^2\rangle = \langle X^2\rangle - \langle X\rangle^2$. The uncertainty relation for a more general state $\langle q|\mu\rangle = \mu(q)$ with nonzero $\langle q\rangle$ and $\langle p\rangle$ can be easily obtained. Let us write this more general state as

$$\mu(q) = \mathrm{e}^{i\frac{\bar{p}q}{\hbar}}\tilde{\mu}(q - \bar{q}), \tag{2.33}$$

where the identification $\bar{X} = \langle X\rangle$ was made and $\langle(q-\bar{q})|\tilde{\mu}\rangle = \tilde{\mu}(q-\bar{q})$ is adequate. Thus, since $\langle(q-\bar{q})^2\rangle_\mu = \langle q^2\rangle_{\tilde{\mu}}$ and $\langle(p-\bar{p})^2\rangle_\mu = \langle p^2\rangle_{\tilde{\mu}}$, we can write

$$\langle\delta q^2\rangle_\mu\langle\delta p^2\rangle_\mu = \langle q^2\rangle_{\tilde{\mu}}\langle p^2\rangle_{\tilde{\mu}} \geq \frac{1}{4}\hbar^2, \tag{2.34}$$

the uncertainty relation from an arbitrary state.

If there exists a state that satisfy $\bar{q} = \bar{p} = 0$, and if it minimizes the uncertainty relation, there will be a unique root α_0 of the equation $J(\alpha_0) = 0$. We can find this state as a function of a_0 since it will be solution of

$$(\alpha_0 q + ip)\mu(q) = \alpha_0 q\mu(q) + \hbar\mu(q)' = 0. \tag{2.35}$$

Solving this differential equation, we get $\mu(q) = C\mathrm{e}^{-\frac{\alpha_0 q^2}{2\hbar}}$. Normalizing to one and remembering the Gaussian integral, we get that $C = \left(\frac{a_0}{\hbar\pi}\right)^{1/4}$. These general states that minimize $\langle\delta q^2\rangle\langle\delta p^2\rangle$ are called *squeezed states*, but minimizing the uncertainty is not the only property of such states. In fact, a squeezed state also makes one of the uncertainties ($\langle\delta q^2\rangle$ or $\langle\delta p^2\rangle$) lower than the threshold, at the same time that it maintains the uncertainty product [36]. We will see more on that now.

2.2.2 Squeezed states and the operators a and $a^\dagger$

Suppose we prepared a squeezed state (2.35) of a harmonic oscillator with $\bar{q} = \bar{p} = 0$, and we want to know its time evolution. The simplest way is to rewrite the uncertainty condition in (2.35) in terms of the annihilation (and creation) operators, since their dynamic is given in the Heisenberg picture as

$$\begin{aligned} a(t) =& \mathrm{e}^{i\mathrm{H}t}a\mathrm{e}^{-i\mathrm{H}t} = \mathrm{e}^{-i\omega t}a, \\ a^\dagger(t) =& \mathrm{e}^{i\mathrm{H}t}a^\dagger\mathrm{e}^{-i\mathrm{H}t} = \mathrm{e}^{i\omega t}a^\dagger. \end{aligned} \tag{2.36}$$

The last equation follows directly from the Heisenberg equation of motion:

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[a, \mathrm{H}] = -\frac{i}{\hbar}\hbar\omega\underbrace{[a, a^\dagger a]}_{a} = -i\omega a(t). \tag{2.37}$$

Using the relations for Q and P in terms of a and $a^\dagger$, we obtain

$$(\alpha Q + iP)\mu = 0 \implies \left[(\alpha\lambda^2 - 1)a + (\alpha\lambda^2 + 1)a^\dagger\right]\mu = 0 \tag{2.38}$$

The time dependent state $\mu(t) = \mathrm{e}^{-i\mathrm{H}t}\mu$ satisfies

$$\begin{aligned}
&\mathrm{e}^{-i\mathrm{H}t}\left[(\alpha\lambda^2 - 1)a + (\alpha\lambda^2 + 1)a^\dagger\right]\mathrm{e}^{i\mathrm{H}t}\mu(t) = \\
&\left[(\alpha\lambda^2 - 1)a(-t) + (\alpha\lambda^2 + 1)a^\dagger(-t)\right]\mu(t) = \\
&\left[(\alpha\lambda^2 - 1)\mathrm{e}^{i\omega t}a + (\alpha\lambda^2 + 1)\mathrm{e}^{-i\omega t}a^\dagger\right]\mu(t) = 0.
\end{aligned} \tag{2.39}$$

This equation has the same form of (2.37), but with time-independent α, which allows us to write

$$\frac{\lambda^2\alpha(t) - 1}{\lambda^2\alpha(t) + 1} = \frac{\lambda^2\alpha - 1}{\lambda^2\alpha + 1}\mathrm{e}^{2i\omega t}. \tag{2.40}$$

Solving for $\alpha(t)$ we get

$$\alpha(t) = \frac{\lambda^2\alpha\cos(\omega t) - i\sin(\omega t)}{\lambda^2\left(\cos(\omega t) - i\lambda^2\alpha\sin(\omega t)\right)}. \tag{2.41}$$

Thus, the wave packet remains Gaussian at all times, while its width oscillates. There is another, more formal, definition of a squeezed state based on the squeeze operators.

Definition 1 *A squeeze operator is a unitary operator which, when applied to the oscillator vacuum state, produce a squeeze state*

One example is

$$U(\theta) = \exp\left[\theta((a)^2 - (a^\dagger)^2)/2\right]. \tag{2.42}$$

This operator has the following properties:

$$\begin{aligned}
U^\dagger(\theta)aU(\theta) &= \cosh(\theta)a - \sinh(\theta)a^\dagger, \\
U^\dagger(\theta)a^\dagger U(\theta) &= \cosh(\theta)a^\dagger - \sinh(\theta)a.
\end{aligned} \tag{2.43}$$

Proof: First, define the skew-hermitian $A \coloneqq (\theta(a^\dagger)^2 - \theta a^2)/2$, so that $U^\dagger = \mathrm{e}^A$. The left hand side of the first equality is then $\mathrm{e}^A a\mathrm{e}^{-A}$. Recalling the Campbell identity, which is a specific case of the Baker–Campbell–Hausdorff formula:

$$\mathrm{e}^A B\mathrm{e}^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}[\underbrace{A, [A, \ldots, [A,}_{\text{n times}} B]\cdots]], \tag{2.44}$$

which is true for any pair of operators A and B. As can be easily verified, we have

$$\begin{aligned}
[A, a] &= -\theta a^\dagger, \\
[A, a^\dagger] &= -\theta a,
\end{aligned} \tag{2.45}$$

and using these equalities, we have

$$\underbrace{[A,[A,\ldots,[A,}_{\text{n times}}B]\cdots]] = \begin{cases} \theta^n a, & \text{for n even,} \\ -\theta^n a^\dagger, & \text{for n odd.} \end{cases} \tag{2.46}$$

So we get

$$U^\dagger(\theta) a U(\theta) = a\sum_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!} - a^\dagger \sum_{n=0}^{\infty}\frac{\theta^{2n+1}}{(2n+1)!} = \cosh(\theta)a - \sinh(\theta)a^\dagger. \tag{2.47}$$

The second equality follows in a similar way, and its valid to stress that θ may be promoted to complex, and will have a similar relation, and all derivations follows.

We also have

$$\begin{aligned} U^\dagger(\theta) q U(\theta) &= U^\dagger(\theta)\frac{\lambda}{\sqrt{2}}(a + a^\dagger)U(\theta) = \mathrm{e}^{-\theta}q, \\ U^\dagger(\theta) p U(\theta) &= U^\dagger(\theta)\frac{i\hbar}{\sqrt{2}\lambda}(a - a^\dagger)U(\theta) = \mathrm{e}^{\theta}p. \end{aligned} \tag{2.48}$$

Now, we can prove that $U(\theta)$ is a squeeze operator. In fact, put $U(\theta)|0\rangle$ in (2.35). It is easy to see that the following equation can be obtained

$$\left(\frac{\mathrm{e}^{-\theta}\lambda}{\sqrt{2}} - \frac{\mathrm{e}^{\theta}\hbar}{\alpha\sqrt{2}\lambda}\right)a^\dagger|0\rangle = 0 \implies \alpha\lambda^2 = \hbar \mathrm{e}^{2\theta}. \tag{2.49}$$

Hence $U(\theta)$ is a squeeze operator with the given θ.

2.2.3 Squeezed states from time evolution

A squeezed state rises naturally from the vacuum state provided that the harmonic oscillator Hamiltonian is time dependent. In fact, consider the Schrödinger dynamic

$$i\hbar\partial_t\psi = \mathrm{H}(t)\psi. \tag{2.50}$$

and that this time dependence is on the frequency $\omega(t)$ and mass $m(t)$. Since we have a time dependent Hamiltonian, we integrate it formally in the interval $[0,t]$, in the Riemann sense:

$$\psi(t) = S(t)\psi_0 = \lim_{N\to\infty}\prod_{j=1}^{\infty}\mathrm{e}^{-\frac{i}{\hbar}\mathrm{H}(t_j)\Delta t}\psi_0, \tag{2.51}$$

with the ordinary partition.

Proposition 1 *The evolving state in* (2.51) *is a squeezed state.*

Proof: Define the quantity

$$\hat{A}(t) = \hat{S}(t)\left(P(0)\hat{q} - Q(0)\hat{p}\right)\hat{S}^\dagger(t), \tag{2.52}$$

where $P(t)$ and $Q(t)$ are functions yet to be found and $\hat{S}(t)$ is the same operator as in (2.51). $S(t)$ is unitary, and can be easily seen if we submit it to the Schrödinger equation, alongside with $S^\dagger(t)$. Thus, at all times

$$\hat{A}(t)\psi(t) = \hat{S}(t)\left(P(0)\hat{q} - Q(0)\hat{p}\right)\hat{S}^\dagger(t)\hat{S}(t)\psi_0 = 0, \tag{2.53}$$

since ψ_0 is a state of the harmonic oscillator and therefore is a squeezed state. Now, if we submit $A(t)$ to the equation of motion, i.e.,

$$\hbar\partial_t A(t) = i[A_0, \mathrm{H}] \tag{2.54}$$

Since the H is quadratic in q and p, if $A(t)$ is a polynomial of degree n in q and p, then $[A, \mathrm{H}]$ is still of degree n. Thus, as $[A_0, \mathrm{H}]$ starts as a linear function, it is natural to suppose that A will be linear in q and p as well, and we can write $A(t) = P(t)\hat{q} - Q(t)\hat{p}$. Using the equation of motion, we find

$$\hbar A(t) = i[(P\hat{q} - Q\hat{p}), \left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2\right)] = -\hbar\left(\frac{P}{m}\hat{p} + m\omega^2 Q\hat{q}\right), \tag{2.55}$$

and since $\partial_t A(t) = \dot{P}\hat{q} - \dot{Q}\hat{p}$, we conclude

$$\dot{Q} = P/m, \quad \dot{P} = -m\omega^2 Q, \tag{2.56}$$

which coincides with the classical Hamiltonian equations for the harmonic oscillator. Thus, our supposition was correct and since $A(t)\psi(t) = 0$ for all times, we put $\alpha(t) = -iP(t)/Q(t)$ and $A(t)\psi(t)$ have the form of (2.35). ■

2.3 Boguliubov Transformation

A Boguliubov transformation can be seen as an isomorphism between canonical commutative (or anticomumtative) algebra. Physically, it is a linear transformation of creation/annihilation operators that preserve the algebraic relations among them [37, 38]. Instead of going into mathematical details and things alike, let us write two examples: one for the fermionic and other for the bosonic case.

2.3.1 Fermionic Case

Consider, for fermion operators, the Hamiltonian

$$\mathrm{H} = \alpha(c_1^\dagger c_1 + c_2^\dagger c_2) + \beta(c_1^\dagger c_2^\dagger + c_2 c_1), \tag{2.57}$$

which arises, for example, in the BCS theory of superconductivity [39]. Note that β must be real for the Hamiltonian to be hermitian.

The fermionic Boguliubov transformation is

$$\begin{aligned} c_1^\dagger &= u d_1^\dagger + v d_2 \\ c_2^\dagger &= u d_2^\dagger - v d_1, \end{aligned} \tag{2.58}$$

where u, v are real numbers since we restricted ourselves to real β, and called Boguliubov coefficients. Now, we suppose that the relation of anticommutation (since we are working with fermions) works on both sets of operators. Then, we have

$$\{c_1^\dagger, c_1\} = u^2\{d_1^\dagger, d_1\} + v^2\{d_2^\dagger, d_2\}, \tag{2.59}$$

which implies $u^2 + v^2 = 1$, suggesting the parametrization $u = \cos(\theta)$ and $v = \sin(\theta)$. The remaining step is to diagonalize the Hamiltonian. First, we write as

$$\mathrm{H} = \frac{1}{2}\begin{pmatrix} c_1^\dagger & c_2 & c_2^\dagger & c_1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & -\alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & -\beta & -\alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \\ c_2 \\ c_1^\dagger \end{pmatrix} + \alpha, \tag{2.60}$$

where the relation $\{c_i, c_i^\dagger\} = 1$ was used. Now, consider the upper block

$$\begin{pmatrix} c_1^\dagger & c_2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix}, \tag{2.61}$$

together with the Boguliubov transformation

$$\begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2^\dagger \end{pmatrix}. \tag{2.62}$$

Then, we have

$$\begin{aligned} &\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \\ &= \begin{pmatrix} \alpha\cos(2\theta) - \beta\sin(2\theta) & \beta\cos(2\theta) + \alpha\sin(2\theta) \\ \beta\cos(2\theta) + \alpha\sin(2\theta) & \beta\sin(2\theta) - \alpha\cos(2\theta) \end{pmatrix}, \end{aligned} \tag{2.63}$$

where the matrices $\begin{pmatrix} d_1 & d_2^\dagger \end{pmatrix}$ and alike were omitted. If we choose θ such that $\tan(2\theta) = -\frac{\beta}{\alpha}$, we will have $\cos(2\theta) = \frac{\alpha^2}{\sqrt{\alpha^2+\beta^2}}$ and (2.63) is equal to

$$\begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \tag{2.64}$$

where $\gamma = \sqrt{\alpha^2 + \beta^2}$. Including the other block of the Hamiltonian, we conclude

$$\mathrm{H} = \gamma(d_1 d_1^\dagger + d_2 d_2^\dagger) + \alpha - \gamma\,. \tag{2.65}$$

2.3.2 Bosonic Case

We use the same Hamiltonian, but the Boguliubov transformation reads as

$$\begin{aligned} c_1^\dagger &= u d_1^\dagger + v d_2 \\ c_2^\dagger &= u d_2^\dagger + v d_1 . \end{aligned} \tag{2.66}$$

Note that the sign was chosen differently to ensure that the commutation relations for d_1 and d_2 imply the result $[c_1, c_2] = 0$. We also require

$$[c_1, c_1^\dagger] = u^2[d_1, d_1^\dagger] - v^2[d_2, d_2^\dagger] = 1 \implies u^2 - v^2 = 1, \tag{2.67}$$

suggesting the parametrization $u = \cosh(\theta)$ and $v = \sinh(\theta)$. Now, we introduce the Hamiltonian matrix notation as before:

$$\text{H} = \frac{1}{2}\begin{pmatrix} c_1^\dagger & c_2 & c_2^\dagger & c_1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \\ c_2 \\ c_1^\dagger \end{pmatrix} - \alpha, \tag{2.68}$$

where, for bosons, we used $[c_i, c_i^\dagger]$=1. Again, focusing only on the 2x2 upper block and using the Boguliubov transformation

$$\begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2^\dagger \end{pmatrix}, \tag{2.69}$$

we get

$$\begin{aligned} &\begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} = \\ &\qquad = \begin{pmatrix} \alpha\cosh(2\theta) + \beta\sinh(2\theta) & \beta\cosh(2\theta) + \alpha\sinh(2\theta) \\ \beta\cosh(2\theta) + \alpha\sinh(2\theta) & \alpha\cosh(2\theta) + \beta\sinh(2\theta) \end{pmatrix}, \end{aligned} \tag{2.70}$$

where (again) the matrices $\begin{pmatrix} d_1 & d_2^\dagger \end{pmatrix}$ and alike were omitted. If we choose θ such that $\tanh(2\theta) = -\frac{\beta}{\alpha}$, we will have $\cos(2\theta) = \frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}}$ and (2.70) is equal to

$$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \tag{2.71}$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. Note that $\alpha > \beta$, and if this is not the case, the Hamiltonian has not a stable equilibrium, but instead represents a system at an unstable point. Including the other block of the Hamiltonian, we conclude

$$\text{H} = \gamma(d_1^\dagger d_1 + d_2^\dagger d_2) - \alpha + \gamma\,. \tag{2.72}$$

2.4 Cauchy Theory

In this section, we want to stress two mathematical tools that were widely used in this project: the residue theorem and the Sokhotski-Plemelj formula. The latter is often not mentioned in standard quantum field theory books, although being of great use. We start this section with some definitions and important formulas from complex analysis. It is not our objective to be completely rigorous and/or mathematically formal, but some level of serious argumentation will be used throughout this section.

2.4.1 Holomorphic and analytic functions

Let C be a smooth curve[2] in the complex plane. If $f(z)$ is continuous on $\mathbb{C}$, then the complex integral

$$\int_C f(z)\mathrm{d}z \tag{2.73}$$

can be defined and expressed in terms of real integrals. In fact, write

$$f(z) = u(x,y) + i(x,y), \quad \mathrm{d}z = \mathrm{d}x + i\mathrm{d}y \implies \tag{2.74}$$

$$\implies \int_C f(z)\mathrm{d}z = \int_C (u\mathrm{d}x - v\mathrm{d}y) + i\int_C (v\mathrm{d}x + u\mathrm{d}y) \tag{2.75}$$

and the integrals in the rhs are known to exist. Our next concept is the derivative of a complex function:

Definition 2 *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a continuous functions, where U is an open set of $\mathbb{C}$. We say that f is holomorphic in $z_0 \in U$ if the limit*

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}, \tag{2.76}$$

exists. The complex number $f'(z_0)$ is called derivative of f in z_0. If f is holomorphic in all points of U, we say that f is holomorphic in U.

If we set $f(z) = u(x,y) + iv(x,y)$, (2.76) can be written as

$$f'(z) = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}. \tag{2.77}$$

It should be clear that, in order to ensure the existence of the derivative, the rhs have the same value for arbitrary $\Delta z \to 0$. If we set $\Delta z = \Delta x$, then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{2.78}$$

Alternatively, if we set $\Delta z = i\Delta y$, we get

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \tag{2.79}$$

[2] It can be a piecewise smooth curve. Just refer to the Riemann-Stieltjes integral

and it naturally follows that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.80}$$

These are the *Cauchy-Riemann conditions*, and they follow directly from the definition of the derivative. The inverse theorem also holds:

Theorem 1 *If $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives satisfying Cauchy-Riemann conditions in some neighborhood of z, then $f(z) = u + iv$ is holomorphic at z [40] .*

Now, one may ask about the existence of partial derivatives, given the way we treated holomorphic functions. This calls for one more definition:

Definition 3 *Let $U \subset \mathbb{C}$ be an open set. We say that a function $f : U \to \mathbb{C}$ is analytic if, for all $z_0 \in U$, there exists a power series with a convergence radius $\rho > 0$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z_0)(z - z_0)^n \tag{2.81}$$

for all $z \in U$ that satisfies $|z - z_0| < \rho$.

In complex analysis, a holomorphic function is an analytic function, although they are defined in different ways. That analytic functions are holomorphic can be verified by definition, but the converse also holds true [41].

2.4.2 Cauchy theorem and other theorems

Integrals of analytic functions possess some very important properties, and (perhaps) the most fundamental one is expressed by the Cauchy theorem:

Theorem 2 *If $f(z)$ is analytic in a simply connected domain D, and γ is a (piecewise smooth) simple closed curve in D, then*

$$\oint_\gamma f(z)\mathrm{d}z = 0. \tag{2.82}$$

Proof:

We present a derivation based on the Stokes Theorem, and a general proof of this important theorem can be found in any complex analysis book. First, recall that if γ is a simple closed curve in a simply connected domain D and if $P(x,y)$ and $Q(x,y)$ have continuous first partial derivatives in D, then

$$\oint_\gamma (P\mathrm{d}x + Q\mathrm{d}y) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}S, \tag{2.83}$$

where S is the area bounded by γ. One may recognize this as a special case of the Stokes Theorem, called Green's Theorem. Since $f(z)$ is analytic, the continuity of partial derivatives of u and v hold true, and the Green's theorem therefore can be applied. And, if we use Cauchy-Riemann conditions, the Cauchy theorem is proved. In fact,

$$\oint_\gamma (u\mathrm{d}x - v\mathrm{d}y) = \iint_S \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \mathrm{d}x\mathrm{d}y = 0, \tag{2.84}$$

$$\oint_\gamma (v\mathrm{d}x + u\mathrm{d}y) = \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y = 0. \tag{2.85}$$

■

There is a converse of this theorem and it is called Morera theorem, which is valid mentioning:

Theorem 3 *Morera theorem: if $f(z)$ is continuous in a domain D and if $\oint f(z)\mathrm{d}z = 0$ for every simple closed path in D with its interior also in D, then $f(z)$ is analytic in D.*

The vanishing of a contour integral is closely related to the independence of path of an integral. One last theorem that is very important to our analysis, and that I will not prove, is the following:

Suppose that z_0 is a fixed point. If the integral $\int_{z_0}^{z} f(\xi)\mathrm{d}\xi$ is independent of the integration path, then it must represent a function of z. Let us call it $F(z)$. Then, this function is a primitive function of $f(z)$, as follows from the fundamental theorem of integral calculus: if $f(z)$ is analytic in a simply connected domain D, then the function

$$F(z) = \int_{z_0}^{z} f(\xi)\mathrm{d}\xi \tag{2.86}$$

is also analytic in D and $f(z) = \frac{\mathrm{d}}{\mathrm{d}z}F(z)$.

The Cauchy theorem can be used in many forms, and the most basic one is the Cauchy integral formula:

Proposition 2 *(Cauchy Integral Formula) If $f(z)$ is analytic inside and on γ and if the point $z = a$ is in the interior of γ, then*

$$\oint \frac{f(z)\mathrm{d}z}{z-a} = 2\pi i f(a). \tag{2.87}$$

Proof:

Construct a circle $\tilde{\gamma}$ about $z = a$ with an arbitrary small radius r such that this circle is within γ. Now, connect the two circles using two straight lines which do not intersect themselves and leave the point a out of the total closed loop, as can be seen in figure 1:

Following this scheme, we find by the use of the Cauchy Integral Theorem

$$0 = -\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z + \int_{\tilde{\gamma}} \frac{f(z)}{z-a}\mathrm{d}z, \tag{2.88}$$

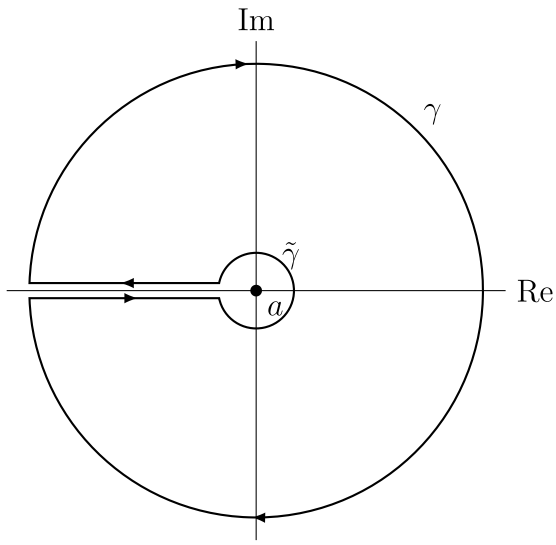

Figure 1 – Determination of the loop $\tilde{\gamma}$

since the other terms come from the straight lines, which are obtained from one another by just reversing the orientation, and the minus sign comes from the orientation that was chosen for the interal in γ. Then

$$\int_{\gamma} \frac{f(z)}{z-a} \mathrm{d}z = \int_{\tilde{\gamma}} \frac{f(z)}{z-a} \mathrm{d}z. \tag{2.89}$$

The integral in the rhs can be easily evaluated by the parametrization $\tilde{\gamma}(\theta) = a + r\mathrm{e}^{i\theta}$, $\theta \in [0, 2\pi]$, because the inner circle was oriented counterclockwise. Thus

$$\int_{\gamma} \frac{f(z)}{z-a} \mathrm{d}z = \int_{0}^{2\pi} \mathrm{d}\theta \frac{f(a + r\mathrm{e}^{i\theta})}{r\mathrm{e}^{i\theta}} ir\mathrm{e}^{i\theta} = i \int_{0}^{2\pi} \mathrm{d}\theta f(a + r\mathrm{e}^{i\theta}). \tag{2.90}$$

For last, if we take the limit $r \to 0$ in the equation above, the proposition follows. ∎

The Cauchy integral formula shows a remarkable property of analytic functions: if one knows the valued of a function in a closed contour, then the value at an arbitrary point inside the contour can be known. To emphasize this, its common to find in the literature the following representation:

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\xi)\mathrm{d}\xi}{\xi - z}. \tag{2.91}$$

Since the Leibniz rule can be used, one can calculate the $n-$th derivative of a function by a induction process:

$$\frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n} = \frac{n!}{2\pi i} \oint \frac{f(\xi)\mathrm{d}\xi}{(\xi - z)^{n+1}}. \tag{2.92}$$

2.4.3 Laurent series

The so called Taylor series in real analysis is a very important concept, either theoretically or practically speaking. In complex analysis, this theorem also holds [41], but other very crucial series expansion can be derived, and it plays a key role in the so called Residue theorem (section 2.4.4):

Theorem 4 *Every function f(z) analytic in an annulus*

$$R_1 < |z - z_0| < R_2, \tag{2.93}$$

can be expanded in a series of positive and negative powers of $(z - z_0)$*, namely*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n. \tag{2.94}$$

This series is known as the Laurent series, it is unique for a given annulus, and

$$c_n = \frac{1}{2\pi i} \oint_{c_i} \frac{f(z)\mathrm{d}z}{(z - z_0)^{n+1}}, \tag{2.95}$$

where c_i *is a circle of radius* $R_1 < r_i < R_2$.

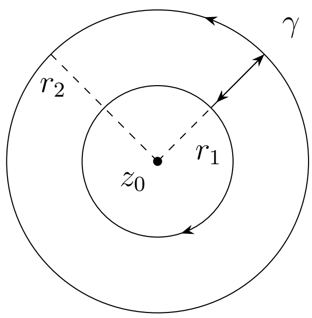

Figure 2 – The radii r_1 and r_2 are such that $R_1 < r_1 < r_2 < R_2$. Thus, f is analytic in the region delimited by γ.

Proof: Consider the Figure 2. Then, following the theorem notation we get

$$f(z) = \frac{1}{2\pi i} \oint_{c_2} \frac{f(\xi)\mathrm{d}\xi}{\xi - z} + \frac{1}{2\pi i} \oint_{c_1} \frac{f(\xi)\mathrm{d}\xi}{(\xi - z)}. \tag{2.96}$$

The first integral can be treated as

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{c_2} \frac{f(\xi)\mathrm{d}\xi}{\xi - z} &= \frac{1}{2\pi i} \oint_{c_2} \frac{f(\xi)}{\xi - z_0 - z + z_0}\mathrm{d}\xi \\
&= \frac{1}{2\pi i} \oint_{c_2} \frac{f(\xi)}{\xi - z_0} \frac{1}{1 - (z - z_0)/(\xi - z_0)}\mathrm{d}\xi \\
&= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \oint_{c_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\mathrm{d}\xi,
\end{aligned} \tag{2.97}$$

whereas the second integral is treated by expanding $1/(\xi - z)$ in slightly different geometric series:

$$\frac{1}{\xi - z} = -\frac{1}{z - z_0} \frac{1}{1 - (\xi - z_0)/(z - z_0)} = -\sum_{m=0}^{\infty} \frac{(\xi - z_0)^m}{(z - z_0)^{m+1}}, \tag{2.98}$$

which is convergent by the ratio test. Then

$$\begin{aligned}\frac{1}{2\pi i}\oint_{c_1}\frac{f(\xi)\mathrm{d}\xi}{(\xi-z)} &= -\frac{1}{2\pi i}\oint_{c_1}\frac{f(\xi)\mathrm{d}\xi}{(\xi-z)} \\ &= \sum_{m=0}^{\infty}\frac{1}{(z-z_0)^{m+1}}\frac{1}{2\pi i}\oint_{c_1} f(\xi)(\xi-z_0)^m\mathrm{d}\xi.\end{aligned} \tag{2.99}$$

Replace m by $-(n+1)$, with $n<0$, and rewrite the above equation as

$$\frac{1}{2\pi i}\oint_{c_1}\frac{f(\xi)\mathrm{d}\xi}{(\xi-z)} = \sum_{n=-1}^{-\infty}(z-z_0)^n\frac{1}{2\pi i}\oint_{c_1}\frac{f(\xi)}{(\xi-z_0)^{n+1}}\mathrm{d}\xi. \tag{2.100}$$

Finally, the integrals in (2.97) and (2.99) may be just as well evaluated over a common circle c, concentric with c_1 and c_2, lying inside the annulus.

To prove the uniqueness, assume that an expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \tag{2.101}$$

exists and is valid in the annulus $R_1 < |z-z_0| < R_2$. Choose an arbitrary integer k, multiply both sides of this equation by $(z-z_0)^{-(k+1)}$ and integrate around a circle c such that a is enclosed by this circle, inside the annulus[3]:

$$\oint_c \frac{f(z)\mathrm{d}z}{(z-z_0)^{k+1}} = \sum_{n=-\infty}^{\infty} c_n\oint_c\frac{\mathrm{d}z}{(z-z_0)^{k+1-n}}. \tag{2.102}$$

All the integrals in the rhs will vanish except for $n=k$. In fact, given the integral

$$I = \oint\frac{\mathrm{d}z}{(z-z_0)^n} \tag{2.103}$$

with the contour $z = z_0 + r\mathrm{e}^{i\theta}$, we have

$$I = \int_{-\pi}^{\pi}\mathrm{d}\theta\, i r^{1-n}\mathrm{e}^{i(1-n)\theta} = \frac{r^{1-n}}{1-n}\mathrm{e}^{i(1-n)\theta}\Bigg|_{-\pi}^{\pi} = 0, \tag{2.104}$$

if $n\neq 1$, whose value is $2\pi i$. Therefore

$$\oint_c\frac{f(z)\mathrm{d}z}{(z-z_0)^{k+1}} = c_k 2\pi i. \tag{2.105}$$

■

A good discussion about the meaning and differences of the Laurent and Taylor series is presented by [40]: the part of the Laurent series consisting of positive powers of $(z-a)$ is called the *regular part*, and the other part is the *principal part*. The regular part reminds us of the the Taylor series, but it should be emphasized that the nth coefficient cannot be associated, in general, with $f^{(n)}(z_0)$ because the latter may not exist.[4]

[3] Therefore, the radius of this circle is limited.

[4] Of course, if the principal part is identically zero, then $f(z)$ is analytic at $z=z_0$, and the Laurent series is identical with the Taylor series.

2.4.4 Zeros, singularities and the residue theorem

Let us start this section by recalling a simple definition: A point $z = z_0$ is called a zero, or a root, of the function $f(z)$ if $f(z_0) = 0$. If $f(z)$ is analytic at $z = z_0$, the its Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \tag{2.106}$$

must have $c_0 = 0$. Let c_m be the first non-vanishing coefficient, then the zero is said to be of order m. If $m = 1$, is said to be a simple zero. In practical terms, the order of a zero can be found by calculating

$$\lim_{z \to z_0} \frac{f(z)}{(z-a)^n}, \tag{2.107}$$

and the first n for which this limit do not vanish will be the order, naturally.

If a function $f(z)$ is analytic in some neighborhood of z_0, except at z_0 itself, then it is said to have an isolated singularity at this point. It is customary to distinguish isolated singularities by the following types of behavior of $f(z)$:

- $|f(z)| < M \in \mathbb{R}$, i.e, remains bounded when $z \to z_0$;
- $f(z)$ is not bounded and $|f(z)|$ approaches infinity;
- Neither of the two cases above; in plain terms, $f(z)$ oscillates.

Examples of these types are

1. $f(z) = \sin(z)/z$
2. $f(z) = 1/\sin(z)$
3. $f(z) = \mathrm{e}^{1/z}$

Now, let z_0 be an isolated singularity. Let γ be a simple closed path surrounding z_0. Then the integral

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)\mathrm{d}z = \frac{1}{2\pi i} \sum_{n} c_n \oint_{\gamma} (z - z_0)^n \mathrm{d}z = c_{-1}, \tag{2.108}$$

is called the residue of f at z_0, i.e., the residue is the coefficient of the term $1/(z - z_0)$ appearing in the Laurent series of f. Furthermore, a function $f : U \subset \mathbb{C} \to \mathbb{C}$ is said to be *meromorphic* in u if the set of singular points $\mathbf{S}$ is discrete. The residues of a function at its isolated singularities find their application in the evaluation of integrals, complex or real, and the basis of this statement is the

Theorem 5 *(The Residue Theorem) Let $f(z): U \subset \mathbb{C} \to \mathbb{C}$ be a meromorphic function, and let γ be any simple closed curve oriented counterclockwise in U that does not meet any of $f(z)$ poles. Then*

$$\oint_{\gamma} f(z)\mathrm{d}z = 2\pi i \sum_{z \in \mathbf{S}_{\gamma}} R(f, z), \tag{2.109}$$

where $\mathbf{S}_{\gamma}$ is the set of singular points of f that lie inside of the trace of γ, and $R(f,z)$ is the residue of f at z

Proof:

Since the region delimited by γ is compact, there is only a finite number of residues inside γ. In figure 3, we have drawn a new contour, that also delimits a simple connected region, but where $f(z)$ is analytic. We do that for each pole in $\mathbf{S}_{\gamma}$. Thus, if we let c_j denotes the circle that closes the pole z_j, it follows from the Cauchy theorem that

$$0 = \oint_{\tilde{\gamma}} f(z)\mathrm{d}z = \oint_{\gamma} f(z)\mathrm{d}z - \sum_{z_j \in \mathbf{S}_{\gamma}} \oint_{c_j} f(z)\mathrm{d}z, \tag{2.110}$$

and using (2.108), we obtain the required result. See that although we have not explicitly chosen an orientation for the new $\tilde{\gamma}$, it should be clear that in either cases (the outer curve clockwise or not), the signs on (2.110) will remain the same. ■

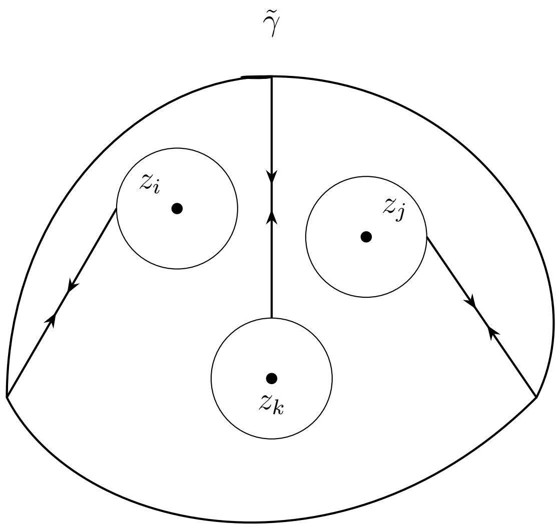

Figure 3 – New contour integration that encircles the poles of $f(z)$.

2.4.5 The Sokhotski-Plemelj theorem

It is customary to encounter real integrands which have poles in the integration contour. For instance, consider

$$\int_0^1 \frac{\mathrm{d}x}{x - x_0}. \tag{2.111}$$

If $x_0 \notin [0,1]$, this integral in well defined in the Riemann sense. However, if $x_0 \in (0,1)$, then this integral is not well defined in the Riemann sense. If such a problem appears in physical problems, we need to state how it appeared and its interpretation, and try to extract a finite value of it. One possibility is the Cauchy principal value:

Definition 4 *If $f(x)$ is integrated over an interval $[a,b]$ that contains a pole x_0, we define the principal value as*

$$P\int_a^b f(x)\mathrm{d}x = \lim_{\delta\to 0^+}\left(\int_a^{x_0-\delta} f(x)\mathrm{d}x + \int_{x_0+\delta}^b f(x)\mathrm{d}x\right) \tag{2.112}$$

if such a limit exists. The letter P in the left of the integral indicates a principal value.

One might ask if exists a connection between the Cauchy principal value and the residue theorem, since both of them deals with singularities. The answer is an important theorem, called Sokhotski-Plemelj theorem:

Theorem 6 *Suppose f is analytic in a neighborhood of $x_0 \in \mathbf{R}$, then*

$$\lim_{\epsilon\to 0^+}\int_a^b \frac{f(x)}{x-x_0 \mp i\epsilon}\mathrm{d}x = P\int_a^b \frac{f(x)}{(x-x_0)}\mathrm{d}x \pm i\pi f(x_0), \tag{2.113}$$

or even

$$\lim_{\epsilon\to 0^+}\frac{1}{x-x_0\mp i\epsilon} = \frac{1}{x-x_0} \pm i\pi\delta(x-x_0). \tag{2.114}$$

Proof:

Define the two limiting processes

$$\phi^{\pm}(x_0) = \lim_{\epsilon\to 0^+}\int_a^b \frac{f(x)}{x-x_0\mp i\epsilon}\mathrm{d}x. \tag{2.115}$$

We start by analyzing $\phi^+(x_0)$. Since $f(x)$ is analytic in a ball of radius R centered in x_0, let $0<\delta<R$. Then

$$\int_a^b \frac{f(x)}{(x-x_0-i\epsilon)}\mathrm{d}x = \int_{\gamma_\delta}\frac{f(z)}{x_0 - i\epsilon}\mathrm{d}z \tag{2.116}$$

where C_δ is the contour shown in Figure 4.

Therefore, we may take $\epsilon \to 0^+$ and we get

$$\phi^+(x_0) = \int_{\gamma_\delta}\frac{f(z)}{z-x_0}\mathrm{d}z, \tag{2.117}$$

and since $\phi^+(x_0)$ is independent of δ, we may write

$$\begin{aligned}\phi^+(x_0) &= \lim \delta \to 0^+\left(\int_a^{x_0-\delta}\frac{f(x)}{x-x_0}\mathrm{d}x + \int_{x_0+\delta}^b\frac{f(x)}{x-x_0}\mathrm{d}x + \int_{C_\delta}\frac{f(z)}{z-x_0}\mathrm{d}z\right)\\ &= P\int_a^b\frac{f(x)}{x-x_0}\mathrm{d}x + \lim_{\delta\to 0^+}\int_{C\delta}\frac{f(z)}{z-x_0}.\end{aligned} \tag{2.118}$$

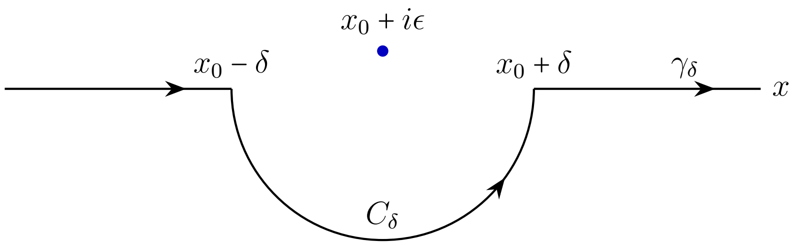

Figure 4 – The contour γ_δ. Now, ϵ can be taken to zero alongise this path and the integral will still exists.

But given that $f(z)$ is analytic in x_0, we find that

$$\frac{f(z)}{z - x_0} = \frac{1}{z - x_0} \sum_{n=0}^{\infty} a_n (z - x_0)^n = \sum_{n=-1}^{\infty} a_n (z - x_0)^n, \tag{2.119}$$

with $a_{-1} = f(x_0)$. Thus, if we use $C_\delta = x_0 + \mathrm{e}^{i\theta}$, $\theta \in [\pi, 2\pi]$, we get

$$\int_{C\delta} \frac{f(z)}{z - x_0} \mathrm{d}z = i\pi f(x_0) + 2 \sum_{n=0}^{\infty} \frac{a_{2n} \delta^{2n+1}}{2n + 1}. \tag{2.120}$$

The first term is easily obtained, just substitute the analytic expansion for $f(z)/(z - x_0)$, use the said parametrization and integrate for $n = -1$. Now, the second term will read as

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \int_{C_\delta} \mathrm{d}z (z - x0)^n &= \sum_{n=0}^{\infty} i a_n \delta^{n+1} \int_{\pi}^{2\pi} \mathrm{d}\theta \mathrm{e}^{i\theta(n+1)} = \\ &= \sum_{n} \frac{a_n \delta^{n+1}}{n + 1} [1 - \cos((n + 1)\pi)], \end{aligned} \tag{2.121}$$

which is 2 if n is even and 0 otherwise. Thus,

$$\lim_{\delta \to 0^+} \int_{C_\delta} \frac{f(z)}{z - x_0} \mathrm{d}z = i\pi f(x_0). \tag{2.122}$$

Finally,

$$\phi^+(x_0) = P \int_a^b \frac{f(x)}{x - x_0} \mathrm{d}x + i\pi f(x_0), \tag{2.123}$$

and as this hold true for any function f, we may write

$$\lim_{\epsilon \to 0^+} \frac{1}{x - x_0 - i\epsilon} = \frac{1}{x - x_0} + i\pi \delta(x - x_0). \tag{2.124}$$

A similar analysis for $\phi^-(x_0)$ concludes the theorem.

■

3 QUANTUM BROWNIAN MOTION

3.1 The Model Lagrangian

We consider a single non-relativistic particle of mass m in a harmonic potential of characteristic frequency ω_0 with Lagrangian

$$L_{\rm p} = \frac{m}{2}\dot{x}^2 - \frac{m\omega_0^2}{2}x^2, \tag{3.1}$$

such that $x = x(t)$ is the particle position, and hereafter a dot over a quantity means time-derivative. We assume that this particle interacts with a reservoir modeled by a continuum of oscillators labeled by ν, $\nu > 0$, with Lagrangian

$$L_{\rm R} = \frac{\mu}{2}\int_0^\infty {\rm d}\nu \left(\dot{R}^2 - \nu^2 R^2\right). \tag{3.2}$$

The reservoir variables are such that $R = R(t,\nu)$. We take the possibly time-dependent interaction between the test oscillator and the reservoir to be mediated by

$$L_{\rm int} = x\int_0^\infty {\rm d}\nu \beta \dot{R}, \tag{3.3}$$

where the coupling constant, $\beta = \beta(t,\nu)$, is assumed to be different from zero for all reservoir frequencies ν. Also, β is taken to satisfy $\beta(t,-\nu) = \beta(t,\nu)$. This type of coupling is well-known and it is the essence, for instance, of the Hopfield dielectric model [16]. Hence, the total Lagrangian is

$$L = L_{\rm p} + L_{\rm R} + L_{\rm int} = \frac{m}{2}\left(\dot{x}^2 - \omega_0^2 x^2\right) + \frac{\mu}{2}\int_0^\infty {\rm d}\nu\left(\dot{R}^2 - \nu^2R^2\right) + x\int_0^\infty {\rm d}\nu\beta\dot{R}. \tag{3.4}$$

Now, if a Lagrangian can be expressed as $L(t, x_i, \dot{x}_i)$, then the Euler-Lagrange equations read as

$$\frac{\delta L}{\delta x_i} - \frac{\rm d}{{\rm d}t}\frac{\delta L}{\delta \dot{x}_i} = 0, \tag{3.5}$$

where $\frac{\delta L}{\delta \dot{x}_i}$ is called conjugate momenta. Thus, applying these equations to the (3.4), we get

$$\frac{\partial L}{\partial x} = -m\omega_0^2 x + \int_0^\infty {\rm d}\nu\beta\dot{R}, \quad p \coloneqq \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{3.6}$$

which implies

$$\ddot{x} + \omega_0^2 x = \frac{1}{m}\int_0^\infty {\rm d}\nu\beta\dot{R}. \tag{3.7}$$

Now, for the reservoir, we get

$$\frac{\delta L}{\delta R} = -\mu\nu^2 R, \quad Q \coloneqq \frac{\delta L}{\delta \dot{R}} = \mu\dot{R} + x\beta, \tag{3.8}$$

which permits us to conclude

$$\ddot{R} + \nu^2 R = -\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}(x\beta), \tag{3.9}$$

and these are the equations of motion of the given system. We see that if $\beta \neq 0$, Eq. (3.7) describes a harmonic oscillator driven by the external force $F = \int_0^\infty \mathrm{d}\nu \beta \dot{R}$, which in turn depends on the oscillator velocity $v = \dot{x}$ by means of Eq. (3.9). Therefore, the coupling given by Eq. (3.3) models some sort of viscous medium for the oscillator under study. Note also that this coupling is distinct from the coupling of [42, 43], but is included in the generalized model of [44].

Notice that when the reservoir oscillators are quantized, the force F undergoes quantum fluctuations, giving rise to a sort of Quantum Brownian motion [45]. In order to gain a deeper insight on how this occurs, let us consider the (weak coupling) limit where $\beta \to 0$, for which the R.H.S. of Eq. (3.7) approaches zero. In physical terms, this corresponds to the case where the oscillator under study is a test particle that cannot interfere with the reservoir in an appreciable manner. In this limit, the momentum canonically conjugated to R is

$$Q = \mu\dot{R} + \beta x \approx \mu\dot{R}, \tag{3.10}$$

and let us assume first that only the reservoir oscillators are quantized, i.e., an operator-valued distribution for $R(t,\nu)$ subjected to the commutation relation

$$[R(t,\nu), Q(t,\nu')] = i\delta(\nu - \nu') \tag{3.11}$$

is known. Then the force F is also an operator, since it is defined in terms of R, and we find that Eq. (3.7) becomes a Langevin equation for x leading to $\langle(\Delta v)^2\rangle \neq 0$ by means of the fluctuations of F. Here, the expectation value is taken with respect to the reservoir quantum state. This is precisely the regime considered in the quantum Brownian motion of [13], where the force F was sourced by Casimir stresses on a charged test particle.

In our model, though, we do not assume the weak coupling regime and we also consider the particle's own "quantumness." Thus uncertainties in measuring the particle velocity exist independently of the particle's interaction with the reservoir. Therefore, from the knowledge of $\langle(\Delta v)^2\rangle$ as a function of time, one can, in principle, determine how the interaction with the reservoir eventually affects the particle's Brownian motion.

The considerations above are general and hold for arbitrary time-dependencies of the coupling constant. Nevertheless, in our work we are interested in transient effects when the particle's interaction with the reservoir starts at a given time, say, $t = 0$. Specifically, we consider the case for which $\beta = 0$ for $t < 0$, henceforth called the non-interacting period, and β assumes a time-independent value after $t > 0$, the interacting regime. This assumption has the advantage of leading to analytical solutions for the quantum correlations.

3.2 Energy Conservation

The major advantage of adopting microscopic models, which are notoriously convoluted, over Langevin methods is the level of control over the conservation laws. Following [46], we take the particle mechanical and kinetic energy as probes of the quantum fluctuations on the system. To obtain the systems Hamiltonian, we first recall the conjugate momenta:

$$\begin{aligned} p &= m\dot{x}, \\ Q &= \mu\dot{R} + x\beta, \end{aligned} \tag{3.12}$$

since the variables $\dot{x}$ and $\dot{R}$ can be written in terms of p and Q, we may calculate the Hamiltonian:

$$\begin{aligned} \mathrm{H} &= p\dot{x} + \int_0^\infty \mathrm{d}\nu Q\dot{R} - L \\ &= \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 + \frac{1}{2\mu}\int_0^\infty \mathrm{d}\nu \left(Q^2 + x^2\beta^2 + \mu^2\nu^2R^2 - 2x\beta Q\right) \\ &= \frac{p^2}{2m} + m\omega_\mathrm{e}^2x^2 - \frac{x}{\mu}\int_0^\infty \mathrm{d}\nu\beta Q + \frac{1}{2}\int_0^\infty \mathrm{d}\nu\left(\frac{Q^2}{\mu} + \mu\nu^2R^2\right) \end{aligned} \tag{3.13}$$

where the time-dependent "effective" frequency ω_e is defined by

$$\omega_\mathrm{e}^2 = \omega_0^2 + \frac{1}{m\mu}\int_0^\infty \mathrm{d}\nu\beta^2. \tag{3.14}$$

We note that because the system is interacting there is some degree of freedom in interpreting the particle's energy, the effective frequency, the interaction energy, and the reservoir energy, because only the full Hamiltonian, H, is unambiguously defined. The quantity ω_e only coincides with the oscillator effective frequency in certain cases, where a careful limit of the system parameters should be observed. We cite [47] for a recent study of a particle in a time-dependent harmonic potential.

For our purposes, and because we work in the Heisenberg picture, it is instructive to express the Hamiltonian in terms of velocities rather than momenta as in Eq. (3.13), which will allow for the identification of how the particle mechanical energy changes as the interaction is turned on. We find that

$$\begin{aligned} \mathrm{H} = &\frac{m}{2}\dot{x}^2 + \frac{m\omega_0^2}{2}x^2 + \frac{1}{2\mu}\int_0^\infty \mathrm{d}\nu\beta^2x^2 - \frac{x}{\mu}\int_0^\infty \mathrm{d}\nu\left(\beta\mu\dot{R} + x\beta^2\right) + \\ &+ \frac{1}{2}\int_0^\infty \mathrm{d}\nu\left(\mu\dot{R}^2 + 2\dot{R}x\beta + \frac{x^2\beta^2}{\mu} + \nu^2\mu R^2\right) \\ = &\underbrace{\frac{m}{2}\dot{x}^2 + \frac{m\omega_0^2}{2}x^2}_{\mathrm{H_p}} + \underbrace{\frac{\mu}{2}\int_0^\infty \mathrm{d}\nu\left(\dot{R}^2 + \nu^2R^2\right)}_{\mathrm{H_R}}, \end{aligned} \tag{3.15}$$

where the first two terms were identified as $\mathrm{H_p}$, while the latter is $\mathrm{H_R}$. We note that H_p coincides with the particle's energy before the interaction is turned on, at $t = 0$. Let us analyze the conservation of energy. We know that

$$\frac{\mathrm{dH_p}}{\mathrm{d}t} = m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x}, \quad \frac{\mathrm{dH_R}}{\mathrm{d}t} = \mu \int_0^\infty \mathrm{d}\nu \left(\dot{R}\ddot{R} + \nu^2 R\dot{R}\right). \tag{3.16}$$

Using equation (3.7) and substituting in $\frac{\mathrm{dH_p}}{\mathrm{d}t}$, we get

$$\frac{\mathrm{dH_p}}{\mathrm{d}t} = m\dot{x}\left(\frac{1}{m}\int_0^\infty \mathrm{d}\nu\beta R - \omega_0^2 x\right) + m\omega_0^2 x\dot{x} = \dot{x}\int_0^\infty \mathrm{d}\nu\beta \dot{R}. \tag{3.17}$$

Writing (3.9) as

$$\ddot{R} = -\frac{1}{\mu}\dot{\beta}x - \frac{1}{\mu}\beta\dot{x} - \nu^2 R, \tag{3.18}$$

hence

$$\begin{aligned} \frac{\mathrm{dH_R}}{\mathrm{d}t} &= \int_0^\infty \mathrm{d}\nu \left(\mu\dot{R}\ddot{R} + \mu\nu^2 R\dot{R}\right) = \\ &= \int_0^\infty \mathrm{d}\nu \left(-\dot{\beta}\dot{R}x - \beta\dot{x}\dot{R} - \mu\nu^2 R\dot{R} + \mu\nu^2 R\dot{R}\right) \\ &= \int_0^\infty \mathrm{d}\nu - (\beta\dot{x}\dot{R} + \dot{\beta}x\dot{R}). \end{aligned} \tag{3.19}$$

Therefore,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -x\int_0^\infty \mathrm{d}\nu\dot{\beta}\dot{R}, \tag{3.20}$$

and thus $\mathrm{d}H/\mathrm{d}t = 0$ for $t \neq 0$, when β is time-independent. Accordingly, changes in H_p are followed by changes in H_R for $t \neq 0$ ensuring that H remains constant. We will adopt the observable H_p as a measure of how the particle energy changes as it enters the interacting regime.

In what follows, the quantization of this theory is worked out for all possible choice of parameters in order to determine the fluctuations of x and $\dot{x}$, the probes of the environment-induced Brownian motion on the particle. We are interested in studying how $\langle H_\mathrm{p}\rangle$ changes as the interactions are turned on. We also discuss changes in $\langle T\rangle$, where $T = mv^2/2$ is the particle kinetic energy. It should be stressed that the identification of the particle's energy is one of the main strengths of the microscopic model over Langevin equation methods, for which in general it is not possible to write down the system Hamiltonian and it is not always clear how to treat time-dependent scenarios. This is similar to what occurs in discussing energy in effective models for polarization and magnetization in electrodynamics [48].

3.3 Canonical Quantization

We work in the Heisenberg picture, for which quantization can be obtained from the expansion of the quantities x and R in a complete set of mode functions followed by imposition of the canonical commutation relations on the Fourier coefficients. Therefore, we start by solving for the most general solution of the equations of motion.

We define the two-component "field" Ψ, such that $\Psi_1 = \Psi_1(t) = x(t)$ and $\Psi_2 = \Psi_2(t,\nu) = R(t,\nu)$. We note that ν plays the same coordinate role as t. With this definition, Eqs. (3.7) and (3.9) combine into a single field equation for Ψ, which in turn implies the following proposition

Proposition 3 *Under equations* (3.7) *and* (3.9)*, the following quantity is time-independent for* ***any two solutions*** *of the field equation:*

$$\begin{aligned}\langle\Psi,\Psi'\rangle = i\Bigg[& m(\Psi_1^*\partial_t\Psi_1' - \Psi_1'\partial_t\Psi_1^*) + \mu\int_0^\infty \mathrm{d}\nu\left(\Psi_2^*\partial_t\Psi_2' - \Psi_2'\partial_t\Psi_2^*\right) \\ & - \int_0^\infty \mathrm{d}\nu\beta(\Psi_1^*\Psi_2' - \Psi_1'\Psi_2^*)\Bigg],\end{aligned} \tag{3.21}$$

and therefore this is the defined sesquilinear form for our field. Being time independent is fundamental since it assures that orthogonal modes in a given time t_0 will remain orthogonal.

Proof:

It is easier to look to the "old" motion equations using the given field notation. In fact, we get

$$\begin{aligned}\Psi_1^*\left(\partial_t^2 + \omega_0^2\right)\Psi_1' &= \Psi_1^*\frac{1}{m}\int_0^\infty \mathrm{d}\nu\beta\dot{\Psi}_2' \\ \Psi_1'\left(\partial_t^2 + \omega_0^2\right)\Psi_1^* &= \Psi_1'\frac{1}{m}\int_0^\infty \mathrm{d}\nu\beta\dot{\Psi}_2^*,\end{aligned} \tag{3.22}$$

and if we sum the two equations above, we get

$$\frac{\partial}{\partial t}m\left(\Psi_1^*\partial_t\Psi_1' - \Psi_1'\partial_t\Psi_1^*\right) = \int_0^\infty \mathrm{d}\nu\beta\left(\Psi_1^*\partial_t\Psi_2' - \Psi_1'\partial_t\Psi_2^*\right). \tag{3.23}$$

The treatment for the (3.9) is the same and we obtain

$$-\frac{\partial}{\partial t}\int_0^\infty \mathrm{d}\nu\mu\left(\Psi_2^*\partial_t\Psi_2' - \Psi_2'\partial_t\Psi_2^*\right) = \int_0^\infty \Psi_2^*\partial_t(\beta\Psi_1') - \Psi_2'\partial_t(\beta\Psi_1^*). \tag{3.24}$$

If we write the second part of (3.23) as

$$\frac{\partial}{\partial t}\int_0^\infty \mathrm{d}\nu\beta\Psi_1^*\Psi_2' - \beta\Psi_1'\Psi_2^* + \int_0^\infty \Psi_2^*\partial_t(\beta\Psi_1') - \Psi_2'\partial_t(\beta\Psi_1^*), \tag{3.25}$$

we recognize as the rhs of (3.24) and, after factorizing the global time derivative, we get

$$\begin{aligned}\frac{\partial}{\partial t}\Bigg[& m\left(\Psi_1^*\partial_t\Psi_1' - \Psi_1'\partial_t\Psi_1^*\right) + \mu\int_0^\infty \mathrm{d}\nu\left(\Psi_2^*\partial_t\Psi_2' - \Psi_2'\partial_t\Psi_2^*\right) \\ & - \int_0^\infty \mathrm{d}\nu\beta\left(\Psi_1^*\Psi_2' - \Psi_2^*\Psi_1'\right)\Bigg] = 0,\end{aligned} \tag{3.26}$$

which concludes the proof. The imaginary unit i in the formula comes from the following observation: since this equation should work for every solution, it does so for a simple wave solution $e^{-i\omega t}C_\omega$, where C_ω is a normalization constant. Putting this solution in the above formula, we get a complex quantity, and hence the necessity of the imaginary unity.

■

We can use this scalar product to find a complete set of positive norm field modes $\{\Psi_\omega\}$, i.e., $\langle\Psi_\omega, \Psi_{\omega'}\rangle = \delta_{\omega\omega'}$ such that

$$\Psi = \sum_\omega \left[a_\omega \Psi_\omega + a^*_\omega \Psi^*_\omega\right], \tag{3.27}$$

and $a_\omega = \langle\Psi_\omega, \Psi\rangle$. We note that ω here is a generic index that might assume continuous and/or discrete values. Also, it follows from Eq. (3.21) that if Ψ_ω is a positive norm solution, then Ψ^*_ω has negative norm, and both positive and negative norm modes are necessary to span the whole space of solutions. Finally, the Fourier coefficients a_ω are time-independent and uniquely determined once Ψ, $\partial_t\Psi$ are given in some initial instant of time, i.e., we have a well-defined Cauchy problem.

As customary in any field theory, different sets of field modes lead to physically distinct physical vacua. In our model, a privileged choice can be made in the non-interacting regime, for which the theory vacuum corresponds to *all* the oscillators in their fundamental states. Specifically, for $t < 0$, we look for field modes indexed by their positive frequency ω such that $\Psi_\omega \propto \exp(-i\omega t)$. These functions comprise a complete set of positive norm field modes.

Indeed, the first obvious field mode is given $\omega = \omega_0$, with $\Psi_{\omega_0,2} = 0$, and

$$\Psi_{\omega_0,1}(t) = \frac{1}{\sqrt{2\omega_0 m}} e^{-i\omega_0 t}, \tag{3.28}$$

which is already normalized: $\langle\Psi_{\omega_0}, \Psi_{\omega_0}\rangle = 1$. The second family of mode functions correspond to the case where only one reservoir oscillator is excited. If we denote by Φ_ω such functions, then, for each $\omega > 0$, we find $\Phi_{\omega,1} = 0$, and

$$\Phi_{\omega,2}(t,\nu) = \frac{\delta(\nu-\omega)}{\sqrt{2\nu\mu}} e^{-i\nu t}. \tag{3.29}$$

Therefore, for $t < 0$, we find the general expression

$$\Psi = a_{\omega_0}\Psi_{\omega_0} + a^*_{\omega_0}\Psi^*_{\omega_0} + \int_0^\infty d\omega (b_\omega \Phi_\omega + b^*_\omega \Phi^*_\omega). \tag{3.30}$$

In particular, because $x = \Psi_1$, we find for $t < 0$ that

$$x(t) = \frac{1}{\sqrt{2\omega_0 m}} (a_{\omega_0} e^{-i\omega_0 t} + a^*_{\omega_0} e^{i\omega_0 t}). \tag{3.31}$$

We observe that each mode function is a solution of the field equation. Thus, if the evolution of Ψ_{ω_0} and Φ_ω is known, Eq. (3.30) furnishes x at all times. In order to find the evolution of Ψ_{ω_0}, we note that if $\{\Gamma_\omega\}$ is a complete set of positive norm field modes in the interaction period, then, for $t > 0$, we can write

$$\Psi_{\omega_0}(t) = \int_0^\infty \mathrm{d}\omega[c_\omega \Gamma_\omega(t) + d^*_\omega \Gamma^*_\omega(t)], \tag{3.32}$$

where $c_\omega = \langle \Gamma_\omega, \Psi_{\omega_0}\rangle|_{t\to 0^+}$, $d^*_\omega = -\langle \Gamma^*_\omega, \Psi_{\omega_0}\rangle|_{t\to 0^+}$. Note that as the interacting period starts, the second entry of Ψ_{ω_0}, that vanishes in the non-interacting period, can be non-zero. Similarly, we find that, for $t > 0$,

$$\Phi_\omega(t) = \int_0^\infty \mathrm{d}\omega'[c_{\omega,\omega'}\Gamma_{\omega'}(t) + d^*_{\omega,\omega'}\Gamma^*_{\omega'}(t)], \tag{3.33}$$

and $c_{\omega,\omega'} = \langle \Gamma_{\omega'}, \Phi_\omega\rangle|_{t\to 0^+}$, $d^*_{\omega,\omega'} = -\langle \Gamma^*_{\omega'}, \Phi_\omega\rangle|_{t\to 0^+}$.

The complete set $\{\Gamma_\omega\}$ can be obtained as follows. By assuming a time-dependence in the form $\Gamma_\omega(t) = \exp(-i\omega t)\Gamma^0_\omega$ with $\omega > 0$, we find that

$$(\omega_0^2 - \omega^2)\Gamma^0_{\omega,1} + \frac{i\omega}{m}\int_0^\infty \mathrm{d}\nu \beta \Gamma^0_{\omega,2} = 0, \tag{3.34}$$

$$(\nu^2 - \omega^2)\Gamma^0_{\omega,2} - \frac{i\omega}{\mu}\beta\Gamma^0_{\omega,1} = 0. \tag{3.35}$$

Now, the general solution of Eq. (3.35) is

$$\Gamma^0_{\omega,2}(\nu) = \frac{i\omega}{\nu^2 - \omega^2}\frac{\beta(\nu)}{\mu}\Gamma^0_{\omega,1} + A_\omega \delta(\nu - \omega), \tag{3.36}$$

where A_ω is an arbitrary constant. Thus, Eq. (3.34) implies that

$$\begin{aligned}(\omega_0^2 - \omega^2)\Gamma^0_{\omega,1} + \frac{i\omega}{m}\left(\int_0^\infty \mathrm{d}\nu \frac{i\omega}{\nu^2-\omega^2}\frac{\beta^2(\nu)}{\mu}\Gamma^0_{\omega,1} + A_\omega \int_0^\infty \mathrm{d}\nu \beta(\nu)\delta(\nu-\omega)\right) = \\ = \underbrace{\left[\omega_0^2 - \omega^2 - \frac{\omega^2}{m\mu}\int_0^\infty \mathrm{d}\nu \frac{\beta^2(\nu)}{\nu^2-\omega^2}\right]}_{\zeta_r(\omega)}\Gamma^0_{\omega,1} + \frac{i\omega}{m}A_\omega \beta(\omega) = 0,\end{aligned} \tag{3.37}$$

from where we conclude

$$A_\omega = \frac{im}{\omega\beta(\omega)}\zeta_r(\omega)\Gamma^0_{\omega,1}. \tag{3.38}$$

Although we used the Greek letter ζ for this function, it should be stressed that this definition has nothing to do with the Riemann zeta function. The subscript "r" in the definition is due to the following proposition:

Proposition 4 *The defined function $\zeta_r(\omega)$ is the real part of a complex function $\zeta(\omega)$, i.e., $\zeta_r(\omega) = Re[\zeta(\omega)]$, and*

$$\zeta(\omega) = \omega_0^2 - \omega^2 - \frac{\omega}{2m\mu}\int_{-\infty}^\infty \mathrm{d}\nu \frac{\beta^2(\nu)}{\nu - \omega + i\epsilon}. \tag{3.39}$$

Here, $\epsilon > 0$ is a small parameter to be taken to zero at the end of the calculations. Clearly, when $\omega > 0$,

$$\zeta_i(\omega) = Im[\zeta(\omega)] = \pi\frac{\omega\beta^2(\omega)}{2m\mu} > 0. \tag{3.40}$$

Proof:

First, we begin by separating the integral of ζ_r in partial fractions:

$$\frac{1}{\nu^2-\omega^2} = \lim_{\epsilon\to 0}\frac{1}{2\omega}\left(\frac{1}{\nu-\omega+i\epsilon} - \frac{1}{\nu+\omega-i\epsilon}\right), \tag{3.41}$$

then, its clear that

$$\zeta_r(\omega) = \mathrm{Re}[\zeta(\omega)] = \mathrm{Re}\left[\omega_0^2 - \omega^2 - \frac{\omega}{4m\mu}\int_0^\infty \mathrm{d}\nu\beta^2(\nu)\left(\frac{1}{\nu-\omega+i\epsilon} - \frac{1}{\nu+\omega-i\epsilon}\right)\right]. \tag{3.42}$$

Now, we may separate the integral in the sum and do the substitution $\nu \to -\nu$, and using that $\beta(\nu)$ is an even function, we finish the first part of the proposition. To verify the last part, just remember the Sokhotski-Plemelj (see section 2.4.5):

$$\lim_{\epsilon\to 0}\frac{1}{x-x_0\pm i\epsilon} = \frac{1}{x-x_0} \mp i\pi\delta(x-x_0) \implies \zeta_i(\omega) = \mathrm{Im}[\zeta(\omega)] = \pi\frac{\omega\beta^2(\omega)}{2m\mu}. \tag{3.43}$$

■

The quantity $\Gamma^0_{\omega,1}$ is a normalization constant, which can be determined through the scalar product. In fact, since the scalar product was shown to be time independent, we may just focus our analysis on the sole time independent term in $\langle\Gamma_\omega, \Gamma_{\omega'}\rangle$, since the other terms will cancel out after some algebra. We obtain

$$\langle\Gamma_\omega, \Gamma_{\omega'}\rangle = \frac{2\mu m^2}{\omega\beta^2(\omega)}|\zeta(w)|^2|\Gamma^0_{\omega,1}|^2\delta(\omega-\omega'), \tag{3.44}$$

but we know that

$$\zeta_i(\omega) = \frac{\pi\omega\beta^2(\omega)}{2m\mu} \implies \frac{2\mu m^2}{\omega\beta^2(\omega)} = \frac{\pi m}{\zeta_i(\omega)}, \tag{3.45}$$

hence

$$\langle\Gamma_\omega, \Gamma_{\omega'}\rangle = \left|\Gamma^0_{\omega,1}\right|^2\frac{m\pi|\zeta(\omega)|^2}{\zeta_i(\omega)}\delta(\omega-\omega'), \tag{3.46}$$

and thus

$$\Gamma^0_{\omega,1} = \frac{1}{\sqrt{m\pi}}\frac{\sqrt{\zeta_i(\omega)}}{\zeta(\omega)} \tag{3.47}$$

finishes the construction of the complete set of positive norm mode functions $\{\Gamma_\omega\}$.

We are ready to determine the Fourier coefficients in Eqs. (3.32) and (3.33). Taking

$$\Psi_{w_0} = \begin{pmatrix}\frac{\mathrm{e}^{-i\omega_0 t}}{\sqrt{2\omega_0 m}}\\ 0\end{pmatrix}, \quad \Phi_\omega = \begin{pmatrix}0\\ \frac{\delta(\nu-\omega)\mathrm{e}^{-i\nu t}}{\sqrt{2\nu\mu}},\end{pmatrix} \tag{3.48}$$

and the previous relations, we find for c_ω

$$\begin{aligned} c_\omega = \langle \Gamma_\omega, \Psi_{\omega_0} \rangle|_{t\to 0^+} = \sqrt{\frac{m}{2\omega_0}} \left(\omega_0 \Gamma^*_{\omega,1} - i\partial_t \Gamma^*_{\omega,1}\right)\Big|_{t=0} + \\ + i\mu \int_0^\infty \mathrm{d}\nu \partial_t \Psi_{\omega_0,2} \Gamma_{\omega,2}(\nu)^* + i \int_0^\infty \beta(\nu) \frac{\mathrm{e}^{-i\omega_0 t}}{\sqrt{2\omega_0 m}} \Gamma_{\omega,2}(\nu)^*. \end{aligned} \tag{3.49}$$

From the motion equation (3.9), we obtain

$$\dot{\Psi}_{\omega_0,2} = -\frac{\beta(\nu)}{\mu} \frac{\mathrm{e}^{-i\omega_0 t}}{\sqrt{2\omega_0 m}}, \tag{3.50}$$

in such a way that when we plug it in the (3.49), it cancels out with the third term and we get

$$c_\omega = \sqrt{\frac{m}{2\omega_0}} \left(\omega_0 \Gamma^*_{\omega,1} - i\partial_t \Gamma^*_{\omega,1}\right)\Big|_{t=0}. \tag{3.51}$$

This may seen odd at a first glance, since this term came from $\partial_t \Psi_{\omega_0,2}$, and $\Psi_{\omega_0,2} = 0$. But the motion equation implies that the derivative is not, and can easily be seen from integrating $\lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \mathrm{d}t$. The coefficient d^*_ω is obtained in the same way, with the same reasoning. For the coefficient $c_{\omega,\omega'} = \langle \Gamma_{\omega'}, \Phi_\omega \rangle|_{t\to 0^+}$ we have

$$c_{\omega,\omega'} = im\Gamma^*_{\omega'} \partial_t \Phi_{\omega,1} + \sqrt{\frac{\mu}{2\omega}} \left[(\omega - i\partial_t) \Gamma^*_{\omega',2}(t,\omega) - \frac{i\beta(\omega)}{\mu} \Gamma^*_{\omega',1} \right]\Bigg|_{t=0}, \tag{3.52}$$

but when we use the motion equation (3.7) for the first term and do the same procedure as we did before, i.e., integrate $\lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \mathrm{d}t$, we obtain zero. Thus,

$$c_{\omega,\omega'} = \sqrt{\frac{\mu}{2\omega}} \left[(\omega - i\partial_t) \Gamma^*_{\omega',2}(t,\omega) - \frac{i\beta(\omega)}{\mu} \Gamma^*_{\omega',1} \right]\Bigg|_{t=0}. \tag{3.53}$$

Therefore, the various Fourier components appearing in Eqs. (3.32), (3.33) are given by

$$\begin{aligned} c_\omega &= \sqrt{\frac{m}{2\omega_0}} \left(\omega_0 \Gamma^*_{\omega,1} - i\partial_t \Gamma^*_{\omega,1}\right)\Big|_{t=0}, \\ d^*_\omega &= -\sqrt{\frac{m}{2\omega_0}} \left(\omega_0 \Gamma_{\omega,1} - i\partial_t \Gamma_{\omega,1}\right)\big|_{t=0}, \\ c_{\omega,\omega'} &= \sqrt{\frac{\mu}{2\omega}} \left[(\omega - i\partial_t) \Gamma^*_{\omega',2}(t,\omega) - \frac{i\beta(\omega)}{\mu} \Gamma^*_{\omega',1} \right]\Bigg|_{t=0}, \\ d^*_{\omega,\omega'} &= \sqrt{\frac{\mu}{2\omega}} \left[(i\partial_t - \omega) \Gamma_{\omega',2}(t,\omega) + \frac{i\beta(\omega)}{\mu} \Gamma_{\omega',1} \right]\Bigg|_{t=0}. \end{aligned} \tag{3.54}$$

Finally, the quantization of the system is concluded by promoting the c-numbers a_{ω_0}, b_ω of the expansion (3.30) to operators subjected to the commutation relations $[a_{\omega_0}, b_\omega] = 0, [b_\omega, b_{\omega'}] = 0$, and

$$[a_{\omega_0}, a^\dagger_{\omega_0}] = 1, \tag{3.55}$$

$$[b_\omega, b^\dagger_{\omega'}] = \delta(\omega - \omega'). \tag{3.56}$$

Accordingly, the system vacuum state, $|0\rangle$, defined by $a_{\omega_0}|0\rangle = b_\omega|0\rangle = 0$ for all $\omega > 0$, corresponds to the case where all the oscillators are in their fundamental states before the interaction is turned on.

We note that by plugging the mode expansions (3.32), (3.33) back into Eq. (3.30) we find that, for $t > 0$,

$$\Psi(t) = \int_0^\infty d\omega \left[\gamma_\omega \Gamma_\omega(t) + \gamma_\omega^\dagger \Gamma_\omega^*(t)\right], \tag{3.57}$$

where the operators γ_ω are related to the non-interacting creation operators via the Bogoliubov transformation (see section 2.3)

$$\gamma_\omega = c_\omega a_{\omega_0} + d_\omega a_{\omega_0}^\dagger + \int_0^\infty d\nu \left(c_{\nu,\omega} b_\nu + d_{\nu,\omega} b_\nu^\dagger\right). \tag{3.58}$$

A lengthy computation then reveals that the operators γ_ω satisfy

$$\left[\gamma_\omega, \gamma_{\omega'}\right] = 0, \tag{3.59}$$

$$\left[\gamma_\omega, \gamma_{\omega'}^\dagger\right] = \delta(\omega - \omega'). \tag{3.60}$$

In fact, let us explicitly verify the first relation. Using (3.58) and the linearity of the commutator, we get

$$\left[\gamma_\omega, \gamma_{\omega'}\right] = c_\omega d_{\omega'} - d_\omega c_{\omega'} + \int d\nu (c_{\nu,\omega} d_{\nu,\omega'} - d_{\nu,\omega} c_{\nu,\omega'}). \tag{3.61}$$

The first two terms can be computed easily using (3.54) and reduce to [1]

$$c_\omega d_{\omega'} - d_\omega c_{\omega'} = -im\left(\Gamma_{\omega,1}^* \overleftrightarrow{\partial_t} \Gamma_{\omega',1}^*\right). \tag{3.62}$$

The last two terms in the integrand can be written like

$$\begin{aligned} c_{\nu,\omega} d_{\nu,\omega'} - d_{\nu,\omega} c_{\nu,\omega'} &= -\frac{\mu}{\nu}\left(\nu \Gamma_{\omega,2}^* i\partial_t \Gamma_{\omega',2}^* + \nu \Gamma_{\omega,2}^* i \frac{\beta(\nu)}{\mu} \Gamma_{\omega',1}^* - i\partial_t \Gamma_{\omega,2}^* \nu \Gamma_{\omega',2}^* - i\frac{\beta(\nu)}{\mu} \Gamma_{\omega,1}^* \nu \Gamma_{\omega',2}^*\right) \\ &= -i\left[\mu \int d\nu \Gamma_{\omega,2}^* \overleftrightarrow{\partial_t} \Gamma_{\omega',2}^* - \int d\nu \beta(\nu) \left(\Gamma_{\omega,1}^* \Gamma_{\omega',2}^* - \Gamma_{\omega',1}^* \Gamma_{\omega,2}^*\right)\right], \end{aligned} \tag{3.63}$$

hence

$$\begin{aligned} \left[\gamma_\omega, \gamma_{\omega'}\right] &= -i\left[m\left(\Gamma_{\omega,1}^* \overleftrightarrow{\partial_t} \Gamma_{\omega',1}^*\right) + \mu \int d\nu \Gamma_{\omega,2}^* \overleftrightarrow{\partial_t} \Gamma_{\omega',2}^* - \int d\nu \beta(\nu) \left(\Gamma_{\omega,1}^* \Gamma_{\omega',2}^* - \Gamma_{\omega',1}^* \Gamma_{\omega,2}^*\right)\right] \\ &= -\langle \Gamma_\omega, \Gamma_{\omega'}^* \rangle = 0. \end{aligned} \tag{3.64}$$

By the same procedure, one finds $\left[\gamma_\omega, \gamma_{\omega'}^*\right] = \langle \Gamma_\omega, \Gamma_{\omega'} \rangle = \delta(\omega - \omega')$.

1 The notation $A\overleftrightarrow{\partial_t} B$ means $A\partial_t B - B\partial_t A$.

The commutation relations above are of great importance to our analysis. Indeed, if the system under study were quantized already in the interacting period, one would obtain the expansion (3.57) with the corresponding (instantaneous) vacuum state, $|0\rangle_{\rm qp}$, of the theory defined as $\gamma_\omega|0\rangle_{\rm qp} = 0$ for all ω. This choice of quasiparticle vacuum gives rise to the well-known model explored in [45, 32]. In our case, however, $|0\rangle_{\rm qp} \neq |0\rangle$, and thus Eq. (3.58) is the Bogoliubov transformation relating the two quantum field representations. We discuss the relation between the two vacua in the next section.

We finish this section with a remark regarding the canonical commutation relation for the Heisenberg operator [see Eq. (3.57)]

$$x(t) = \frac{1}{\sqrt{m\pi}}\int_0^\infty \mathrm{d}\omega\sqrt{\zeta_i(\omega)}\left[\gamma_\omega\frac{\mathrm{e}^{-i\omega t}}{\zeta(\omega)} + H.c.\right]. \tag{3.65}$$

Thus it follows that

Proposition 5

$$[x(t), p(t)] = -\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{\omega}{\zeta(\omega)}, \tag{3.66}$$

where we used the definition $p = m\dot{x}$.

Proof:

In fact, let us see the product $x(t)p(t)$:

$$x(t)p(t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\mathrm{d}\omega'\sqrt{\zeta_i(\omega)\zeta_i(\omega')}\left(\frac{\gamma_\omega\gamma^*_{\omega'}i\omega'\mathrm{e}^{-i(\omega-\omega')t}}{\zeta(\omega)\zeta^*(\omega')} - \frac{\gamma_\omega\gamma_{\omega'}i\omega'\mathrm{e}^{-i(\omega+\omega')t}}{\zeta(\omega)\zeta(\omega')} + \right.$$
$$\left. + \frac{\gamma^*_\omega\gamma^*_{\omega'}i\omega'\mathrm{e}^{i(\omega+\omega')t}}{\zeta^*(\omega)\zeta^*(\omega')} - \frac{\gamma^*_\omega\gamma_{\omega'}i\omega'\mathrm{e}^{-i(\omega'-\omega)t}}{\zeta^*(\omega)\zeta(\omega')}\right). \tag{3.67}$$

Thus,

$$[x, p] = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\frac{|\zeta_i(\omega)|}{|\zeta(\omega)|^2}2i\omega = \frac{1}{\pi}\int_0^\infty \omega\left(\frac{1}{\zeta^*(\omega)} - \frac{1}{\zeta(\omega)}\right)\mathrm{d}\omega. \tag{3.68}$$

Now, we introduce a useful lemma:

Lemma 1 *If $f(z)$ is a complex holomorphic function, the real component of $f(z)$ is even and the imaginary part is odd, then $f^*(z) = f(-z)$.*

Making use of this lemma, we get

$$\int_0^\infty \mathrm{d}\omega\frac{\omega}{\zeta^*(\omega)} = -\int_{-\infty}^0 \mathrm{d}\omega\frac{\omega}{\zeta(\omega)}. \tag{3.69}$$

Formula (3.66) will be obtained if we prove that ζ satisfies the hypothesis of our lemma. The imaginary part is odd, as can be seen by (3.43).

We verify the real part by definition:

$$\zeta_r(-\omega) = \omega_0^2 - \omega^2 + \frac{\omega}{2m\mu}\int_{-\infty}^{\infty}\mathrm{d}\nu\frac{\beta^2(\nu)}{\nu+\omega} \overset{\nu\to-\nu}{=} \omega_0^2 - \omega^2 - \frac{\omega}{2m\mu}\int_{-\infty}^{\infty}\mathrm{d}\nu\frac{\beta^2(\nu)}{\nu-\omega} = \zeta_r(\omega). \tag{3.70}$$

■

We note that the function $\zeta(\omega)$ defined in Eq. (3.39) is an analytic function in the lower half complex ω plane. Furthermore, $\zeta(\omega) \neq 0$ for all ω satisfying $\mathrm{Im}(\omega) \leq 0$. This can be seen as follows. If we write $\omega = \omega_r - i\omega_i$, with $\omega_i > 0$, we find that $\mathrm{Im}[\zeta(\omega)] = 0$ only if $\omega_r = 0$, whereas

$$\zeta(-i\omega_i) = \omega_0^2 + \omega_i^2\left(1 + \frac{1}{m\mu}\int_0^\infty \mathrm{d}\nu \frac{\beta^2(\nu)}{\nu^2 + \omega_i^2}\right), \tag{3.71}$$

is always positive for $\omega_i > 0$. We conclude from this that $1/\zeta(\omega)$ is also analytic in the lower half-plane. This can be used to evaluate the integral in Eq. (3.66) by closing the integration contour in the lower half complex plane to obtain

$$[x(t), p(t)] = -\frac{i}{\pi}\lim_{r\to\infty}\int_\pi^{2\pi} \mathrm{d}\theta \frac{r^2 \mathrm{e}^{2i\theta}}{\zeta(r\mathrm{e}^{i\theta})} = i, \tag{3.72}$$

where we used the property $\zeta(\omega) \to -\omega^2$ for $|\omega| \to \infty$, which holds as long as $\beta(\nu) \to 0$ for $\nu \to \infty$.

3.4 Two Point Correlation Function For An Exactly Solvable Case

We now turn our attention to the two-point function $\langle x(t)x(t')\rangle$, which can be used to calculate the oscillator quantities of interest here.

Proposition 6 *Using the analytical properties of $1/\zeta(\omega)$, the correlation function assumes the form*

$$\langle x(t)x(t')\rangle = \langle x(t)x(t')\rangle_{\mathrm{tr}} + \langle x(t)x(t')\rangle_{\mathrm{qp}}, \tag{3.73}$$

valid for $t, t' \geq 0$, where

$$\begin{aligned}\langle x(t)x(t')\rangle_{\mathrm{tr}} = \frac{1}{2\omega_0 m}\Big\{&[(\omega_0 + i\partial_t)A(t)][(\omega_0 - i\partial_{t'})A^*(t')]\\ + \frac{2\omega_0}{\pi}\int_0^\infty \mathrm{d}\omega\zeta_i(\omega)&[i\dot{I}_\omega(0)\mathrm{e}^{-i\omega t} - (\omega + i\partial_t)I_\omega(t)][i\dot{I}_\omega(0)\mathrm{e}^{i\omega t'} - (\omega - i\partial_{t'})I_\omega^*(t')]\Big\},\end{aligned} \tag{3.74}$$

is the transient correlation and

$$\langle x(t)x(t')\rangle_{\mathrm{qp}} = \frac{1}{m\pi}\int_0^\infty \mathrm{d}\omega \frac{\zeta_i(\omega)}{|\zeta(\omega)|^2}\mathrm{e}^{-i\omega\Delta t}, \tag{3.75}$$

is the two-point function with respect to the instantaneous quasiparticle vacuum state. Here we defined the auxiliary functions

$$A(t) = \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\omega \frac{\sin(\omega t)}{\zeta(\omega)}, \tag{3.76}$$

$$I_\omega(t) = \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\omega' \frac{\sin(\omega' t)}{\zeta(\omega')(\omega'^2 - \omega^2)}. \tag{3.77}$$

Proof:

From the equation (3.57), we take the expected value in the vacuum $|0\rangle_{\text{qp}}$:

$$\langle\Psi(t)\Psi(t')\rangle_{\text{qp}} = \int_0^\infty \mathrm{d}\omega\Gamma_\omega(t)\Gamma^*_\omega(t') = \frac{1}{m\pi}\int_0^\infty \mathrm{d}\omega \mathrm{e}^{-i\omega t}\frac{\zeta_i(\omega)}{|\zeta(\omega)|^2}, \tag{3.78}$$

if we consider the case $\Psi_1(t)$, which is our interest. Naturally, this expression must appear if we take the expected value in the old vacuum $|0\rangle$ with expression (3.30), and some other additional terms, that the new vacuum does not "see". This other expected value reads as

$$\langle\Psi(t)\Psi(t')\rangle = \Psi_{\omega_0}(t)\Psi^*_{\omega_0}(t') + \int_0^\infty \mathrm{d}\omega\Phi_\omega(t)\Phi(t')^*. \tag{3.79}$$

Let us examine possible representations of these products. We should look for a representation that uses general characteristics of $\zeta(\omega)$, in a way that the assertions that we make remains valid for any even $\beta(\nu)$ functions. We know that

$$\Psi_{\omega_0,1}(t) = \int_0^\infty \mathrm{d}\omega[c_\omega\Gamma_{\omega,1}(t) + d^*_\omega\Gamma^*_{\omega,1}(t)], \tag{3.80}$$

and using (3.54), we get

$$\Psi_{\omega_0,1}(t) = \int_0^\infty \mathrm{d}\omega\sqrt{\frac{m}{2\omega_0}}|\Gamma^0_{\omega,1}|^2\left[(\omega_0+\omega)\mathrm{e}^{-i\omega t} - (\omega_0-\omega)\mathrm{e}^{i\omega t}\right]. \tag{3.81}$$

Using that

$$\begin{aligned}(\omega_0+\omega)\mathrm{e}^{-i\omega t} - (\omega_0-\omega)\mathrm{e}^{i\omega t} = 2\omega cos(\omega t) - 2i\omega_0\sin(\omega t) \quad \text{and}\\ |\Gamma^0_{\omega,1}|^2 = \frac{1}{m\pi}\frac{\zeta_i(\omega)}{|\zeta(\omega)|^2} = \frac{1}{2i}\left(\frac{1}{\zeta^*(\omega)} - \frac{1}{\zeta(\omega)}\right)\frac{1}{m\pi},\end{aligned} \tag{3.82}$$

we get

$$\Psi_{\omega_0,1}(t) = \frac{1}{\pi\sqrt{2m\omega_0}}\int_0^\infty \mathrm{d}\omega\left(\frac{1}{\zeta} - \frac{1}{\zeta^*}\right)(i\omega\cos(\omega t) + \omega_0\sin(\omega t)). \tag{3.83}$$

Define the functions

$$A_1(t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\frac{\sin(\omega t)}{\zeta(\omega)} \quad \text{and} \quad A_2(t) = -\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\frac{\sin(\omega t)}{\zeta^*(\omega)}, \tag{3.84}$$

and using the lemma 1, we have

$$A_2(t) = \frac{1}{\pi}\int_{-\infty}^0 \mathrm{d}\omega\frac{\sin(\omega t)}{\zeta(\omega)}, \tag{3.85}$$

in such a way that

$$\Psi_{\omega_0,1}(t) = \frac{1}{\sqrt{2m\omega_0}}(\omega_0 + i\partial_t)A(t), \quad \text{where} \quad A := \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\omega\frac{\sin(\omega t)}{\zeta(\omega)}. \tag{3.86}$$

Now, we analyze $\Phi_{\omega,1}(t) = \int_0^\infty \mathrm{d}\omega' [c_{\omega,\omega'}\Gamma_{\omega',1}(t) + d^*_{\omega,\omega'}\Gamma^*_{\omega',1}(t)]$. Using (3.54), we have

$$\begin{aligned} c_{\omega,\omega'}\Gamma_{\omega',1} &= \sqrt{\frac{\mu}{2\omega}}\left[(\omega+\omega')\Gamma^{0\,*}_{\omega',2}(\omega)\Gamma^0_{\omega',1} - \frac{i\beta(\omega)}{\mu}|\Gamma^0_{\omega',1}|^2\right]\mathrm{e}^{-i\omega' t}, \\ d^*_{\omega,\omega'}\Gamma^*_{\omega',1} &= -\sqrt{\frac{\mu}{2\omega}}\left[(\omega-\omega')\Gamma^0_{\omega',2}(\omega)\Gamma^{0\,*}_{\omega',1} - \frac{i\beta(\omega)}{\mu}|\Gamma^0_{\omega',1}|^2\right]\mathrm{e}^{i\omega' t}, \end{aligned} \tag{3.87}$$

but

$$\Gamma^{0\,*}_{\omega',2}(\omega)\Gamma^0_{\omega',1} = -\Gamma^0_{\omega',2}(\omega)\Gamma^{0\,*}_{\omega',1} = \left(\frac{i\omega'}{\omega^2-\omega'^2}\frac{\beta(\omega)}{\mu} + \frac{im}{\omega'\beta(\omega')}\zeta_r(\omega')\delta(\omega-\omega')\right)|\Gamma^0_{\omega',1}|^2, \tag{3.88}$$

hence

$$\begin{aligned} \Phi_{\omega,1}(t) = -\sqrt{\frac{\mu}{2\omega}}\int_0^\infty \mathrm{d}\omega' |\Gamma^0_{\omega',1}|^2 \Bigg\{ &\left(\frac{i\omega'}{\omega^2-\omega'^2}\frac{\beta(\omega)}{\mu} + \frac{im}{\omega'\beta(\omega')}\zeta_r(\omega')\delta(\omega-\omega')\right)\times \\ &\times\left[(\omega+\omega')\mathrm{e}^{-i\omega' t} + (\omega-\omega')\mathrm{e}^{i\omega' t}\right] + \frac{i\beta(\omega)}{\mu}(\mathrm{e}^{-i\omega' t} - \mathrm{e}^{i\omega' t})\Bigg\}. \end{aligned} \tag{3.89}$$

Writing the exponential in sine and cosine functions, integrating the delta terms and recalling the definition of $|\Gamma^0_{\omega',1}|^2$ we have

$$\begin{aligned} \Phi_{\omega,1}(t) = &-\sqrt{\frac{\mu}{2\omega}}\frac{2i}{\pi\beta(\omega)}\frac{\zeta_i(\omega)}{|\zeta(\omega)|^2}\zeta_r(\omega)\mathrm{e}^{-i\omega t} + \left(\sqrt{\frac{\mu}{2\omega}}\frac{2i\omega}{m\pi}\int_0^\infty \mathrm{d}\omega' \frac{\zeta_i(\omega')}{|\zeta(\omega')|^2}\frac{\omega'\cos(\omega' t)}{\omega'^2-\omega^2}\right. \\ &\left.+\sqrt{\frac{\mu}{2\omega}}\frac{2\omega^2}{m\pi}\int_0^\infty \mathrm{d}\omega' \frac{\zeta_i(\omega')}{|\zeta(\omega')|^2}\frac{\sin(\omega' t)}{\omega'^2-\omega^2}\right)\frac{\beta(\omega)}{\mu}. \end{aligned} \tag{3.90}$$

We may proceed as in the first case. First, write $\frac{\zeta_i(\omega)}{|\zeta(\omega)|^2} = \frac{1}{2i}\left(\frac{1}{\zeta^*} - \frac{1}{\zeta}\right)$ in the argument of the integrals, and identify

$$I_1 = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega' \frac{\sin(\omega' t)}{\zeta(\omega')(\omega'^2-\omega^2)} \quad \text{and} \quad I_2 = -\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega' \frac{\sin(\omega' t)}{\zeta^*(\omega')(\omega'^2-\omega^2)}. \tag{3.91}$$

Then, using lemma 1, we may write

$$\Phi_{\omega,1}(t) = -\frac{2i}{\pi}\sqrt{\frac{\mu}{2\omega}}\frac{\zeta_i(\omega)}{\beta(\omega)}\left[\frac{\zeta_r(\omega)}{|\zeta(\omega)|^2}\mathrm{e}^{-i\omega t} - (\omega + i\partial_t)I_\omega(t)\right], \tag{3.92}$$

where

$$I_\omega(t) \coloneqq \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\omega' \frac{\sin(\omega' t)}{\zeta(\omega')(\omega'^2-\omega^2)}. \tag{3.93}$$

For last, we may note that

$$\dot{I}_\omega(0) = \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{d}\omega' \frac{\omega'}{\zeta(\omega')(\omega'^2-\omega^2)} = \frac{1}{2\pi}\left(\int_{-\infty}^\infty \mathrm{d}\omega' \frac{1}{\zeta(\omega')(\omega'-\omega)} + \int_{-\infty}^\infty \mathrm{d}\omega' \frac{1}{\zeta(\omega')(\omega'+\omega)}\right). \tag{3.94}$$

By the Cauchy Theorem (see section 2.4), if a complex function f is holomorphic, we have

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)\mathrm{d}z}{z-a}. \tag{3.95}$$

We know that $\zeta(\omega)$ is holomorphic in the lower half complex ω plane. Then, taking a semi-circle contour integral in this plane, with clockwise orientation, we get

$$-\frac{i}{2}\frac{1}{\zeta(\omega)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{1}{\zeta(\omega')(\omega'-\omega)} \quad \text{and} \quad -\frac{i}{2}\frac{1}{\zeta^*(\omega)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{1}{\zeta(\omega')(\omega'+\omega)}. \tag{3.96}$$

Thus $\dot{I}_\omega(0) = -i\frac{\zeta_r(\omega)}{|\zeta(\omega)|^2}$ and, using that $\beta(\omega) = \sqrt{\frac{2m\mu\zeta_i(\omega)}{\omega\pi}}$, we get

$$\Phi_{\omega,1}(t) = -\frac{i}{\sqrt{2m\omega_0}}\sqrt{\frac{2\omega_0}{\pi}}\sqrt{\zeta_i(\omega)}\left[i\dot{I}_\omega(0)\mathrm{e}^{-i\omega t} - (\omega + i\partial_t)I_\omega(t)\right]. \tag{3.97}$$

■

This proposition furnishes us with a natural way of understanding qualitatively the two point function, but is very expensive when we need to compute and make numerical analysis. There is a way, which we will see now, that is better from the computational point of view:

Proposition 7 *The transient two point function can be written as*

$$\langle x(t)x(t')\rangle_{\text{tr}} = \frac{1}{2\omega_0 m}\Bigg\{[(\omega_0 + i\partial_t)A(t)][(\omega_0 - i\partial_{t'})A^*(t')] + \frac{2\omega_0}{\pi}\int_0^\infty \mathrm{d}\omega\zeta_i(\omega)\left[\frac{\mathrm{e}^{-i\omega t}}{\zeta^*(\omega)}B^*_\omega(t') + B_\omega(t)\frac{\mathrm{e}^{i\omega t'}}{\zeta(\omega)} + B_\omega(t)B^*_\omega(t')\right]\Bigg\}, \tag{3.98}$$

where

$$B_\omega(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\mathrm{e}^{i\omega' t}}{\zeta(\omega')}\frac{1}{\omega'+\omega+i\epsilon}. \tag{3.99}$$

Proof:

It is enough to show that our defined function $I_\omega(t)$ can be written in terms of the new function $B_\omega(t)$. First, consider

$$i\dot{I}_\omega(0)\mathrm{e}^{-i\omega t} - (\omega + i\partial_t)I_\omega(t) = \frac{\zeta_r(\omega)}{|\zeta(\omega)|^2}\mathrm{e}^{-i\omega t} - \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{\omega\sin(\omega' t) + i\omega'\cos(\omega' t)}{\zeta(\omega')(\omega'^2 - \omega^2)}. \tag{3.100}$$

The numerator of the integrand can be expressed as $-\frac{1}{2i}\left[(\omega'-\omega)\mathrm{e}^{i\omega' t} + (\omega'+\omega)\mathrm{e}^{-i\omega' t}\right]$, then

$$\cdots = \frac{\zeta_r(\omega)}{|\zeta(\omega)|^2}\mathrm{e}^{-i\omega t} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\omega'\frac{1}{\zeta(\omega')}\left(\frac{\mathrm{e}^{i\omega' t}}{\omega'+\omega} + \frac{\mathrm{e}^{-i\omega' t}}{\omega'-\omega}\right). \tag{3.101}$$

Since $\zeta(\omega)$ is holomorphic in the lower half plane, the integral with the denominator $\frac{1}{\omega'-\omega}$ will have just one pole, and the Cauchy theorem applies. The other integral should be treated with more caution, and we will use the Sokhotski-Plemelj theorem. Thus, making a semi circle contour to the Cauchy theorem, we have

$$-\frac{e^{-i\omega t}}{2\zeta(\omega)} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{e^{-i\omega' t}}{\zeta(\omega')(\omega'-\omega)}, \tag{3.102}$$

and the Sokhotski-Plemelj theorem

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{e^{i\omega' t}}{\zeta(\omega')(\omega'+\omega)} &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \left(\frac{e^{i\omega' t}}{\zeta(\omega')(\omega'+\omega+i\epsilon)} + i\pi\delta(\omega'+\omega)\right)\\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{e^{i\omega' t}}{\zeta(\omega')(\omega'+\omega+i\epsilon)} + \frac{1}{2}\frac{e^{-i\omega t}}{\zeta^*(\omega)}.\end{aligned} \tag{3.103}$$

Thus, we have

$$\begin{aligned}\ldots &= \frac{\zeta_r(\omega)}{|\zeta(\omega)|^2}e^{-i\omega t} - \underbrace{\frac{e^{-i\omega t}}{2}\left(\frac{1}{\zeta(\omega)} - \frac{1}{\zeta^*(\omega)}\right)}_{\frac{+i\zeta_i(\omega)}{|\zeta(\omega)|^2}e^{-i\omega t}} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{e^{i\omega' t}}{\zeta(\omega')(\omega'+\omega+i\epsilon)}\\ &= \frac{e^{-i\omega t}}{\zeta^*(\omega)} + B_\omega(t),\end{aligned} \tag{3.104}$$

where the Cauchy principal value is understood.

■

It is valid to notice that the term involving the product of $\frac{1}{\zeta^*(\omega)}\frac{1}{\zeta(\omega)}$ is the static part $\langle x(t)x(t')\rangle_{\text{qp}}$, and thus is excluded from the transient equation. In general, depending on the coupling constant $\beta(\nu)$, the analytical properties of $1/\zeta(\omega)$ in the upper half complex plane can be convoluted. For instance, we note that for $\beta(\nu)^2 \propto \Theta(\nu_0^2 - \nu^2)$, $1/\zeta(\omega)$ will have logarithmic dependence on ω. An interesting scenario that can be exactly integrated and produces a meromorphic $1/\zeta(\omega)$ occurs when the coupling constant β is the combination of two Lorentzians:

$$\frac{\beta^2(\omega_0\eta)}{\omega_0} = \frac{\sigma^2 m\mu}{\pi}\left[\frac{\eta_0}{(\eta-\eta_r)^2+\eta_0^2} + \frac{\eta_0}{(\eta+\eta_r)^2+\eta_0^2}\right], \tag{3.105}$$

which represents a coupling with maximum intensity at $\omega_0\eta = \omega_0\eta_r$. Here σ is a dimensionless constant that measures the magnitude of the interaction. Also, $\eta_0 \to 0$ implies

$$\frac{\beta^2(\omega_0\eta)}{\omega_0} \to \sigma^2 m\mu\left[\delta(\eta-\eta_r) + \delta(\eta+\eta_r)\right]. \tag{3.106}$$

The corresponding oscillator effective squared frequency is [cf. Eq. (3.14)]

$$\omega_e^2 = \omega_0^2(1+\sigma^2), \tag{3.107}$$

and the ζ function can be expressed as

$$\frac{\zeta(\omega_0\eta)}{\omega_0^2} = 1 - \eta^2 + \eta\sigma^2\frac{\eta - i\eta_0}{(\eta-i\eta_0)^2 - \eta_r^2}. \tag{3.108}$$

In fact, from the (3.39), we know that

$$\frac{\zeta(\omega_0\eta)}{\omega_0^2} = 1 - \eta^2 - \frac{\eta}{2m\mu\omega_0}\int_{-\infty}^{\infty} \mathrm{d}\eta' \frac{\beta^2(\omega_0\eta')}{\eta' - \eta + i\epsilon'}. \tag{3.109}$$

Using (3.106), we get

$$\frac{\zeta(\omega_0\eta)}{\omega_0^2} = 1 - \eta^2 - \frac{\sigma^2\eta}{2\pi}(I_1 + I_2), \tag{3.110}$$

where we defined

$$I_1(\eta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta + i\epsilon'} \frac{\eta_0}{(\eta' - \eta_r)^2 + \eta_0^2}, \quad \text{and} \quad I_2(\eta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta + i\epsilon'} \frac{\eta_0}{(\eta' + \eta_r)^2 + \eta_0^2}. \tag{3.111}$$

If we find a closed form for one of the integrals above, the other will follow from symmetry. So, let us analyze $I_1(\eta)$. First, we use Sokhotski-Plemelj:

$$I_1 = P\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta} \frac{\eta_0}{(\eta' - \eta_r)^2 + \eta_0^2} - i\pi \frac{\eta_0}{(\eta - \eta_r)^2 + \eta_0^2}, \tag{3.112}$$

using

$$P\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta} \frac{\eta_0}{(\eta' - \eta_r)^2 + \eta_0^2} = -\pi \frac{\eta - \eta_r}{(\eta - \eta_r)^2 + \eta_0^2}, \tag{3.113}$$

which we will show later on, we get

$$I_1 = \frac{-\pi}{\eta + i\eta_0 - \eta_r}, \tag{3.114}$$

and by the same means

$$I_2 = \frac{-\pi}{\eta + i\eta_0 + \eta_r}. \tag{3.115}$$

If we put these results in (3.110), we have (3.108). Now, we pay our debt:

Lemma 2 *Let*

$$H[L](\eta) := P\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta} L(\eta') \tag{3.116}$$

be a functional in our function space[2]. *If* $L(\eta') = \frac{\eta_0}{(\eta' - \eta_r)^2 + \eta_0^2}$, *we get*

$$H[L](\eta) = -\pi \frac{\eta - \eta_r}{(\eta - \eta_r)^2 + \eta_0^2}. \tag{3.117}$$

Proof:

Separating into partial fractions, we get

$$H[L](\eta) = \frac{1}{2i}(I_+ - I_-), \quad \text{where} \quad I_\pm(\eta) := P\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{(\eta' - \eta_r \mp i\eta_0)(\eta' - \eta)}. \tag{3.118}$$

[2] One might recognize our defined functional as the Hilbert transformation [49].

Thus, using partial fractions once more in the integrals above, we have

$$I_{\pm}(\eta) = \frac{1}{\eta - \eta_r \mp i\eta_0} \cancelto{0}{P\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta}} - \frac{1}{\eta - \eta_r \mp i\eta_0} P\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\eta' - \eta_r \mp i\eta_0}, \tag{3.119}$$

but

$$P\int_{-R}^{R} \frac{\mathrm{d}\eta'}{\eta' - z} = \ln\frac{R - z}{-R - z} = \ln(-1), \quad \text{if} \quad R \to \infty. \tag{3.120}$$

However, this is a complex logarithm, and thus a multivalued function [41], and will only be well defined if we select a branch. In fact, $-1 = \mathrm{e}^{\pm i\pi}$, then if $\mathrm{Im}[z] > 0$, we choose $+\pi$ in the argument of the exponential; else, $-\pi$. Hence

$$P\int_{-R}^{R} \frac{\mathrm{d}\eta'}{\eta' - z} = i\pi\,\mathrm{sign}(\mathrm{Im}[z]), \tag{3.121}$$

and thus

$$H[L](\eta) = -\frac{1}{2i}\left(\frac{i\pi}{\eta - \eta_r - i\eta_0} + \frac{i\pi}{\eta - \eta_r + i\eta_0}\right) = -\pi\frac{\eta - \eta_r}{(\eta - \eta_r)^2 + \eta_0^2}. \tag{3.122}$$

■

The advantage of the functional form of Eq. (3.108) is that $1/\zeta(\omega)$ is a meromorphic function with exactly four simple poles in the upper half plane, determined by the three independent parameters σ, η_r, η_0. Furthermore, these poles are the zeros of $\zeta(\omega_0\eta)$, which are determined by a degree four polynomial equation in η. We let η_i, $i = 1,2,3,4$, be these complex roots, and $\mathcal{R}_{\eta_i}$ be the corresponding residue of $\omega_0^2/\zeta(\omega_0\eta)$ at η_i. Also, the reflection property $\zeta(\omega)^* = \zeta(-\omega)$ implies that $-\eta_i^*$ is also a root with residue $\mathcal{R}_{-\eta_i^*} = -\mathcal{R}_{\eta_i}^*$. These properties hold for any meromorphic $1/\zeta$ with simple poles. For such functions,

$$\begin{aligned}\frac{\omega_0^2}{\zeta(\omega_0\eta)} &= -\frac{\omega_0^2}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{d}\eta'}{\zeta(\omega_0\eta')(\eta' - \eta + i\epsilon)} \\ &= \sum_i \frac{\mathcal{R}_{\eta_i}}{\eta - \eta_i},\end{aligned} \tag{3.123}$$

which is a consequence of the Residue Theorem (see section 2.4.4) and can be used to write the stationary two-point function (3.75) as

$$\begin{aligned}\langle x(t)x(t')\rangle_{\mathrm{qp}} = \frac{-i}{\omega_0 m\pi}\sum_j \mathcal{R}_{\eta_j}\Big[&\sin(\eta_j\omega_0\Delta t)\mathrm{Si}(\eta_j\omega_0\Delta t) \\ &+ \cos(\eta_j\omega_0\Delta t)\mathrm{Ci}(\eta_j\omega_0\Delta t) - \frac{\pi}{2}\sin(\eta_j\omega_0\Delta t)\Big],\end{aligned} \tag{3.124}$$

where Si and Ci are the sine and cosine integral functions, defined as

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(t)}{t}\mathrm{d}t = \frac{\pi}{2} - \int_z^{\infty} \frac{\sin(t)}{t}\mathrm{d}t, \tag{3.125}$$

$$\mathrm{Ci}(z) = -\int_z^{\infty} \frac{\cos(t)}{t}\mathrm{d}t. \tag{3.126}$$

Also, we set $\Delta t = t - t' - i\epsilon$, where $\epsilon > 0$ should be taken to zero later on, at the end of the calculations. To prove this equation, write

$$\langle x(t)x(t')\rangle_{\text{qp}} = \frac{1}{2i\omega_0 m\pi}\int_0^\infty \mathrm{d}\eta \left(\frac{\omega_0^2}{\zeta^*(\omega_0\eta)} - \frac{\omega_0^2}{\zeta(\omega_0\eta)}\right)\mathrm{e}^{-i\omega_0\eta\Delta t}. \tag{3.127}$$

Since Δt can be positive or negative, none of the integrals above can be set to zero right away. In fact, if $\Delta t > 0$, for example, then the integrand with $\frac{1}{\zeta(\omega)}$ will be zero since the contour, fixed by convergence, should be in the upper half plane, where $\zeta(\omega)$ does not have roots. Thus, using the reflection property

$$\frac{\omega_0^2}{\zeta^*(\omega_0\eta)} = \frac{\omega_0^2}{\zeta(-\omega_0\eta)} = -\sum_{\eta_i}\frac{\mathcal{R}_{\eta_i}}{\eta+\eta_i}, \tag{3.128}$$

we have

$$\langle x(t)x(t')\rangle_{\text{qp}} = -\frac{1}{i\omega_0 m\pi}\sum_{\eta_i}\mathcal{R}_{\eta_i}\int_0^\infty \mathrm{d}\eta\frac{\eta}{\eta^2-\eta_i^2}\mathrm{e}^{-i\omega_0\eta\Delta t}. \tag{3.129}$$

If we use partial fractions and a suitable translation in the variables, we get [50]

$$\int_0^\infty \mathrm{d}\eta\frac{\eta}{\eta^2-\eta_i^2}\mathrm{e}^{-i\omega_0\eta\Delta t} = -\frac{1}{2}[\mathrm{e}^{-i\omega_0\eta_i\Delta t}\mathrm{E}_i(i\omega_0\eta_i\Delta t) + \mathrm{e}^{i\omega_0\eta_i\Delta t}\mathrm{E}_i(-i\omega_0\eta_i\Delta t)], \tag{3.130}$$

where E_i is the exponential integral, defined by

$$\mathrm{E}_i(x) = -\int_{-x}^\infty \mathrm{d}t\frac{\mathrm{e}^{-t}}{t}, \tag{3.131}$$

and from the relations

$$\begin{aligned}\mathrm{Ci}(z) &= \frac{1}{2}[\mathrm{E}_i(ix) + \mathrm{E}_i(-ix)]\\ \mathrm{Si}(z) &= \frac{1}{2i}[\mathrm{E}_i(ix) - \mathrm{E}_i(-ix)] + \frac{\pi}{2},\end{aligned} \tag{3.132}$$

we conclude (3.124).

Finally, the transient part of the two-point function is exactly given by

$$\begin{aligned}\langle x(t)x(t')\rangle_{\text{tr}} =& \frac{1}{2\omega_0 m}\sum_{k,j}\mathcal{R}^*_{\eta_k}\mathcal{R}_{\eta_j}\times\Bigg\{\left[(\eta_k^*+1)(\eta_j+1) - \frac{\sigma^2}{\pi}F_{kj}(0)\right]\mathrm{e}^{-i\eta_k^*\omega_0 t + i\eta_j\omega_0 t'}\\ &+\frac{\sigma^2}{\pi}\left[\mathrm{e}^{i\eta_j\omega_0 t'}F_{kj}(\omega_0 t) + \mathrm{e}^{-i\eta_k^*\omega_0 t}F^*_{jk}(\omega_0 t')\right]\Bigg\},\end{aligned} \tag{3.133}$$

where the various auxiliary functions are defined as

$$g(\alpha) \coloneqq \mathrm{e}^{i\alpha}\left[\frac{i\pi}{2} + \mathrm{Ci}(\alpha) - i\mathrm{Si}(\alpha)\right], \tag{3.134}$$

$$G_{kj}(c,\alpha) \coloneqq \frac{cg(c\alpha) + \eta_k^* g(-\eta_k^*\alpha)}{(\eta_j - \eta_k^*)(\eta_k^* + c)} + (\eta_j \leftrightarrow \eta_k^*), \tag{3.135}$$

$$\begin{aligned}F_{kj}(\alpha) \coloneqq& \frac{i}{2}[G_{kj}(-\eta_r - i\eta_0, \alpha) + G_{kj}(\eta_r - i\eta_0, \alpha)]\\ &- (\eta_0 \leftrightarrow -\eta_0).\end{aligned} \tag{3.136}$$

These equations are tricky to be seen, and some manipulation is needed. The best way to tackle this derivation is to divide it by parts: first, let us analyze $(w_0 + i\partial_t)A(t)$:

$$\left(1+\frac{i}{\omega_0}\partial_t\right)\frac{\omega_0^2}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\eta\frac{\mathrm{e}^{i\omega_0\eta t}}{\zeta(\omega_0\eta)} \overset{\eta\to-\eta}{=} \left(1+\frac{i}{\omega_0}\partial_t\right)\frac{\omega_0^2}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}\eta\frac{\mathrm{e}^{-i\omega_0\eta t}}{\zeta^*(\omega_0\eta)} = \sum_k \mathcal{R}_k^*(\eta_k^*+1)\mathrm{e}^{-i\omega_0\eta_k^* t}. \tag{3.137}$$

One may note that we ignored one of the exponential in the first step. In fact, this is due to the contour that we used. The exponential that was ignored had negative argument, and would not converge in the lower half plane. Thus, if we choose a upper semi circle, the function $\zeta(\omega_0\eta)$ has no roots, and therefore the residue is zero.

For the other integrals, we use the following identity, which can be easily derived just by using partial fractions and the special functions $\mathrm{E}_i(ix)$, $\mathrm{Si}(z)$ and $\mathrm{Ci}(z)$:

$$\begin{aligned}\int_0^{\infty}\mathrm{d}\eta\frac{\eta}{(\eta-a)(\eta+b)}\frac{\mathrm{e}^{-i\eta t}}{\eta+c} &= \frac{1}{(a+b)(a+c)}\Bigg[c\,\mathrm{e}^{ict}\left(-\mathrm{Ci}(ct)-i\frac{\pi}{2}+i\mathrm{Si}(ct)\right)+\\ &\quad + a\,\mathrm{e}^{-iat}\left(-\mathrm{Ci}(-at)-i\frac{\pi}{2}+i\mathrm{Si}(-at)\right)\Bigg]+\\ &\quad + \frac{1}{(a+b)(b-c)}\Bigg[c\,\mathrm{e}^{ict}\left(-\mathrm{Ci}(ct)-i\frac{\pi}{2}+i\mathrm{Si}(ct)\right)-\\ &\quad - b\,\mathrm{e}^{ibt}\left(-\mathrm{Ci}(bt)-i\frac{\pi}{2}+i\mathrm{Si}(bt)\right)\Bigg]\\ &= \frac{-cg(ct)+ag(-at)}{(a+b)(a+c)}+\frac{-cg(ct)-bg(bt)}{(a+b)(b-c)}.\end{aligned} \tag{3.138}$$

Before we tackle the remaining integrals, let us analyze one more property of $\zeta(\omega_0\eta)$ that will be useful when we deal with (3.99):

$$\frac{\omega_0^2}{\zeta^*(\omega_0\eta)} = \frac{\omega_0^2}{\zeta(-\omega_0\eta)} = \frac{-\omega_0^2}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta'}{\zeta(\omega_0\eta')(\eta'+\eta+i\epsilon)} = \sum_i \frac{\mathcal{R}_{\eta_i}^*}{\eta-\eta_i^*}, \tag{3.139}$$

and, in the same way, one finds a similar relation for its conjugate:

$$\frac{\omega_0^2}{\zeta(\omega_0\eta)} = \frac{\omega_0^2}{\zeta^*(-\omega_0\eta)} = \frac{\omega_0^2}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta'}{\zeta^*(\omega_0\eta')(\eta'+\eta+i\epsilon)} = \sum_i \frac{\mathcal{R}_{\eta_i}}{\eta-\eta_i}, \tag{3.140}$$

in such a way that

$$\begin{aligned}\omega_0^2 B_{\omega_0\eta}(t) &= \sum_i \frac{\mathcal{R}_{\eta_i}^*}{\eta-\eta_i^*}\mathrm{e}^{-i\omega_0\eta_i^* t}\\ \omega_0^2 B_{\omega_0\eta}^*(t) &= \sum_i \frac{\mathcal{R}_{\eta_i}}{\eta-\eta_i}\mathrm{e}^{i\omega_0\eta_i t}.\end{aligned} \tag{3.141}$$

Let us now analyze

$$\frac{2\omega_0}{\pi}\int_0^{\infty}\zeta_i(\omega)\frac{\mathrm{e}^{-i\omega t}}{\zeta^*(\omega)}B_{\omega}^*(t'). \tag{3.142}$$

By recalling the value of the imaginary part of $\zeta(\omega)$:

$$\zeta_i(\omega) = \frac{\pi\omega\beta^2(\omega)}{2m\mu}, \tag{3.143}$$

substituting $\beta^2(\omega_0\eta)$, separating it by partial fractions and using the expressions that we just derived, we have

$$\begin{aligned}\cdots = -\frac{\sigma^2}{\pi}\sum_{i,j}\mathcal{R}^*_{\eta_i}\mathcal{R}_{\eta_j}\mathrm{e}^{i\eta_j\omega_0 t'}\frac{1}{2i}\int_0^\infty \mathrm{d}\eta\frac{\eta\mathrm{e}^{-i\eta\omega_0 t}}{(\eta-\eta_i^*)(\eta-\eta_j)}\Bigg(\frac{1}{\eta-\eta_r-i\eta_0}-\frac{1}{\eta-\eta_r+i\eta_0}+\\ +\frac{1}{\eta+\eta_r-i\eta_0}-\frac{1}{\eta+\eta_r+i\eta_0}\Bigg).\end{aligned} \tag{3.144}$$

In hands of (3.138) and our auxiliary functions (3.136), we get

$$\frac{2\omega_0}{\pi}\int_0^\infty \zeta_i(\omega)\frac{\mathrm{e}^{-i\omega t}}{\zeta^*(\omega)}B^*_\omega(t') = \frac{\sigma^2}{\pi}\sum_{i,j}\mathcal{R}^*_{\eta_i}\mathcal{R}_{\eta_j}\mathrm{e}^{i\eta_j\omega_0 t'}F_{ij}(\eta_r,\eta_0,\omega_0 t), \tag{3.145}$$

and by the same reasoning

$$\frac{2\omega_0}{\pi}\int_0^\infty \zeta_i(\omega)\frac{\mathrm{e}^{i\omega t'}}{\zeta(\omega)}B_\omega(t) = \frac{\sigma^2}{\pi}\sum_{i,j}\mathcal{R}_{\eta_i}\mathcal{R}^*_{\eta_j}\mathrm{e}^{-i\eta_j^*\omega_0 t}F^*_{ij}(\eta_r,\eta_0,\omega_0 t'). \tag{3.146}$$

The last term follows from (3.141):

$$\begin{aligned}\frac{2\omega_0}{\pi}\int_0^\infty \mathrm{d}\omega\zeta_i(\omega)B_\omega(t)B^*_\omega(t') = \frac{\sigma^2}{\pi}\sum_{i,j}\mathcal{R}^*_{\eta_i}\mathcal{R}_{\eta_j}\mathrm{e}^{-i\eta_i^*\omega_0 t+i\eta_j\omega_0 t'}\times\\ \frac{1}{2i}\int_0^\infty \mathrm{d}\eta\frac{\eta}{(\eta-\eta_i^*)(\eta-\eta_j)}\Bigg(\frac{1}{\eta-\eta_r-i\eta_0}-\frac{1}{\eta-\eta_r+i\eta_0}+\\ +\frac{1}{\eta+\eta_r-i\eta_0}-\frac{1}{\eta+\eta_r+i\eta_0}\Bigg) = \frac{\sigma^2}{\pi}\sum_{i,j}\mathcal{R}^*_{\eta_i}\mathcal{R}_{\eta_j}\mathrm{e}^{-i\eta_i^*\omega_0 t+i\eta_j\omega_0 t'}F_{ij}(\eta_r\eta_0,0),\end{aligned} \tag{3.147}$$

which concludes our derivation.

Equations (3.124) and (3.133) represent one of the main results in our work. We stress that although the damped harmonic oscillator is one of the most studied systems in quantum optics, the above equations represent an exact solution for the Wightman function in such systems, that can be used to unveil various quantum features of interest.

3.5 Quantum Brownian Motion

We are now able to discuss in detail the quantum Brownian motion. For definiteness, we assume in this work that before the quench the particle and the reservoir oscillators are in their fundamental state, at zero temperature. Accordingly, $\langle x\rangle \equiv 0$ and $\langle v\rangle \equiv 0$ throughout the system evolution. We are interested in determine how the oscillator changes as it starts to interact with the environment. Thus, if $\mathcal{E}_0$ is the particle energy at $t < 0$, then $\Delta\mathcal{E} = \langle H_\mathrm{p}\rangle - \mathcal{E}_0$ probes how much the particle energy changed. Similarly, if $\mathcal{T}_0$ is the particle kinetic energy at $t < 0$, $\Delta\mathcal{T} = \langle T\rangle - \mathcal{T}_0$ is the corresponding change due to the reservoir.

3.5.1 Late-time regime

The parameters η_0, η_r and σ determine how the particle achieves the late-time regime, when transients effects already took place. According to Eq. (3.15), the change in particle energy, on average, after the system reaches its equilibrium is $\Delta\mathcal{E}_{\text{asy}} = \lim_{t\to\infty} \Delta\mathcal{E}$, which reads, with the aid of the correlation function (3.124),

$$\begin{aligned}\Delta\mathcal{E}_{\text{asy}} =& \frac{m}{2}\lim_{t'\to t}(\partial_t\partial_{t'} + \omega_0^2)\langle x(t)x(t')\rangle_{\text{qp}} - \frac{\omega_0}{2}\\ =& \frac{\omega_0}{2}\left[\frac{-i}{\pi}\sum_j \mathcal{R}_{\eta_j}(1+\eta_j^2)\ln(-i\eta_j) - 1\right],\end{aligned} \tag{3.148}$$

whereas the change in the particle kinetic energy, $\Delta\mathcal{T}_{\text{asy}} = \lim_{t\to\infty}\Delta\mathcal{T}$, reads

$$\Delta\mathcal{T}_{\text{asy}} = \frac{\omega_0}{2}\left[\frac{-i}{\pi}\sum_j \mathcal{R}_{\eta_j}\eta_j^2\ln(-i\eta_j) - \frac{1}{2}\right]. \tag{3.149}$$

Figures 5 and 6 depict $\Delta\mathcal{E}_{\text{asy}}$ and $\Delta\mathcal{T}_{\text{asy}}$, respectively, as functions of η_0 and η_r, for a fixed $\sigma = 1$. Figure 5 shows that the particle energy always increases due to the interaction with the reservoir and this effect is more pronounced for $(\eta_0, \eta_r) \sim (0, 1)$, i.e., near the resonance [cf. Eq. (3.106)].

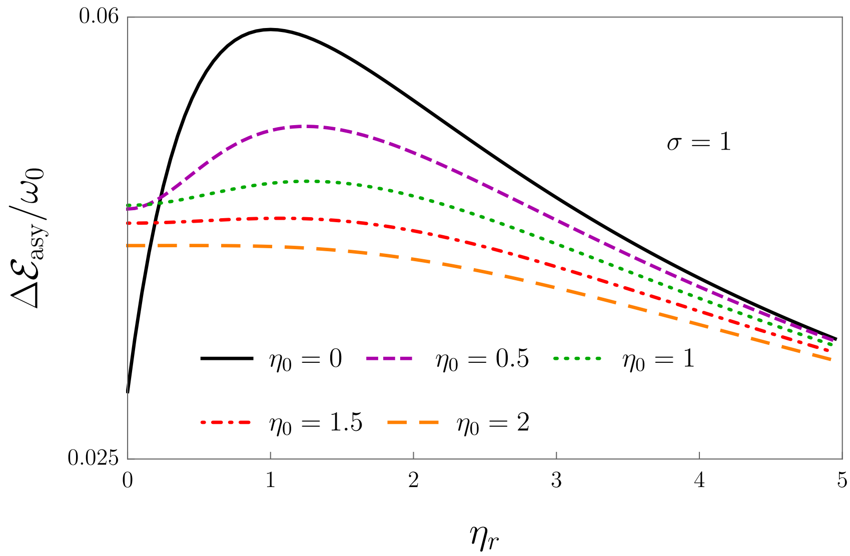

Figure 5 – Late-time behavior of the particle energy. Here it is assumed that $\sigma = 1$. Notice that the particle energy gain is maximum near $(\eta_0, \eta_r) \sim (0, 1)$, which corresponds to a resonance with the reservoir oscillator of frequency $\nu = \omega_0$. Also, note that the energy exchange is minimum for $\eta_0 = \eta_r \sim 0$, the regime in which the particle exchange energy only with the low energy reservoir oscillators.

Also, although it is fairly intuitive that the particle energy gain is maximum around the resonance, Fig. 5 also reveals that the energy gain is minimum (for the parameter window shown in the figure) near $\eta_0 = \eta_r \sim 0$. We attribute this to the fact that in this regime the particle is allowed to exchange energy mainly with the low energy reservoir oscillators.

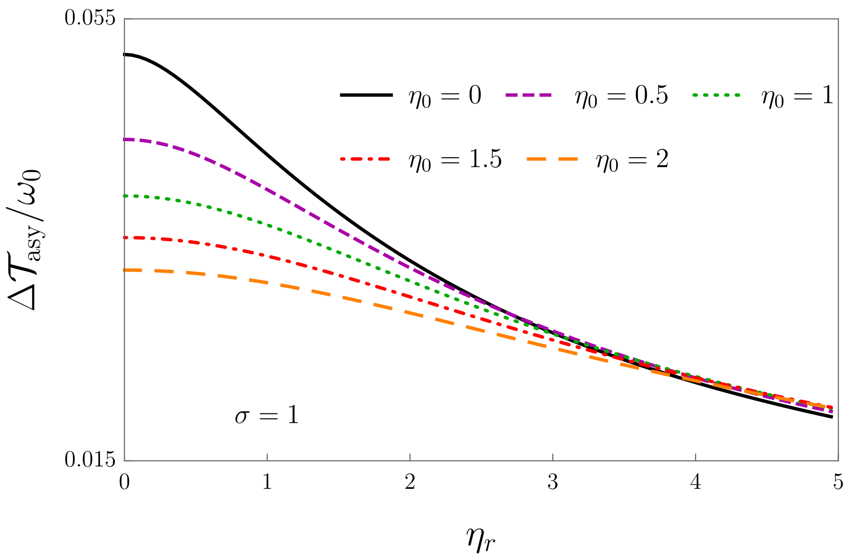

Figure 6 – Late-time behavior of the particle kinetic energy, for $\sigma = 1$. Notice that, differently from the total energy, the particle kinetic energy is maximum at around $\eta_0 = \eta_r \sim 0$, and it is insensitive to the resonance at $\eta_r = 1$.

Figure 6 depicts the late-time behavior of the particle kinetic energy exchange. Similarly to the total particle energy, the kinetic energy always increases due to the interaction with the reservoir. Moreover, interesting features are observed in sharp distinction to the total energy: the particle kinetic energy is insensitive to the resonance at $\eta_r = 1$ and it is maximum near $\eta_0 = \eta_r \sim 0$, where the particle interacts mostly with low energy reservoir oscillators.

3.5.2 Transient regime

We now consider the transient regime after the quantum quench. We start with some remarks about the transient time duration as function of the parameters σ, η_0 and η_r. Inspection of Eq. (3.133) shows that the relaxation times are determined by the imaginary parts of the roots η_j, and these roots are solution of the degree four polynomial equation

$$(1 - \eta^2)[(\eta - i\eta_0)^2 - \eta_r^2] + \sigma^2\eta(\eta - i\eta_0) = 0. \tag{3.150}$$

Moreover, the contribution of a given root η_j to the transient correlation is modulated by the residue $\mathcal{R}_{\eta_j}$, and thus an interesting interplay between these quantities occur.

Proposition 8 *For $\sigma \to 0$ (weak coupling regime), the solutions of Eq. (3.150) read*

$$\eta = \pm 1 + \frac{\sigma^2}{2}\frac{\pm 1 - i\eta_0}{(\pm 1 - i\eta_0)^2 - \eta_r^2} + \mathcal{O}(\sigma^4), \tag{3.151}$$

$$\eta = \pm\eta_r + i\eta_0 + \frac{\sigma^2}{2}\frac{\pm\eta_r + i\eta_0}{(\pm\eta_r + i\eta_0)^2 - 1} + \mathcal{O}(\sigma^4), \tag{3.152}$$

whereas for $\sigma \to \infty$ (strong coupling), we find

$$\eta = \pm\sigma + \frac{i\eta_0}{2} + \mathcal{O}(\sigma^{-1}), \tag{3.153}$$

$$\eta = \frac{i}{\sigma^2}\frac{\eta_0^2 + \eta_r^2}{\eta_0} + \mathcal{O}(\sigma^{-3}), \tag{3.154}$$

$$\eta = i\eta_0 - \frac{i}{\sigma^2}\frac{\eta_r^2(\eta_0^2 + 1)}{\eta_0} + \mathcal{O}(\sigma^{-3}). \tag{3.155}$$

Proof:

We begin with $\sigma \to 0$. The unperturbed equation, i.e., with $\sigma = 0$, is

$$(1 - \eta^2)[(\eta - i\eta_0)^2 - \eta_r^2] = 0, \tag{3.156}$$

which give us the four roots $\eta = \pm 1$ and $\eta = i\eta_0 \pm \eta_r$. In order to look for perturbed roots, we write $\eta = \eta^{(0)} + \sigma^2\eta^{(2)} + O(\sigma^4)$. Let

$$g(\eta) = (1 - \eta^2)[(\eta - i\eta_0)^2 - \eta_r^2] \quad \text{and} \quad h(\eta) = \eta(\eta - i\eta_0), \tag{3.157}$$

such that our problem can be written as

$$g(\eta) + \sigma^2 h(\eta) = 0. \tag{3.158}$$

Expanding until order σ^2, we get

$$\cancelto{0}{g(\eta^{(0)})} + \sigma^2\eta^{(2)}g'(\eta^{(0)}) + \sigma^2 h(\eta^{(0)}) = 0 \implies \eta^{(2)} = -\frac{h(\eta^{(0)})}{g'(\eta^{(0)})}. \tag{3.159}$$

Thus, substituting the unperturbed roots, we have

$$\begin{aligned} \eta^{(0)} = \pm 1 &\implies \eta^{(2)} = \frac{\pm 1 - i\eta_0}{2[(\pm 1 - i\eta_0)^2 - \eta_r^2]} \\ \eta^{(0)} = i\eta_0 \pm \eta_r &\implies \eta^{(2)} = \frac{\pm\eta_r + i\eta_0}{2[(\pm\eta_r + i\eta_0)^2 - 1]}. \end{aligned} \tag{3.160}$$

For $\sigma \to \infty$, we could try to derive a general formula as we did for $\sigma \to 0$, but this would not be a good idea, since no convergence would be assured. It is better to look for roots in order σ and $\frac{1}{\sigma^2}$, i.e., do a Laurent series expansion (see section 2.4.3). Thus, let us write $\eta = a\sigma + b$. If we substitute this in the original polynomial and collect σ^4 terms, we end up with

$$\sigma^4 a^2(1 - a^2) = 0 \implies a = \pm 1, \tag{3.161}$$

and collecting σ^3 terms

$$\sigma^3 a(2b - i\eta_0) = 0 \implies b = \frac{i\eta_0}{2}. \tag{3.162}$$

Lesser orders in sigma will give contributions of $O(\sigma^{-1})$. These are called the large roots. If we look for $\eta = \eta^{(0)} + \frac{\eta^{(2)}}{\sigma^2} + \cdots$, where $\eta^{(0)}$ are the roots of the dominant part, i.e.,

$$\eta(\eta - i\eta_0)\Big|_{\eta^{(0)}} = 0 \implies \eta^{(0)} = 0 \quad \text{and} \quad \eta^{(0)} = i\eta_0. \tag{3.163}$$

Now, we just substitute in the polynom and collect σ terms:

$$
\begin{aligned}
\eta^{(0)} = 0 \implies (\eta_0^2 + \eta_r^2) + i\eta_0\eta^{(-2)} = 0 \therefore \eta^{(-2)} = i\frac{\eta_0^2 + \eta_r^2}{\eta_0} \\
\eta^{(0)} = i\eta_0 \implies -\eta_r^2(1 + \eta_0^2) + i\eta_0\eta^{(-2)} = 0 \therefore \eta^{(-2)} = -i\frac{\eta_r^2(1 + \eta_0^2)}{\eta_0}.
\end{aligned}
\tag{3.164}
$$

■

Thus, these formulas can be used to determine the relaxation times in each of the asymptotic regimes. For any $\sigma > 0$, the minimum among the imaginary parts of the η_j, when multiplied by ω_0, determines the system relaxation time. Figure 7 depicts the imaginary parts of the roots for $\sigma = 1$.

Note that, in general, the relaxation time increases as η_0 goes to zero. An example of this behavior is depicted in Fig. 8 upper panel, for $\eta_r = 0$, $\sigma = 1$. One can see from the figure that as the interaction is turned on, the particle energy acquires a damped oscillatory behavior as function of time, with a frequency determined by σ. Figure 8 middle panel depicts the particle energy for the same η_0 and η_r parameters and for the stronger coupling of $\sigma = 5$. Notice that both the frequency of the oscillations and their magnitude increase as σ grows.

We also present in Fig. 8 bottom panel the particle energy as function of time for several values of η_r and for $\eta_0 = 0.5$, $\sigma = 1$, from which one can see that η_r, which determines the frequency of the reservoir oscillators that are in resonance with the particle, also affects the oscillatory pattern.

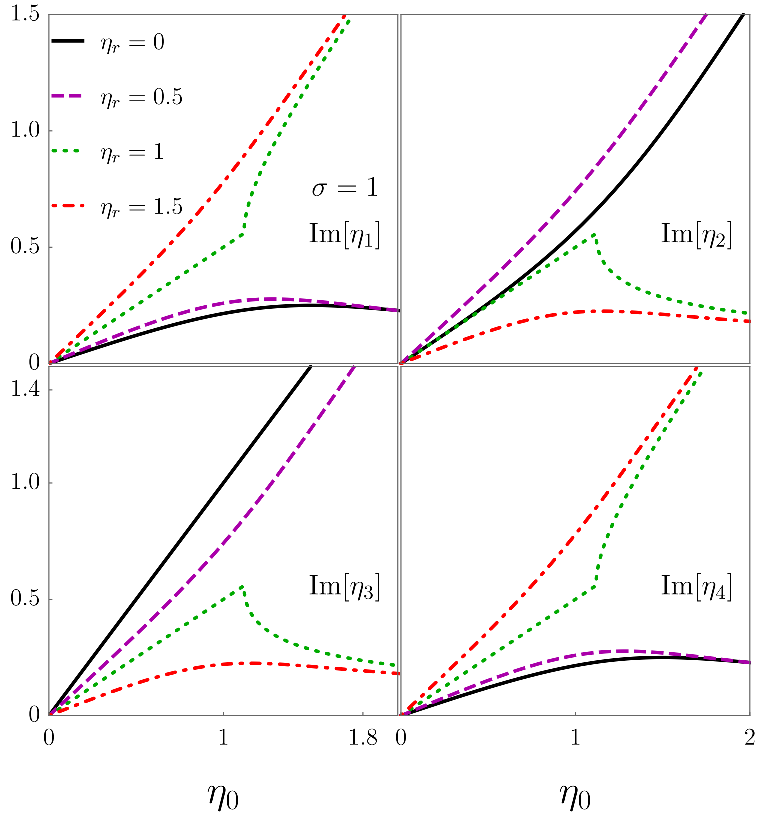

Figure 7 – Imaginary parts of the roots of Eq. (3.150) for $\sigma = 1$. The labels of the roots are not relevant for our analysis. Note that as $\eta_0 \to 0$, the imaginary parts vanish, meaning that the system takes longer to reach equilibrium for smaller η_0.

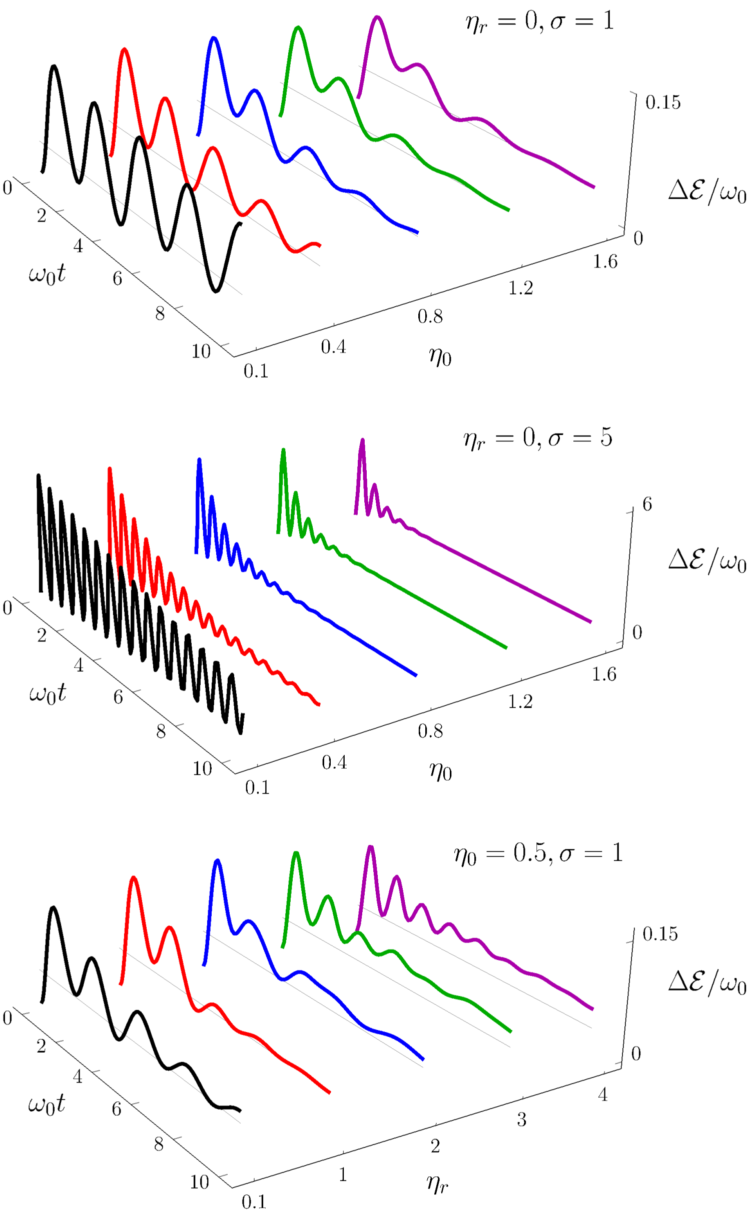

Figure 8 – Particle energy variation as function of time. Top panel: Particle energy for $\eta_r = 0$ and $\sigma = 1$. Notice that the relaxation time decreases with η_0 (for fixed σ). Middle panel: Particle energy for $\eta_r = 0$ and $\sigma = 5$. Notice that the amplitude and the frequency of the oscillations increase with σ. Bottom panel: Particle energy for $\eta_0 = 0.5$ and $\sigma = 1$. The resonance parameter η_r has a non-trivial effect on the particle energy.

As a final application of the analytical correlations, we present in Fig. 9 how the particle kinetic energy changes during the transient regime. Recall that in the semi-classical quantum Brownian motion discussed in [13], the particle kinetic energy, which is assumed to be initially zero, can actually become negative due to the quantum fluctuations of an external field. In our model, which contain an analytical solution to the quantum Brownian motion, the analogue effect is the diminishing of the particle's initial (positive) kinetic energy.

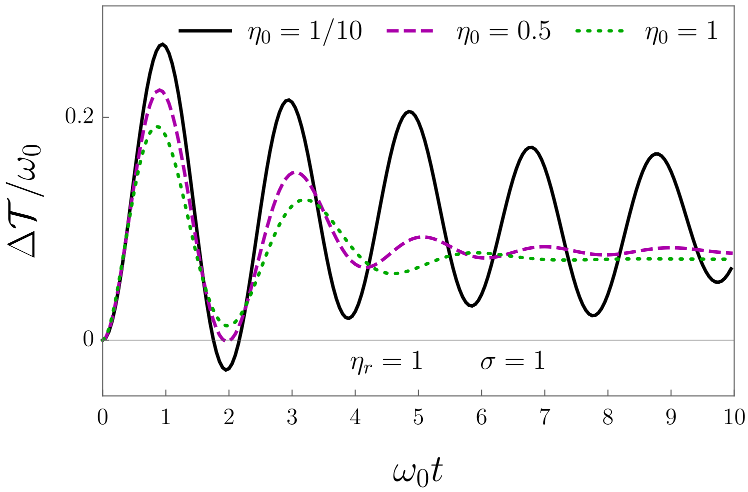

Figure 9 – Particle kinetic energy variation as function of time, for $\eta_r = \sigma = 1$ and several values of η_0. Notice that depending on the system parameters, the particle can actually loose some of its initial kinetic energy, in analogy to the subvacuum effect of [13].

Figure 9 shows a situation where this occurs, for $\eta_0 = 0.1, \eta_r = \sigma = 1$. For these parameters, the particle, after interacting with the reservoir for some period of time, actually looses part of its kinetic energy. We note that in this system this is a true quantum effect, whose origin can be traced back to the phenomenon of quantum squeezing [51]. Indeed, note that the initial kinetic energy decreases if and only if $\langle p^2 \rangle$ decreases, at the expense of increasing $\langle x^2 \rangle$ to ensure the validity of Heisenberg's uncertainty relation.

We finish this section with a remark regarding the conservation of energy in this system. Figure 8 shows that the particle, for the parameters considered in the plots, always gain energy. There are two possible origins to this energy gain, namely, the particle can extract energy from the reservoir and it can gain energy from the external agent that turns the interaction on at $t = 0$. Here, because the microscopic model is known, it is possible to pinpoint exactly the source of the particle energy gain. Indeed, it follows from Eq. (3.20) that the particle and the reservoir energies, $\langle H_{\rm p} \rangle$ and $\langle H_{\rm R} \rangle$, respectively, are related through

$$\frac{\mathrm{d}\langle H_{\rm p} \rangle}{\mathrm{d}t} + \frac{\mathrm{d}\langle H_{\rm R} \rangle}{\mathrm{d}t} = -\int_0^\infty \mathrm{d}\nu \dot{\beta} \langle x \dot{R} \rangle. \tag{3.165}$$

Therefore, because the rhs of the above equation is zero for $t \neq 0$ and $H_{\rm p}$ is a continuous function of time, it follows that the energy gained by the particle comes exclusively from its interaction with the reservoir oscillators.

4 Final Remarks

In this work we studied how a reservoir modifies the quantum Brownian motion of a particle. By starting from a Lagrangian model for the total system, we were able to quantize the system canonically and obtain analytical solutions for the quantum correlations. The major results in our work are the analytical correlations and their application to characterize the quantum Brownian motion of a particle.

A couple of important remarks are in order. We note that the linear coupling between the particle and the reservoir is an important assumption in order to find analytical solutions. In general, for non-linear couplings the system cannot be quantized analytically and methods to find approximate solutions are necessary. For instance, for the quantum Brownian motion of [13] the particle interacts non-linearly with the electric field, and the assumption of negligible particle displacement was implemented in order to find approximate solutions.

Also, we note that the reservoir model here implemented serves as an analogue model to the electric field of [13], to the extent that it can simulate important features like the subvacuum effect and gives information about the late-time regime. However, this analogy is only qualitative, for the system of [13] is a Casimir-like system imprinting velocity fluctuations onto a charged particle, for which, due to its complexity, still remains not thoroughly studied.

We conclude this work with a remark regarding the generality of the two-point function (3.73). The analysis of the quantum Brownian motion was perform for a particular type of interaction given by the two Lorentzians of Eq. (3.105), that leads to analytical solutions. Nevertheless, the correlations (3.73) can be used to study quantum properties of the particle for all types of couplings as long as $\beta \to 0$ when $\nu \to 0$, which is a consistency condition for the function ζ given in Eq. (3.39) to be well-defined for all $\omega > 0$.

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